BELYI FUNCTION ON $X_0(49)$ OF DEGREE 7

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Appendix to: “The Belyi functions and dessin d’enfants corresponding to the non-normal inclusions of triangle groups” by K. Hoshino

KENJI HOSHINO AND HIROAKI NAKAMURA

In [H] §4, we identified two types of the non-normal inclusions of triangle groups (type A and type C from Singerman’s list [S]) as those corresponding to subcovers of the Klein quartic. N. D. Elkies [E] closely studied those subcovers in view of modular curves, i.e., as subcovers under the elliptic modular curve $X_7 = X(7)$ identified with the Klein quartic defined by $X^3Y + Y^3Z + Z^3Y = 0$. Let $\mathbb{C}(X_7)$ be the function field of $X_7$ generated by $y := Y/X$ and $z := Z/X$ with a relation $y + y^3z + z^3 = 0$, and consider two automorphisms of $\mathbb{C}(X_7)$ defined by

$$\sigma : \begin{cases} y \mapsto \zeta^3y, \\ z \mapsto \zeta z, \end{cases} \quad \tau : \begin{cases} y \mapsto 1/z, \\ z \mapsto y/z, \end{cases}$$

where $\zeta := e^{2\pi i/7}$. Then, $\sigma$ and $\tau$ are automorphisms respectively of order 7 and 3, and they form an automorphism group $H$ of $\mathbb{C}(X_7)$ of order 21, a subgroup of the full automorphism group $G$ of order 168 of the Klein quartic. In this respect, the genus zero covers of type A and type C of [S] are respectively $X_1(7) \to X(1)$ and $X_0(7) \to X(1)$ arising from the inclusion relations of $\langle \sigma \rangle \subset H \subset G$. Remarkably, Elkies [E] studied deep arithmetic properties of a genus one subcover $E$ fixed by $\langle \tau \rangle$ with showing that $E$ is $\mathbb{Q}$-isomorphic to $X_0(49)$. Especially, he explicitly presented its function field $\mathbb{C}(E)$ as $\mathbb{C}(E) = \mathbb{C}(u, v)$ with $v^2 = 4u^3 + 21u^2 + 28u$, where

$$u = -\frac{(y + z + yz)^2}{(1 + y + z)yz}, \quad v = -\frac{(2 - y - z + 2y^2 - yz + 2z^2)(y + z + yz)}{yz(1 + y + z)}$$

Recalling also from [E] that standard coordinates of $X_1(7) \cong \mathbb{P}^1_t$ and $X_0(7) \cong \mathbb{P}^1_s$ may be given as

$$t := -y^2z, \quad s := t + \frac{1}{1-t} + \frac{t-1}{t},$$

we would like to interpret the degree 7 cover $E \to X_0(7)$ arising from $\langle \tau \rangle \subset H$ by expressing $s$ by $u, v$ explicitly. In this note, we show

**Proposition A.** Notations being as above, the covering of $\mathbb{P}^1_t$ by the elliptic curve $E : v^2 = 4u^3 + 21u^2 + 28u$ is ramified only above $s = 3\rho, 3\rho^{-1}, \infty$ (where $\rho = e^{2\pi i/6}$), and the equation is given by

$$s = \frac{1}{2} \left\{ (u^2 + 7u + 7)v + (7u^3 + 35u^2 + 49u + 16) \right\}.$$
Thus \( \beta = \frac{s-3\rho}{3\rho-1-3\rho} \) gives a Belyi function (i.e., unramified outside \( \beta = 0, 1, \infty \)) of degree 7 on \( E \) with valency list \([331, 331, 7]\).

In effect, one can induce an isomorphism of covers

\[
X_0(49) \sim \rightarrow E \\
\downarrow \\
X_0(7) \sim \rightarrow X_0(7)
\]

from the conjugacy by the ‘Fricke involution’ \( \pm \frac{1}{\sqrt{7}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) between the modular group \( \Gamma_0(49) \) and \( \{ \pm(\zeta_d^b) \in \text{PSL}_2(\mathbb{Z}) : b \equiv c \equiv 0 \mod 7 \} \) in \( \text{PSL}_2(\mathbb{R}) \) (cf. [E] p.90). Therefore, the computation of \( E \rightarrow X_0(7) \) may be reduced to combining classically well known equations that relate \( X_0(7), X_0(49) \) with the \( J \)-line \( X(1) \) found in, e.g., [F] pp.395–403. Here, however, we shall employ an alternative enjoyable discussion following the Elkies scheme:

\[
X_1(7) = \mathbb{P}^1_s \leftarrow \frac{f}{(\sigma)} X_7 \\
\downarrow g \\
\downarrow \langle \tau \rangle q \\
X_0(7) = \mathbb{P}^1_s \leftarrow \frac{p}{E}.
\]

First of all, since \( \frac{ds}{dt} = \frac{(t^2-t+1)^2}{(t-1)^2t^2} \), the 3-cyclic cover \( g \) is ramified only over \( s = 3\rho, 3\rho^{-1} \) at \( t = \rho, \rho^{-1} \), and the fiber over \( s = \infty \) is formed by the three points \( t = 0, 1, \infty \). We next chase the fibers of \( f \) over these points and their images in \( E \) by \( q \). Noticing that \( \mathbb{C}(X_7) = \mathbb{C}(y, z) = \mathbb{C}(y, t) \) with \( y^7 = \frac{t^3}{1-t} \), we see that \( f \) is totally ramified over \( t = 0, 1, \infty \), and their images by \( q \) coincide at the infinity point on \( E : v^2 = 4u^3 + 21u^2 + 28u \). From this follows that \( p : E \rightarrow \mathbb{P}^1_s \) is totally ramified at the infinity point of \( E \) over \( s = \infty \), hence \( s \) is of the form \( s = F(u) + G(u)v \) with \( F, G \in \mathbb{C}[u], \deg(F) = 3, \deg(G) = 2 \).

The fiber of \( f \) over \( t = \rho \) forms one orbit under the action of \( \sigma (y \mapsto zy) \) whose points are represented by the set of their \( y \)-coordinates \( S_\rho := \{ \xi^2, \xi^5, \xi^8, \xi^{11}, \xi^{14}, \xi^{17}, \xi^{20} \} \), where \( \xi := e^{2\pi i/21} \). The action of \( \tau \) preserves \( \sigma \), and transforms those \( y \)-coordinates (over \( t = \rho \)) as \( y \mapsto y^2\xi^{-14} \), hence decomposes \( S_\rho \) into the three \( \tau \)-orbits \( S^1_\rho := \{ \xi^2, \xi^{11}, \xi^8 \}, S^2_\rho := \{ \xi^5, \xi^{17}, \xi^{20} \} \), and \( S^3_\rho := \{ \xi^{14} = \rho^4 \} \) (This also explains the branch type of \( p \) over \( s = 3\rho \) is ‘331’). Then we compute the \( u \)-coordinates \( u_1, u_2, u_3 \) of the images of these orbits by \( q \) after the formula (A2); it turns out that \( u_1 = -\xi^2 - \xi^{11} - \xi^8 - 1, u_2 = -\xi^7 + \xi^2 + \xi^{11} + \xi^8 - 2 \) and \( u_3 = 3\rho - 1 \). Eliminating \( v \) from \( F(u) + \)
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$G(u)v = 3\rho$, we should then have an equation of the form

$$(4u^3 + 21u^2 + 28u)G(u)^2 - (F(u) - 3\rho)^2 = (u - u_1)^3(u - u_2)^3(u - u_3).$$

Using a symbolic computation software such as MAPLE to compare the coefficients of the both sides above (plus a slight more consideration of signs), we conclude $F(u) = \frac{1}{2}(7u^3 + 35u^2 + 49u + 16)$, $G(u) = \frac{1}{2}(u^2 + 7u + 7)$ as stated in Proposition A. □

Remark. The monodromy representation associated with the above Belyi function $\beta$ is given by $x = (142)(356)(7)$, $y = (175)(346)(2)$, $z = (1234567)$. The following picture illustrates the uniformization of this Grothendieck dessin.

REFERENCES


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