On an AACDMZ Question

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Let $D$ be a (commutative) integral domain with quotient field $K$. Let $F(D)$ denote the set of nonzero fractional ideals of $D$ and let $f(D)$ be the subset of finitely generated members of $F(D)$. For each $A \in F(D)$, we set $D_i A = A^{-1}$ and $(A^{-1})^{-1} = A_v$. The function on $F(D)$ defined by $A \mapsto A_v$ is called the $v$-operation on $D$. If for each $A \in f(D)$, there exists a $B \in F(D)$ with $(AB)_v = D$, then $D$ is called a $v$-domain. If there is a set of prime ideals $\{P_i | i \in I\}$ of $D$ such that $D = \bigcap_{i \in I} D_{P_i}$ and each $D_{P_i}$ is a valuation domain, then $D$ is called an essential domain. [1] investigated characterizations of $v$-domains and related properties. Among other Theorems it proved the following,

**Theorem 1** ([1, Theorem 7]).
(1) If $D$ is an essential domain, then

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$

for all $A_1, \ldots, A_n \in f(D)$.

(2) If $D$ is integrally closed and

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$

for all $A_1, \ldots, A_n \in f(D)$, then $D$ is a $v$-domain.

Relating with Theorem 1 it posed the following,

**Question** ([1, p.7]). Does any $v$-domain $D$ satisfy

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$

for all $A_i \in f(D)$?

The aim of this paper is to give an affirmative answer to the question. We will prove the following,

**Theorem 2.** Let $D$ be a $v$-domain. Then we have

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$
for all $A_i \in f(D)$.

First we recall the definition and some properties of the Kronecker function ring of $D$ with respect to the v-operation. Let $D[X]$ be the polynomial ring of an indeterminate $X$ over $D$. For each $f \in K[X]$, we denote the fractional ideal of $D$ generated by the coefficients of $f$ by $c(f)$.

**Lemma 3** (cf. [2,(32.7)]). Let $D$ be a v-domain. Set

$$D^v = \{0\} \cup \{f/g \mid f, g \in D[X] - \{0\}\} \quad \text{and} \quad c(f)_v \subseteq c(g)_v.$$  

Then,

1. $D^v$ is a domain with quotient field $K(X)$.
2. If $A$ is a nonzero finitely generated ideal of $D$, then $AD^v \cap K = A_v$.

$D^v$ is called the Kronecker function ring of $D$ with respect to the v-operation.

**Lemma 4.** Let $D$ be a v-domain. Let $a \in K - \{0\}$ and $C \in F(D)$. If $aA_v \subseteq B_v$ and $BA^{-1} \subseteq C$ are satisfied for some $A \in f(D)$ and some $B \in F(D)$, then $a \in C_v$.

**Proof.** We note that $(AA^{-1})_v = D$, since $D$ is a v-domain. Then we have

$$a \in a( AA^{-1} )_v = (aA_v A^{-1} )_v \subseteq (B_v A^{-1} )_v = (BA^{-1} )_v \subseteq C_v.$$  

**Proof of Theorem 2.** Let $D$ be a v-domain with quotient field $K$. Let $D^v$ be the Kronecker function ring of $D$ with respect to the v-operation. Let $A_1, \ldots, A_n \in f(D)$. Choose elements $a_{i1}, \ldots, a_{ik(i)}$ of $K - \{0\}$ such that $A_i = (a_{i1}, \ldots, a_{ik(i)})D$ for $1 \leq i \leq n$. We set

$$f_i = a_{i1}X + a_{i2}X^2 + \cdots + a_{ik(i)}X^{k(i)}$$  

for $1 \leq i \leq n$. Since, for each $j$, $a_{ij}/f_i \in D^v$, we have $A_i D^v = f_i D^v$ for $1 \leq i \leq n$. Set $h_i = f_1 \cdots f_{i-1}f_{i+1} \cdots f_n$, and let $d(i)$ denote the degree of $h_i$ for $1 \leq i \leq n$. We set

$$h_1 + h_2 X^{d(1)} + h_3 X^{d(1)+d(2)} + \cdots + h_n X^{d(1)+\cdots+d(n-1)} = g.$$  

Since, for each $j, h_j/g \in D^v$, it immediately follows that $(h_1, \ldots, h_n)D^v = gD^v$, and so

$$(1/f_1, \ldots, 1/f_n)D^v = (g/(f_1 \cdots f_n))D^v.$$
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By taking the inverses, we see that

\[ f_1 D^u \cap \cdots \cap f_n D^u = ((f_1 \cdots f_n)/g)D^u. \]

Now let \( 0 \neq a \in (A_1)_v \cap \cdots \cap (A_n)_v \). Then we have

\[ a \in f_1 D^u \cap \cdots \cap f_n D^u = ((f_1 \cdots f_n)/g)D^u. \]

It follows \( ag/(f_1 \cdots f_n) \in D^u \). Hence we have \( ac(g)_v \subset c(f_1 \cdots f_n)_v \). On the other hand, we have

\[ c(f_1, \cdots, f_n) c(g)^{-1} \subset A_1 \cap \cdots \cap A_n, \]

since for each \( i \),

\[
\begin{align*}
    c(f_1, \cdots, f_n) c(g)^{-1} &= c(f_i h_i)(c(h_1) + \cdots + c(h_n))^{-1} \\
    &\subset c(f_i h_i) c(h_i)^{-1} \subset c(f_i) \\
    &= A_i.
\end{align*}
\]

Then Lemma 4 can be applied to obtain \( a \in (A_1 \cap \cdots \cap A_n)_v \). Thus

\[ (A_1)_v \cap \cdots \cap (A_n)_v \subset (A_1 \cap \cdots \cap A_n)_v. \]

Since the reverse containment is obvious, the proof is now complete.

REFERENCES


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(Received August 17, 1992)