The Quasi KO-types of the Stunted Mod 4 Lens Spaces

Zen-Ichi Yosimura*

*Osaka City University

Copyright ©1993 by the authors. Mathematical Journal of Okayama University is produced by The Berkeley Electronic Press (bepress). http://escholarship.lib.okayama-u.ac.jp/mjou
THE QUASI KO*-TYPES OF THE STUNTED MOD 4 LENS SPACES

ZEN-ICHI YOSIMURA

1. Introduction  Let $KU$ and $KO$ be the complex and the real $K$-spectrum respectively. For any CW-spectrum $X$ its $KU$-homology $KU_*X$ is regarded as a $(Z/2$-graded$)$ abelian group with involution because the complex $K$-spectrum $KU$ possesses the conjugation $\psi_C^{-1} : KU \to KU$. Given CW-spectra $X$ and $Y$ we say that $X$ is quasi $KO_*$-equivalent to $Y$ if there exists an equivalence $\tilde{j} : KO\wedge X \to KO\wedge Y$ of $KO$-module spectra (see [10]). If $X$ is quasi $KO_*$-equivalent to $Y$, then $KO_*X$ is isomorphic to $KO_*Y$ as an $KO_*$-module, and in addition $KU_*X$ is isomorphic to $KU_*Y$ as an abelian group with involution. In [12] and [13] we have completely determined the quasi $KO_*$-types of the real projective space $RP^n$ and its stunted projective space $RP_{m+1}^n = RP^n/RP^m$. In this paper we are interested in the standard mod 4 lens space $L^n(4)$ instead of the real projective space $RP^n$. Our purpose of this paper is to determine the quasi $KO_*$-types of the mod 4 lens space $L^n$ and its stunted lens space $L_{m+1}^n = L^n/L^m$ along the line of [12], [13] or [15], where we simply denote by $L^{2k+1}$ the $(2k + 1)$-dimensional standard mod 4 lens space $L^k(4)$ and by $L^{2k}$ its $2k$-skeleton $L_0^k(4)$.

Let $SZ/2^r (r \geq 1)$ be the Moore spectrum of type $Z/2^r$, and $i : \Sigma^0 \to SZ/2^r$ and $j : SZ/2^r \to \Sigma^1$ be the bottom cell inclusion and the top cell projection respectively. The stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$ of order 2 admits an extension $\tilde{\eta} : \Sigma^1 SZ/2^r \to \Sigma^0$ and a coextension $\hat{\eta} : \Sigma^2 \to SZ/2^r$ satisfying $\tilde{\eta}i = \eta$ and $j\tilde{\eta} = \eta$. In [10] and [11] we introduced some elementary spectra $M_r$, $P_r$, $MP_r$, $V_r$, $V'_r$, $W_{r,s}$ and so on ($r, s \geq 1$) constructed as the cofibers of the following maps respectively: $i\eta : \Sigma^1 \to SZ/2^r$, $\tilde{\eta} : \Sigma^2 \to SZ/2^r$, $i\eta \vee \tilde{\eta} : \Sigma^1 \vee \Sigma^2 \to SZ/2^r$, $i\eta : \Sigma^1 SZ/2^r \to \Sigma^1 SZ/2^s \to SZ/2^r$, $ij : \Sigma^1 SZ/2^s \to SZ/2^r$, $i\eta + j\tilde{\eta} : \Sigma^1 SZ/2^s \to SZ/2^r$ and so on, although these elementary spectra $X_r$ and $X_{r,s}$ were written to be $X_{2^r}$ and $X_{2^r, 2^s}$. In particular we note that the elementary spectrum $W_{r,s}$ coincides with the smash product $P \wedge SZ/2^r$ where $P$ denotes the cofiber of the stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$. In this paper we moreover introduce some new small spectra $U_{r,t,s}$, $V_{r,t,s}$, $MU_{r,t,s}$, $PU_{r,t,s}$ and so on ($r, t, s \geq 1$) constructed as the cofibers of the following maps respectively: $(i\tilde{\eta}, j\tilde{\eta})$.
$\Sigma^1 S^2 \to S^2 \vee S^2$, $(i_\eta, i_\nu \eta_j) : \Sigma^1 S^2 \to S^2 \vee V_t$, $i_\eta \nu \eta_j \nu : \Sigma^1 \vee \Sigma^{-1} V_{r,s} \to S^2$, $\eta \nu \eta_j \nu : \Sigma^2 \vee \Sigma^{-1} V_{t,s} \to S^2$ and so on, where $V_t = V_{t-1,1}$ and $i_\nu : S^2 \vee - \to V_t$ is the canonical inclusion, and $j_\nu : V_{r,s} \to \Sigma^2 S^2$ and $j_{t,s} : V_{t,s} \to \Sigma^2 S^2$ are the canonical projections.

Given CW-spectra $X$ and $Y$ we say that $X$ has the same $\approx$-type as $Y$ if $K\Omega_* X$ is isomorphic to $K\Omega_* Y$ as an abelian group with involution (cf. [3, 4.1]). Dualizing the $K\Omega$-cohomology $K\Omega^* L^n$ with the conjugation $\psi^{-1}$ calculated in [5] (or [7]) we can observe the $\approx$-type of the mod 4 lens space $L^n$ (Proposition 5.1).

Proposition 1. The suspended mod 4 lens space $\Sigma^1 L^n(n \geq 2)$ has the same $\approx$-type as the following small spectrum: $U_{t-1,2t+1,t}$, $MU_{t-1,2t+1,t}$, $\Sigma^2 \vee W_{2t+1}$, $\Sigma^2 \vee W_{2t+1,t-1}$, $\Sigma^0 \vee W_{2t+1,t}$ according as $n = 4t$, $4t + 1$, $4t + 2$, $4t + 3$. Here $W_{1,1}$ should be replaced by $\Sigma^2 S^2$.

More generally we can observe the $\approx$-type of the stunted mod 4 lens spaces $L^n_{m+1}$ (Corollary 5.3, Proposition 5.4 and (5.12)).

Proposition 2. i) The stunted mod 4 lens spaces $L_{4m+1}^{2m+n}$ and $L_{4n}^{2m+n}$ have the same $\approx$-types as $L^n$ and $\Sigma^0 \vee L^n$ respectively.

ii) The suspended stunted mod 4 lens space $\Sigma^1 L_{4m+3}^{2m+n+2}(n \geq 2)$ has the same $\approx$-type as the following small spectrum: $U_{2t,t}$, $\Sigma^2 \vee U_{2t,t}$, $Sz/2^{2t+2} \vee W_{t,t}$, $M_{2t+2} \vee W_{t,t}$ according as $n = 4t$, $4t + 1$, $4t + 2$, $4t + 3$.

iii) The stunted mod 4 lens space $L_{4m+2}^{2m+n+2}(n \geq 1)$ has the same $\approx$-type as $\Sigma^2 \vee L_{4m+3}^{2m+n+2}$.

For a CW-spectrum $X$ having the same $\approx$-type as one of the small spectrum appearing in Propositions 1 and 2 ii) we can determine its quasi $KO_*$-type by developing the same method as adopted in [10] or [11] (Proposition 3.1 and Theorem 3.3). Applying this result to the mod 4 lens space $L^n$ we can easily determine its quasi $KO_*$-type (cf. [4] and [9]).

Theorem 3. The suspended mod 4 lens space $\Sigma^1 L^n(n \geq 2)$ is quasi $KO_*$-equivalent to the following small spectrum: $U_{2r-1,4r+1,2r}$, $MU_{2r-1,4r+1,2r}$, $V_{2r} \vee W_{4r+1,2r+1}$, $\Sigma^4 \vee V_{2r} \vee W_{4r+1,2r+1}$, $V_{2r,4r+3,2r+1}$, $MU_{2r,4r+3,2r+1}$, $S^2/2^{2r+1} \vee W_{4r+3,2r+2}$, $\Sigma^0 \vee S^2/2^{2r+1} \vee W_{4r+3,2r+2}$ according as $n = 8r$, $8r + 1$, $\cdots$, $8r + 7$. Here $V_0 \vee W_{1,1}$ should be replaced.
by $\Sigma^2 SZ/4$.

In order to investigate the quasi $KO_\ast$-types of the stunted mod 4 lens spaces $L^{n}_{m+1}$ in general, we discuss separately in the following three cases (cf. [15]): i) $L^{2k+n}_{2k+1}(n \geq 2)$, ii) $L^{2k+2\ell}_{2k}(\ell \geq 1)$ and iii) $L^{2k+2\ell+1}_{2k}(\ell \geq 0)$. By a quite similar argument to the non-stunted case we can also determine the quasi $KO_\ast$-types of $L^{2k+n}_{2k+1}$ and $DL^{2k+2\ell}_{2k}$ where $DX$ denotes the $S$-dual of $X$ (Theorem 5.8). Dualizing our result for $DL^{2k+2\ell}_{2k}$ we can immediately determine the quasi $KO_\ast$-type of $L^{2k+2\ell}_{2k}$ (Theorem 5.9). In order to establish the rest case we construct certain maps $f_{k,\ell}: Y_{k,\ell} \to X_{k,\ell}$ modelled on the bottom cell inclusions $i: \Sigma^{2k-4m+2} \to \Sigma^{-4m+1}L^{2k+2\ell+1}_{2k+1}$ with $k = 2m$ or $2m - 1$, and then prove that the cofiber of each map $f_{k,\ell}$ has the same quasi $KO_\ast$-type as $\Sigma^{-4m+1}L^{2k+2\ell+1}_{2k+1}$. Using this fact we can determine the quasi $KO_\ast$-type of $L^{2k+2\ell+1}_{2k}$, too (Theorem 5.11). Consequently we can obtain the following main result (cf. [6, Theorem 2] and [8, Theorem 2]).

**Theorem 4.** i) $\Sigma^{-4m}L^{4m+n}_{4m+1}$ is quasi $KO_\ast$-equivalent to $L^n$.

ii) $\Sigma^{-4m}L^{4m+n}_{4m}$ is quasi $KO_\ast$-equivalent to the wedge sum $\Sigma^0 \vee L^n$.

iii) $\Sigma^{-4m+1}L^{4m+n-2}_{4m-1}(n \geq 2)$ is quasi $KO_\ast$-equivalent to the following small spectrum : $U_{4r,2r,2r,2r}, \Sigma^0 \vee U_{4r,2r,2r}, SZ/2^{4r+2} \vee W_{2r,2r}, M_{4r+2} \vee W_{2r,2r}, V_{4r+2,2r+1,2r+1}, \Sigma^4 \vee W_{4r+2,2r+1,2r+1}, V_{4r+4} \vee W_{2r+1,2r+1}, M_{4r+4} \vee W_{2r+1,2r+1}$ according as $n = 8r, 8r + 1, \ldots, 8r + 7$.

iv) $\Sigma^{-4m+1}L^{4m+n-2}_{4m-2}(n \geq 1)$ is quasi $KO_\ast$-equivalent to the following small spectrum : $PU_{4r+1,2r,2r}, \Sigma^0 \vee PU_{4r+1,2r,2r}, P_{4r+3} \vee W_{2r,2r}, \Sigma^4 MP_{4r+3,2r+1,2r+1}, \Sigma^4 \vee \Sigma^4 PU_{4r+3,2r+1,2r+1}, \Sigma^4 P_{4r+5} \vee W_{2r+1,2r+1}, \Sigma^4 MP_{4r+5} \vee W_{2r+1,2r+1}$ according as $n = 8r, 8r + 1, \ldots, 8r + 7$ where $PU_{1,0,0} = \Sigma^{-1}$.

Let $\gamma$ be the canonical complex line bundle over $L^{2k+1} = L^k(4)$ or its restriction to $L^{2k} = L^0(4)$, and $r\gamma$ denote its realification. The 4m-dimensional real vector bundle $2m r\gamma$ over $L^n$ is $KO$-orientable and its Thom complex $T(2m r\gamma)$ is homeomorphic to the stunted mod 4 lens space $L^{4m+n}$. So we remark that Theorem 4 ii) may be proved by means of [13, (3.8)] in a different way from ours.

This paper is organized as follows. In §2 we introduce some new small spectra $U_{r,t,s}, V_{r,t,s}, MU_{r,t,s}, PU_{r,t,s}, R^tU_{r,t,s}$ and so on, and compute their $KU$-homologies with the conjugation $\psi_C^{-1}$ and their $KO$-homologies. In
§3 we determine its quasi $KO_*$-type for a $CW$-spectrum $X$ having the same $\varepsilon$-type as $U_{r,t,s}$, $MU_{r,t,s}$ or $\Sigma^9 \vee U_{r,t,s}$ (Theorem 3.3). In §4 we consider several maps $f : \Sigma^{2i} \to X$ to construct desired maps $f_{k,\ell} : Y_{k,\ell} \to X_{k,\ell}$, and study their cofibers $C(f)$ and their induced homomorphisms $f_* : KU_0\Sigma^{2i} \to KU_0X$. In §5 we first investigate the behavior of the conjugations $\psi_C^{-1}$ on $KU_*L_{m+1}^n$ and $KU_*L_{m+1}^{n+1}$, and then prove our main result separately in the three case as stated above (Theorems 5.8, 5.9 and 5.11).

2. Small spectra $U_{r,t,s}$, $V_{r,t,s}$, $M U_{r,t,s}$ and $P U_{r,t,s}$

2.1. We first construct small spectra $U_{r,s+1}^r$ and $U_{r+1,s}^r$ ($r, s \geq 1$) as the cofibers of following maps respectively:

$$(i\vec{n}, \vec{n}) : \Sigma^1SZ/2^s \to SZ/2^r \vee SZ/2 \quad \text{and} \quad \vec{n} \vee \vec{n} : \Sigma^1SZ/2 \vee \Sigma^1SZ/2^s \to SZ/2^r.$$ 

According to [11, Lemma 1.1] these new spectra $U_{r,s+1}^r$ and $U_{r+1,s}^r$ may be given as the cofibers of the following composites maps

$$i\vec{n}j'_V : \Sigma^{-1}V_{s+1}^r \to SZ/2^r \quad \text{and} \quad iv\vec{n}j : \Sigma^1SZ/2^s \to V_{r+1}^r$$

respectively where $V_{s+1}^r = V_{s}^r$, $V_{r+1}^r = V_{r}^r$ and $j'_V : V_{s+1}^r \to \Sigma^2SZ/2^s$ and $iv : SZ/2^r \to V_{r+1}^r$ are the canonical projection and inclusion. For the convenience sake we set $V_{r}^r = SZ/2^{r+1}$ and $U_{r,s}^r = \Sigma^2SZ/2^{s+1}$. Evidently there holds the S-duality $U_{r,s}^r = \Sigma^2DU_{s,r}^r$. It is easily seen that these new spectra $U_{r,s}^r$ and $U_{r,s}^r$ have the same $\varepsilon$-type as $V_{r,s}^r$ and $V_{r,s}^r$, respectively. As is implicitly established in [10, Theorem 5.2] or [11, Theorem 4.2], we can observe more precisely that

(2.1) $U_{r,s}^r$ and $U_{r,s}^r$ have the same quasi $KO_*$-types as $\Sigma^6V_{s-1,r}^r$ and $\Sigma^2V_{s+1,r-1}^r$ respectively, where $V_{0,t}^r = \Sigma^2SZ/2^t$ and $V_{t,0}^r = SZ/2^t$.

We here introduce new small spectra $U_{r,t,s}^r$, $U_{r,t,s}^r$, $V_{r,t+1,s}^r$, $V_{r,t+1,s}^r$ ($r, t, s \geq 1$) constructed as the cofibers of the following maps respectively:

$$(i\vec{n}, \vec{n}) : \Sigma^1SZ/2^s \to SZ/2^r \vee SZ/2^t,$$

$$i\vec{n} \vee \vec{n} : \Sigma^1SZ/2^t \vee \Sigma^1SZ/2^s \to SZ/2^r,$$

(2.2) $$(i\vec{n}, iv\vec{n}) : \Sigma^1SZ/2^s \to SZ/2^r \vee V_{t+1}^r,$$

$$i\vec{n}j'_V \vee \vec{n} : \Sigma^{-1}V_{t+1}^r \vee \Sigma^1SZ/2^s \to SZ/2^r.$$
THE QUASI KO-TYPES OF THE STUNTED MOD 4 LENS SPACES

Of course the new spectra $U_{r,1,s}$ and $U'_{r,1,s}$ coincide with the previous elementary spectra $U_{r,s+1}$ and $U'_{r+1,s}$ respectively. For the convenience sake we set $V_{r,1,s} = V_{r,s+1}$ and $V'_{r,1,s} = V'_{r+1,s}$, and in addition $U_{r,0,s} = U'_{r,0,s} = V_{r,0,s}$, $U_{0,r,s} = U'_{r,0,s} = V_{r,s}$, $V_{0,r,s} = U'_{r,s}$ and $V'_{r,s} = U_{r,s}$. Evidently there hold the $S$-dualities $U_{r,t,s} = \Sigma^3 DU'_{s,t,r}$ and $V_{r,t,s} = \Sigma^3 DV'_{s,t,r}$. By a routine argument we can easily compute the $KU$-homologies with the conjugation $\psi_C^{-1}$ and the $KO$-homologies of these new spectra.

**Proposition 2.1.** When $X = U_{r,t,s}$, $V_{r,t,s}$, $U'_{r,t,s}$ and $V'_{r,t,s}(r,t,s \geq 1)$, $KU_1X = 0$ and $KU_0X$ with the conjugation $\psi_C^{-1}$ are given as follows:

i) "The $X = U_{r,t,s}$ or $V_{r,t,s}$ case"

<table>
<thead>
<tr>
<th>$r &lt; s &lt; t$</th>
<th>$r &lt; s \geq t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0X \cong Z/2^t \oplus Z/2^{s+1} \oplus Z/2^r$</td>
<td>$Z/2^{s+1} \oplus Z/2^{t-1} \oplus Z/2^r$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} 1 &amp; 2^{t-s} &amp; 0 \ 0 &amp; -1 &amp; 0 \ 0 &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -1 &amp; -2^{s-t+2} &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; -2^{s-t+1} &amp; 1 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r \geq s &lt; t$</th>
<th>$r \geq s \geq t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0X \cong Z/2^t \oplus Z/2^{t+1} \oplus Z/2^{s-1}$</td>
<td>$Z/2^{s+1} \oplus Z/2^s \oplus Z/2^{t-1}$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} 1 &amp; -2^{t-s} &amp; 2^{t-s+1} \ 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; -1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 1 &amp; -2^{s-t+1} \ 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

ii) "The $X = U'_{r,t,s}$ or $V'_{r,t,s}$ case"

<table>
<thead>
<tr>
<th>$s &lt; r &lt; t$</th>
<th>$s &lt; r \geq t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0X \cong Z/2^t \oplus Z/2^r \oplus Z/2^s$</td>
<td>$Z/2^{r+1} \oplus Z/2^{t-1} \oplus Z/2^s$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} -1 &amp; 0 &amp; 0 \ -1 &amp; 1 &amp; 2^{r-s} \ 0 &amp; 0 &amp; -1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2^{t-s+1} \ 0 &amp; 0 &amp; -1 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s \geq r &lt; t$</th>
<th>$s \geq r \geq t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0X \cong Z/2^t \oplus Z/2^{s+1} \oplus Z/2^{r-1}$</td>
<td>$Z/2^{s+1} \oplus Z/2^r \oplus Z/2^{t-1}$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} 2^{r-s+1} &amp; -1 &amp; -2^{s-r+2} \ 0 &amp; 1 &amp; 0 \ -1 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -1 &amp; -2^{s-t+1} &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; -1 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

**Proposition 2.2.** For the small spectra $X = U_{r,t,s}$, $V_{r,t,s}$, $U'_{r,t,s}$ and $V'_{r,t,s}(r,t,s \geq 1)$ their $KO$-homologies $KO_iX(0 \leq i \leq 7)$ are tabulated as follows:
where \((*)_{k,1} \cong Z/2^{k+2}\) and \((*)_{k,t} \cong Z/2^{k+1} \oplus Z/2\) if \(t \geq 2\).

2.2. Choose a map \(k_M : M_s \to \Sigma^1\) satisfying \(k_M i_M = j : SZ/2^s \to \Sigma^1\) and \(2^s k_M = \eta j_M : M_s \to \Sigma^1\) such that the sequence

\[
\Sigma^0 \xrightarrow{2^s i_P} P \xrightarrow{i_{s,M}} M_s \xrightarrow{k_M} \Sigma^1
\]

is a cofiber sequence where \(i_P : \Sigma^0 \to P\) is the bottom cell inclusion. For the convenience' sake we set \(M_0 = \Sigma^2\) and \(k_M = \eta : M_0 \to \Sigma^1\). It is easily seen that \([M_s, \Sigma^1] \cong Z/2^{s+1}\) which is generated by the map \(k_M\) for any \(s \geq 0\). We here introduce new small spectra \(MV'_t, s, QV_{r,t}, QU_{r,t+1}, V'_rM_{r,s},\)

\[MU_{r,t,s}, U'M_{r,t,s}\) and \(V'M_{r,t+1, s}(r, s, t \geq 1)\) constructed as the cofibers of the following maps respectively:

\[
\begin{align*}
\eta \vee \tilde{\eta} : & \Sigma^1 \vee \Sigma^1 SZ/2^s \to SZ/2^t, \\
\tilde{\eta} \vee i \tilde{\eta} : & \Sigma^3 \vee \Sigma^1 SZ/2^s \to SZ/2^t, \\
\tilde{\eta} \vee i \tilde{\eta} j'V : & \Sigma^3 \vee \Sigma^{-1}V_{t+1}r \to SZ/2^t, \\
\tilde{\eta}k_M : & \Sigma^1 M_s \to SZ/2^t, \\
i \eta \vee \tilde{\eta} j'V : & \Sigma^1 \vee \Sigma^{-1}V_{r,s} \to SZ/2^t, \\
i V' \tilde{\eta}k_M : & \Sigma^1 M_s \to V_{r,t}, \\
i V' \tilde{\eta}k_M : & \Sigma^1 M_s \to U_{r,t+1}.
\end{align*}
\]

We moreover choose another map \(h'_V : \Sigma^2 \to V'_{r,s}\) satisfying \(j'V h'_V = i : \Sigma^0 \to SZ/2^s\) and \(2^s h'_V = i_V \tilde{\eta} : \Sigma^2 \to V'_{r,s}\) such that the sequence

\[
\Sigma^2 \xrightarrow{h'_V} V'_{r,s} \xrightarrow{j'V,P} P_r \xrightarrow{2^s j_P} \Sigma^3
\]

is a cofiber sequence where \(j_P : P_r \to \Sigma^3\) is the top cell projection. Notice that \([\Sigma^2, V'_{1,s}] \cong Z/2^{s+2}\) which is generated by the map \(h'_V\), and \([\Sigma^2, V'_{r,s}] \cong

\[
\begin{array}{|c|c|c|c|c|}
\hline
X & 0 & 1 & 2 & 3 \\
\hline
\text{i} & Z/2^t \oplus Z/2^t & Z/2 & (*)_{s-1,t} \oplus Z/2 & Z/2 \\
\hline
U_{r,s} & Z/2^t \oplus Z/2^{t-1} & 0 & Z/2^s \oplus Z/2 & Z/2 \\
\hline
V'_r & Z/2^t & 0 & Z/2^{t-1} \oplus Z/2^{s+1} & Z/2 \\
\hline
(*)_{r-1,t} & Z/2 & Z/2^t \oplus Z/2^{s+1} & Z/2 \\
\hline
\end{array}
\]
Z/2^{s+1} \oplus Z/2 whose direct summands are generated by the maps \( h' \) and \( i'_V \eta \) whenever \( r \geq 2 \) (cf. (1.4)). As is easily checked, the cofibers of the maps \( i_V \eta : \Sigma^2 \to V_{r,t} \) and \( i'_V \eta : \Sigma^2 \to U_{r,t} \) coincide with those of the maps \( i_V \eta : \Sigma^2 \to U_{r,t,s} \) and \( i'_V \eta : \Sigma^2 \to V_{r,t,s} \). Therefore the above new spectra \( V'M_{r,s} \), \( U'M_{r,t,s} \) and \( V'M_{r,t+1,s} \) may be given as the cofibers of the following maps respectively:

\[ h' \eta : \Sigma^3 \to V'_{r,t,s} \]
\[ i_V \eta : \Sigma^3 \to U'_{r,t,s} \]

For the convenience' sake we set \( QU_{r,1} = Q_{r+1} \), \( V'M_{r,1,s} = V'M_{r+1,s} \), \( V'M_{r,0} = Q_r \), \( MU_{r,t,s} = QV_{r,t} \), \( MU_{r,t,0} = QU_{r,t} \) where \( Q_r \) denotes the cofiber of the map \( \eta : \Sigma^3 \to S^2/2 \).

Similarly to Propositions 2.1 and 2.2 we can easily compute the \( KU \)-homologies with the conjugation \( \psi_C^{-1} \) and the \( KO \)-homologies of these new spectra.

**Proposition 2.3.** When \( X = V'M_{r,s} \), \( MU_{r,t,s} \), \( U'M_{r,t,s} \) and \( V'M_{r,t,s} \), \( KU_1X = 0 \) and \( KU_0X \) with the conjugation \( \psi_C^{-1} \) are given as follows:

i) "The \( X = V'M_{r,s}(r \geq 1 \text{ and } s \geq 0) \) case"

\[
\begin{array}{c|c|c}
KU_0X & Z \oplus Z/2^s \oplus Z/2^r & Z \oplus Z/2^{s+1} \oplus Z/2^{r-1} \\
\psi_C^{-1} & \begin{pmatrix} 1 & 0 & 0 \\ -2^{r-s-1} & 1 & 2^{r-s} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ -2^{s-r+2} \end{pmatrix} \\
\end{array}
\]

ii) "The \( X = MU_{r,t,s}(r \geq 0 \text{ and } s, t \geq 1) \) case"

\[
\begin{array}{c|c|c}
KU_0X & Z \oplus Z/2^t \oplus Z/2^s \oplus Z/2^r & Z \oplus Z/2^{s+1} \oplus Z/2^{t+1} \oplus Z/2^r \\
\psi_C^{-1} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 2^{t-s} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2^{s-t+1} & -1 & -2^{s-t+2} & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -2^{s-t+1} & 1 \end{pmatrix} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
KU_0X & Z \oplus Z/2^t \oplus Z/2^{r+1} \oplus Z/2^{s-1} & Z \oplus Z/2^{s+1} \oplus Z/2^s \oplus Z/2^{t-1} \\
\psi_C^{-1} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & -2^{t-s} & 2^{l-s+1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2^{s-t} & 1 & -1 & -2^{s-t+1} \\ -1 & 0 & 1 & 0 \end{pmatrix} \\
\end{array}
\]
iii) "The $X = U'M_{r,t,s}$ or $V'M_{r,t,s}(r, t \geq 1$ and $s \geq 0$) case"

\[
\psi_C^{-1} = \begin{cases}
\begin{array}{c}
s < r < t \\
s < r \geq t
\end{array}
\end{cases}
\begin{cases}
\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
-2^{r-s-1} & -1 & 1 & 2^{r-s}
\end{array}
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 0 & -1
\end{array}
\end{cases}
\]

\[
KU_0X = \begin{cases}
\begin{array}{c}
s \geq r < t \\
s \geq r \geq t
\end{array}
\end{cases}
\begin{cases}
\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
1 & 2^{s-r+1} & -1 & -2^{s-r+2}
\end{array}
\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & -2^{s-r+1}
\end{array}
\end{cases}
\]

**Proposition 2.4.** For the small spectra $X = MV'_{t,s}$, $QV_{r,t}$, $QU_{r,t}$, $V'M_{r,s}$, $MU_{r,t,s}$, $U'M_{r,t,s}$ and $V'M_{r,t,s}(r, t, s \geq 1)$ their $KO$-homologies $KO_iX(0 \leq i \leq 7)$ are tabled as follows:

<table>
<thead>
<tr>
<th>$X \setminus i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MV'_{t,s}$</td>
<td>$Z/2^t$</td>
<td>0</td>
<td>$Z \oplus Z/2^{s+1}$</td>
<td>$Z/2$</td>
</tr>
<tr>
<td>$QV_{r,t}$</td>
<td>$Z \oplus Z/2^r$</td>
<td>0</td>
<td>$Z/2^{t-1} \oplus Z/2$</td>
<td>0</td>
</tr>
<tr>
<td>$QU_{r,t}$</td>
<td>$Z \oplus (*)_{r-1,t}$</td>
<td>$Z/2$</td>
<td>$Z/2^t \oplus Z/2$</td>
<td>0</td>
</tr>
<tr>
<td>$V'M_{r,s}$</td>
<td>$Z \oplus Z/2^t$</td>
<td>$Z/2$</td>
<td>$(*)_{s,r}$</td>
<td>0</td>
</tr>
<tr>
<td>$MU_{r,t,s}$</td>
<td>$Z/2^r \oplus Z/2^t$</td>
<td>0</td>
<td>$Z \oplus Z/2^s \oplus Z/2$</td>
<td>$Z/2$</td>
</tr>
<tr>
<td>$U'M_{r,t,s}$</td>
<td>$Z \oplus Z/2^r$</td>
<td>0</td>
<td>$Z/2^{t-1} \oplus Z/2^{s+1}$</td>
<td>0</td>
</tr>
<tr>
<td>$V'M_{r,t,s}$</td>
<td>$Z \oplus (*)_{r-1,t}$</td>
<td>$Z/2$</td>
<td>$Z/2^t \oplus Z/2^{s+1}$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z/2^t \oplus Z/2$</td>
<td>0</td>
<td>$Z \oplus Z/2^s$</td>
<td>0</td>
</tr>
<tr>
<td>$Z \oplus (*)_{r-1,t}$</td>
<td>$Z/2$</td>
<td>$Z/2^t \oplus Z/2$</td>
<td>0</td>
</tr>
<tr>
<td>$Z \oplus Z/2^r$</td>
<td>0</td>
<td>$Z/2^{t-1} \oplus Z/2$</td>
<td>0</td>
</tr>
<tr>
<td>$Z \oplus Z/2^{r-1}$</td>
<td>0</td>
<td>$Z/2^{s+1}$</td>
<td>0</td>
</tr>
<tr>
<td>$Z/2^{r+1} \oplus Z/2^t$</td>
<td>0</td>
<td>$Z \oplus Z/2^s$</td>
<td>0</td>
</tr>
<tr>
<td>$Z \oplus (*)_{r-1,t}$</td>
<td>$Z/2$</td>
<td>$Z/2^t \oplus Z/2^{s+1}$</td>
<td>0</td>
</tr>
<tr>
<td>$Z \oplus Z/2^r$</td>
<td>0</td>
<td>$Z/2^{t-1} \oplus Z/2^{s+1}$</td>
<td>0</td>
</tr>
</tbody>
</table>

where $(*)_{k,1} \cong Z/2^{k+2}$ and $(*)_{k,\ell} \cong Z/2^{k+1} \oplus Z/2$ if $\ell \geq 2$.

### 2.3. We next introduce new small spectra $V'PP'_{r,s}$, $U'PP'_{r,t,s}$ and $V'PP'_{r,t+1,s}(r, t, s \geq 1)$ constructed as the cofibers of the following maps respectively:
(2.6) \((\bar{\eta} j, \bar{\eta}) : \Sigma^1 S\mathbb{Z}/2^s \to S\mathbb{Z}/2^r \vee \Sigma^0, (i\nu \bar{\eta} j, \bar{\eta}) : \Sigma^1 S\mathbb{Z}/2^s \to V_{r,t} \vee \Sigma^0\) and \((i\nu \bar{\eta} j, \bar{\eta}) : \Sigma^1 S\mathbb{Z}/2^s \to U_{r,t+1} \vee \Sigma^0\).

For the convenience' sake we set \(V'P_{r,t,s} = V'P'_{r,t+1,s}\). Similarly to Propositions 2.3 and 2.4 we can easily compute the \(KU\)-homologies with the conjugation \(\psi_C^{-1}\) and the \(KO\)-homologies of these small spectra.

**Proposition 2.5.** i) The small spectra \(V'P_{r,s}, U'P_{r,t,s}\) and \(V'P_{r,t,s}(r,t,s \geq 1)\) have the same \(\varepsilon\) types as \(U'M_{r,t,s-1} = \Sigma^s U'M'_{r,t,s-1}\) respectively, where \(V'M_{r,0} = Q_r, U'M_{r,t,0} = QV_{r,t}\) and \(V'M_{r,t,0} = QU_{r,t}\).

ii) Their \(KO\)-homologies \(KO_1 X(0 \leq i \leq 7)\) are tabulated as follows:

<table>
<thead>
<tr>
<th>(X)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(V'P_{r,s})</td>
<td>(Z \oplus \mathbb{Z}/2^r)</td>
<td>(\mathbb{Z}/2)</td>
<td>((*)_{s-1,r})</td>
<td>0</td>
</tr>
<tr>
<td>(U'P_{r,t,s})</td>
<td>(Z \oplus \mathbb{Z}/2^r)</td>
<td>0</td>
<td>(\mathbb{Z}/2^{t-1} \oplus \mathbb{Z}/2^s)</td>
<td>0</td>
</tr>
<tr>
<td>(V'P_{r,t,s})</td>
<td>(Z \oplus (*)_t)</td>
<td>(\mathbb{Z}/2)</td>
<td>(\mathbb{Z}/2^{t} \oplus \mathbb{Z}/2^s)</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

where \((*)_{k,1} \cong \mathbb{Z}/2^{k+1}\) and \((*)_{k,t} \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2\) if \(\ell \geq 2\).

We here consider the \(S\)-duals \(PV_{r,s}, PU_{r,t,s}\) and \(PV_{r,t+1,s}\) of \(V'P'_{r,s}, U'P'_{r,t,s}\) and \(V'P'_{s,t+1,r}(r,t,s \geq 1)\) obtained as the cofibers of the following maps respectively:

(2.7) \(\bar{\eta} \vee i\nu \bar{\eta} : \Sigma^2 \vee \Sigma^1 S\mathbb{Z}/2^s \to S\mathbb{Z}/2^r, \bar{\eta} \vee i\nu j^t \bar{\eta} : \Sigma^2 \vee \Sigma^{-1} V_{t,s} \to S\mathbb{Z}/2^r\) and \(\bar{\eta} \vee i\nu j^t \bar{\eta} : \Sigma^2 \vee \Sigma^{-1} U_{t+1,s} \to S\mathbb{Z}/2^r\).

For the convenience' sake we set \(PV_{r,1,s} = PV_{r,s+1}\). Evidently the \(S\)-dualities are given as \(PV_{r,s} = \Sigma^3 DV'P'_{r,s}, PU_{r,t,s} = \Sigma^3 DU'P'_{s,t,s}\) and \(PV_{r,t,s} = \Sigma^3 DV'P'_{r,t,s}\). As a dual of Proposition 2.5 we can obtain the \(KU\)-homologies with the conjugation \(\psi_C^{-1}\) and the \(KO\)-homologies for these \(S\)-dual spectra.

**Corollary 2.6.** i) The small spectra \(PV_{r,s}, PU_{r,t,s}\) and \(PV_{r,t,s}\) \((r,t,s \geq 1)\) have the same \(\varepsilon\) types as the wedge sums \(\Sigma^3 \vee V_{r-1,t,s}, \Sigma^3 \vee U_{r-1,t,s}\) and \(\Sigma^3 \vee V_{r-1,t,s}\) respectively, where \(V_{0,s} = \Sigma^2 S\mathbb{Z}/2^s, U_{0,t,s} = V_{t,s}\) and \(V_{0,t,s} = U_{t,s}\).

ii) Their \(KO\)-homologies \(KO_1 X(0 \leq i \leq 7)\) are tabulated as follows:
$$X|_i \begin{array}{c|c|c|c} \hline i & 0 & 1 & 2 \\
\hline PV_{r,s} & \mathbb{Z}/2^r & 0 & \mathbb{Z}/2^{s-1} \begin{array}{c} \mathbb{Z} \\
\hline PU_{r,t,s} & \mathbb{Z}/2^r \oplus \mathbb{Z}/2^t & \mathbb{Z}/2 & (\ast)_{s-1,t} \begin{array}{c} \mathbb{Z} \\
\hline PV_{r,t,s} & \mathbb{Z}/2^r \oplus \mathbb{Z}/2^{t-1} & 0 & \mathbb{Z}/2^s \begin{array}{c} \mathbb{Z} \\
\hline 4 & 5 & 6 & 7 \\
\end{array}
\end{array}
\end{array}$$

where \((\ast)_{k,1} \cong \mathbb{Z}/2^{k+2}\) and \((\ast)_{k,\ell} \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2\) if \(\ell \geq 2\).

2.4. Denote by \(Q'_t\) and \(R'_t\) the elementary spectra constructed as the cofibers of the maps \(\eta_1 : \Sigma^3SZ/2^t \rightarrow \Sigma^0\) and \(\eta^2_1 : \Sigma^3SZ/2^t \rightarrow \Sigma^0\) respectively. There elementary spectra \(Q'_t\) and \(R'_t\) are related via the obvious map \(\lambda_{Q,R} : \Sigma^1Q'_t \rightarrow R'_t\) satisfying \(\lambda_{Q,R}i'_{Q} = i'_{R} \eta : \Sigma^1 \rightarrow R'_t\) and \(j'_{Q} = j'_{R} \lambda_{Q,R} : Q'_t \rightarrow \Sigma^3SZ/2^t\). Choose a unique map \(\tilde{h}_{Q} : \Sigma^5 \rightarrow Q'_t\) satisfying \(\tilde{h}_{Q}h_{Q} = \tilde{\eta} : \Sigma^2 \rightarrow SZ/2^t\), and then set \(\tilde{h}_{R} = \lambda_{Q,R}\tilde{h}_{Q} : \Sigma^6 \rightarrow R'_t\) (see [11, (2.2)]). We here introduce new small spectra \(R'V'_{r,t,s}, V'R'_{r,s}, R'U_{r,t,s}, U'R'_{r,t,s}\) and \(V'R'_{r,t,s+1}(r, t, s \geq 1)\) constructed as the cofibers of the following maps respectively:

\[
\begin{align*}
\tilde{h}_{R}j : \Sigma^5SZ/2^s & \rightarrow R'_t, \\
(\tilde{\eta}j, \eta^2_1) : \Sigma^3SZ/2^s & \rightarrow \Sigma^2SZ/2^r \vee \Sigma^0, \\
(2.8) & \\
\tilde{h}_{R}jj_{V} : \Sigma^3V_{r,s} & \rightarrow R'_t, \\
(i\tilde{\eta}j, \eta^2_1) : \Sigma^3SZ/2^s & \rightarrow \Sigma^2V_{r,t} \vee \Sigma^0, \\
(i\tilde{\eta}j, \eta^2_1) : \Sigma^3SZ/2^s & \rightarrow \Sigma^2U_{r,t+1} \vee \Sigma^0.
\end{align*}
\]

For the convenience' sake we set \(V'R'_{r,t,s} = V'R'_{r,t,s+1}\) and \(R'U_{0,t,s} = R'V'_{t,s}\). Choose a map \(k'_V : \Sigma^2V'_{t,s} \rightarrow \Sigma^0\) satisfying \(k'_Vi'_V = \eta\tilde{\eta} : \Sigma^2SZ/2^t \rightarrow \Sigma^0\), whose cofiber coincides with the cofiber of the map \(h_Qj : \Sigma^4SZ/2^s \rightarrow Q'_t\). Since the cofiber of the obvious map \(\lambda_{Q,R} : \Sigma^1Q'_t \rightarrow R'_t\) is just the cofiber \(P\) of the map \(\eta : \Sigma^1 \rightarrow \Sigma^0\), the above new small spectra \(R'V'_{t,s}\) and \(R'U_{r,t,s}\) may be given as the cofibers of the following maps respectively:

\[
\begin{align*}
\eta k'_V : \Sigma^3V'_{t,s} & \rightarrow \Sigma^0 \quad \text{and} \quad \eta k'_V j_{U,V'} : \Sigma^3U_{r,t,s} \rightarrow \Sigma^0.
\end{align*}
\]

For these new spectra we can easily compute their \(KU\)-homologies with the conjugation \(\psi_C^{-1}\) and their \(KO\)-homologies.
THE QUASI $KO_*$-TYPES OF THE STUNTED MOD 4 LENS SPACES

Proposition 2.7. i) The small spectra $R'R'_{t,s}$, $V'R'_{r,s}$, $R'U_{t,s}$, $U'R'_{r,s}$ and $V'R'_{r,s}$ ($r, t, s \geq 1$) have the same $\mathcal{W}$-type as the wedge sums $\Sigma^0 \vee V'_{t,s}$, $\Sigma^0 \vee \Sigma^2 V'_{r,s}$, $\Sigma^0 \vee U_{r,s}$, $\Sigma^0 \vee \Sigma^2 U_{t,s}$ and $\Sigma^0 \vee \Sigma^2 V'_{r,s}$ respectively.

ii) Their $KO$-homologies $KO_i X (0 \leq i \leq 7)$ are tabulated as follows:

<table>
<thead>
<tr>
<th>$i \backslash X$</th>
<th>$R'R'_{t,s}$</th>
<th>$V'R'_{r,s}$</th>
<th>$R'U_{t,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>2</td>
<td>$(*)_{s-1, t}$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>$(*)_{s-1, t}$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^{t-1}$</td>
<td>$\mathbb{Z} \oplus (*)_{s-1, r}$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^{t-1}$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>6</td>
<td>$(*)_{s, t}$</td>
<td>$\mathbb{Z}/2^{t-1} \oplus \mathbb{Z}/2$</td>
<td>$(*)_{s-1, t} \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$U'R'_{r,s}$</th>
<th>$V'R'_{r,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^t$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^{t-1} \oplus \mathbb{Z}/2^s$</td>
</tr>
<tr>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$\mathbb{Z}/2^t \oplus \mathbb{Z}/2$</td>
<td>$(*)_{r-1, t} \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^{t-1} \oplus \mathbb{Z}/2^s$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^s$</td>
</tr>
<tr>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$(*)_{r-1, t} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2^t \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where $(*)_{k, 1} \cong \mathbb{Z}/2^{k+2}$ and $(*)_{k, t} \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ if $t \geq 2$.

3. The same quasi $KO_*$-type as $U_{r,t,s}$, $MU_{r,t,s}$ or $\Sigma^0 \vee U_{r,t,s}$

3.1. Let $X$ be a CW-spectrum having the same $\mathcal{W}$-type as the wedge sum $Y \vee W_{r,s}$ where $Y = SZ/2^t$, $\Sigma^0 \vee SZ/2^t$ or $M_t$. In this case we note that there exists an isomorphism $KO_{2i+1}^* X \oplus KO_{2i+5}^* X \cong KO_{2i+1}^* Y \oplus KO_{2i+5}^* Y$ for any $i$. Using the same method as adopted in [10, Theorem 5.2] or [11, Theorem 4.2] we can easily determine the quasi $KO_*$-type of such a CW-spectrum $X$.

Proposition 3.1. Let $Y$ be the small spectrum $SZ/2^t$, $\Sigma^0 \vee SZ/2^t$ or $M_t (t \geq 1)$. If a CW-spectrum $X$ has the same $\mathcal{W}$-type as the wedge sum $Y \vee W_{r,s}$, then it is quasi $KO_*$-equivalent to one of the following wedge sum: i) $\Sigma^{4i} SZ/2^t \vee W_{r,s}$ and $\Sigma^{4i} V_t \vee W_{r,s}$; ii) $\Sigma^{4i} \vee \Sigma^{4i} SZ/2^t \vee W_{r,s}$.
Let $K_T$ be the self conjugate $K$-spectrum (which is sometimes denoted by $KC$). For the small spectra $Y = U_{r,t,s}, MU_{r,t,s}$ and $R'U_{r,t,s}(r, t, s \geq 1)$ their $KT$-homologies $KT_iY(0 \leq i \leq 3)$ are easily calculated as follows:

$$
\begin{array}{c|cc}
Y \setminus i & 0 & 1 \\
\hline
U_{r,t,s} & Z/2^r \oplus Z/2^t & (1)_{s-1,t} \\
MU_{r,t,s} & Z/2^r \oplus Z/2^t & Z \oplus Z/2^s \\
R'U_{r,t,s} & Z \oplus Z/2^r \oplus Z/2^t & (1)_{s-1,t} \oplus Z/2 \\
2 & Z/2^r \oplus Z/2 & Z/2^{r+1} \oplus Z/2^{t-1} \\
 & Z \oplus Z/2^r \oplus Z/2 & Z/2^{r+1} \oplus Z/2^t \\
 & Z/2^r \oplus Z/2 & Z \oplus Z/2^{r+1} \oplus Z/2^{t-1}
\end{array}
$$

Let $X$ be a $CW$-spectrum having the same $\mathcal{U}$-type as $\Sigma U_{r,t,s, U}$. Then there exist two isomorphisms $\theta_1 : KO_1 X \oplus KO_5 X \to KO_1 \Sigma^0 \oplus KO_1 U_{r,t,s} \cong Z/2 \oplus Z/2$ and $\theta_3 : K\Omega_3 \oplus K\Omega_7 X \to K\Omega_3 U_{r,t,s} \cong Z/2$. Identify $KT_0X$ and $KT_2X$ with $KT_0 \Sigma^0 \oplus KT_0 U_{r,t,s} \cong Z \oplus Z/2^r \oplus Z/2^t$ and $KT_2 U_{r,t,s} \cong Z/2^r \oplus Z/2$ respectively. Then the composite homomorphisms $\theta_1(-, \tau B_T^{-1})_* : KT_0 X \to KO_1 X \oplus KO_5 X \cong Z/2 \oplus Z/2$ is represented by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
: Z \oplus Z/2^r \oplus Z/2^t \to Z/2 \oplus Z/2,
$$

and $\theta_3(-, \tau B_T^{-1})_* : KT_2 X \to KO_3 X \oplus KO_7 X \cong Z/2$ is given by the second projection

$$
\begin{pmatrix}
0 & 1 \\
\end{pmatrix}
: Z/2^r \oplus Z/2 \to Z/2.
$$

Consider the automorphism $\alpha_T : KT_0 X \to KT_0 X$ represented by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix}
$$

on $Z \oplus Z/2^r \oplus Z/2^t$. By a routine computation we can easily get an automorphism $\alpha_C : KU_0 X \to KU_0 X$ such that $\psi_C^{-1} \alpha_C = \alpha_C \psi_C^{-1} : KU_0 X \to KU_0 X$ and $\alpha_C \zeta_* = \zeta_* \alpha_T : KT_0 X \to KU_0 X$. Therefore we may regard that the induced homomorphism $\theta_1(-, \tau B_T^{-1})_*$ is represented by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

in place of

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
$$

In other words, it may be regarded that the isomorphism $\theta_1$ is represented by one of three kinds of matrices

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix}
$$

on $Z/2 \oplus Z/2$. Hence the induced homomorphism $(-, \tau B_T^{-1})_* : KT_0 X \to KO_1 X \oplus KO_5 X$ is given as one of the homomorphism represented by the following three kinds of matrices:
THE QUASI KO* -TYPES OF THE STUNTED MOD 4 LENS SPACES

\[(3.2) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]

: \(Z \oplus Z/2^r \oplus Z/2^t \to Z/2 \oplus Z/2\).

**Lemma 3.2.** For the small spectra \(Y = U_{r,t,s}, MU_{r,t,s}\) and \(R^tU_{r,t,s}\) \((r, t, s \geq 1)\) the induced homomorphism \(\tau_* : KT_2; Y \to KO_{2t+1}Y\) are represented by the following rows \(M_{2t+1}(Y)\):

i) \(M_1(U_{r,t,s}) = (0 1) : Z/2^r \oplus Z/2^t \to Z/2;\)

\(M_3(U_{r,t,s}) = (0 1) : Z/2^s \oplus Z/2 \to Z/2;\)

ii) \(M_3(MU_{r,t,s}) = (0 0 1) : Z \oplus Z/2^s \oplus Z/2 \to Z/2;\)

iii) \(M_1(R^tU_{r,t,s}) = (1 0 1) : Z \oplus Z/2^r \oplus Z/2^t \to Z/2;\)

\(M_5(R^tU_{r,t,s}) = (0 0 1) : Z \oplus Z/2^r \oplus Z/2^t \to Z/2;\)

\(M_7(R^tU_{r,t,s}) = (0 1) : Z/2^s \oplus Z/2 \to Z/2.\)

**Proof.** We only show our result in the \(Y = R^tU_{r,t,s}\) case. The other cases are very easy. Since the small spectrum \(R^tU_{r,t,s}\) has the same \(\vee\)type as the wedge sum \(\Sigma^0 \vee U_{r,t,s}\), the induced homomorphism \((-\tau, \tau B_{T}^{-1})_* : KT_0R^tU_{r,t,s} \to KO_1R^tU_{r,t,s} \oplus KO_5R^tU_{r,t,s}\) restricted to \(Z/2^r \subset Z \oplus Z/2^r \oplus Z/2^t\) becomes trivial by (3.2), and \(\tau_* : KT_6R^tU_{r,t,s} \to KO_7R^tU_{r,t,s}\) is given by the second projection \((0 1) : Z/2^s \oplus Z/2 \to Z/2\). Consider the commutative diagram

\[
\begin{array}{ccc}
KO_1R_t' & \simeq & KT_0R_t' \\
\downarrow & & \downarrow \tau B_{T}^{-1} \\
KO_5R_t' & \simeq & KO_5R^tU_{r,t,s} \\
\end{array}
\]

where the vertical arrows are induced by the canonical inclusion \(i_{R^tU} : R_t' \to R^tU_{r,t,s}\). The both sides arrows are isomorphisms and the central one is the obvious monomorphism \(i : Z \oplus Z/2^t \to Z \oplus Z/2^r \oplus Z/2^t\). Our result is now immediate from [11, Lemma 3.2 iii)].

3.2. For a \(CW\)-spectrum \(X\) having the same \(\vee\)type as the small spectrum \(Y = U_{r,t,s}, MU_{r,t,s}\) or \(\Sigma^0 \vee R^tU_{r,t,s}(r, t, s \geq 1)\) we determine its quasi \(KO_*\)-type by using the same method as adopted in [10, Theorem 5.2] or [11, Theorem 4.2].

**Theorem 3.3.** Let \(Y\) be the small spectrum \(U_{r,t,s}, MU_{r,t,s}\) or \(\Sigma^0 \vee U_{r,t,s}(r, t, s \geq 1)\). If a \(CW\)-spectrum \(X\) has the same \(\vee\)type as the small spectrum \(Y\), then it is quasi \(KO_*\)-equivalent to one of the following small
spectra: i) $\Sigma^4 U_{r,t,s}$ and $\Sigma^4 V_{r,t,s}$; ii) $\Sigma^4 MU_{r,t,s}$; iii) $\Sigma^4 \vee \Sigma^4 U_{r,t,s}$, $\Sigma^4 \vee \Sigma^4 V_{r,t,s}$ and $\Sigma^4 R^i U_{r,t,s}(i,j = 0 \text{ or } 1)$, according as $Y = U_{r,t,s}$, $MU_{r,t,s}$ or $\Sigma^0 \vee U_{r,t,s}$.

Proof. We may assume that $KO_3 X \cong Z/2$ and $KO_7 X = 0$ since $KO_3 X \oplus KO_7 X \cong Z/2$ in any case.

i) Note that $KO_1 X \oplus KO_5 X \cong Z/2$. Under the assumption that $KO_1 X = KO_7 X = 0$ we first show that $X$ is quasi $KO_*$-equivalent to the small spectrum $V_{r,t,s}$. Choose a map $g_V : V_{r,t,s} \to KT \wedge X$ such that $(\zeta \wedge 1)g_V : V_{r,t,s} \to KU \wedge X$ is a quasi $KU_*$-equivalence. For our purpose it is sufficient to find a map $h_V : V_{r,t,s} \to KO \wedge X$ satisfying $(c \wedge 1)h_V = (\zeta \wedge 1)g_V$ because such a map $h_V$ is fortunately a quasi $KO_*$-equivalence by virtue of [10, Proposition 1.1]. When $t = 1$ our assertion is immediately established as (2.1) is done. When $t \geq 2$ we consider the following cofiber sequence

$$
\Sigma^1 SZ/2^s (i_\eta, i_\nu \tilde{\eta} j) \to SZ/2^r \vee V_t \xrightarrow{i_\nu \vee i_\nu \nu \nu} V_{r,t,s} \xrightarrow{j_\nu} \Sigma^2 SZ/2^s
$$

obtained from (2.2). Recall that the elementary spectrum $V_t$ is obtained as the cofiber of the map $2^{t-1} \tilde{7} : \Sigma^0 \to P'_1$ where $P'_1$ denotes the cofiber of the map $\tilde{7} : \Sigma^0 \to P'_1$ is the bottom cell inclusion. Since the elementary spectrum $P'_1$ has the same quasi $KO_*$-type as $\Sigma^4$, the composite map $(\eta \wedge 1)(\tau B^{-1}_T \wedge 1)g_V i_\nu : V_t \to \Sigma^2 KO \wedge X$ becomes trivial. On the other hand, it follows from Lemma 3.2 i) that the composite map $(\eta \wedge 1)(\tau B^{-1}_T \wedge 1)g_V i_\nu : SZ/2^r \to \Sigma^2 KO \wedge X$ is trivial, too. Now we can apply [10, Lemma 1.3] to get a map $h : SZ/2^s \to \Sigma^1 KO \wedge X$ satisfying $h_\nu = (\tau B^{-1}_T \wedge 1)g_V$ where the map $g_V$ might be replaced by a new one suitably if necessary. Consider the map $k_\nu : V_t \to \Sigma^1$ of order $2^{t+1}$ satisfying $k_\nu i_\nu = j : SZ/2^{t-1} \to \Sigma^1$ and $2^{t-1}k_\nu = \tilde{\eta} j_\nu : V_t \to \Sigma^1$, whose fiber is the elementary spectrum $P'_1$. Since the map $h$ admits a coextension $\tilde{h} : \Sigma^0 \to KO \wedge X$ with $\tilde{h} j = h$, it is immediately seen that $(\eta \wedge 1)h = \tilde{h} k_\nu i_\nu \tilde{\eta} j = (0 \vee \tilde{h} k_\nu)(i_\eta, i_\nu \tilde{\eta} j) : \Sigma^1 SZ/2^s \to SZ/2^r \vee V_t \to \Sigma^1 KO \wedge X$, and hence $(\eta \wedge 1)h_\nu : V_{r,t,s} \to \Sigma^2 KO \wedge X$ is trivial as desired.

By a similar argument to the above we can show that $X$ is quasi $KO_*$-equivalent to the small spectrum $U_{r,t,s}$ under the assumption that $KO_5 X = KO_7 X = 0$, whose proof is simpler than the above case.

ii) Under the assumption that $KO_1 X = KO_5 X = KO_7 X = 0$ it is sufficient to show that $X$ is quasi $KO_*$-equivalent to the small spectrum $MU_{r,t,s}$. Choose a map $g_{MU} : MU_{r,t,s} \to KT \wedge X$ such that $(\zeta \wedge 1)g_{MU} :
THE QUASI $KO_*$-TYPES OF THE STUNTED MOD 4 LENS SPACES

$MU_{r,t,s} \to KU \land X$ is a quasi $KU_*$-equivalence, and then consider the following cofiber sequence

$$\Sigma^1 \land \Sigma^{-1} V_{r,s} \xrightarrow{i_{\eta} \land j_{jV}} SZ/2^t \xrightarrow{i_{MU}} MU_{r,t,s} \xrightarrow{j_{MU}} \Sigma^2 \land V_{r,s}$$

obtained from (2.4). Since the composite map $(\eta \land 1)(\tau B_{T}^{-1} \land 1)g_{MU}i_{MU} : SZ/2^t \to \Sigma^2 KO \land X$ is trivial, we obtain a map $h_1 \lor h_2 : \Sigma^2 \lor V_{r,s} \to \Sigma^3 KO \land X$ satisfying $(h_1 \lor h_2)j_{MU} = (\tau B_{T}^{-1} \land 1)g_{MU}$ where the map $g_{MU}$ might be replaced by a new one as in the above i) and the map $h_1$ is in fact trivial. Evidently there exists a map $h_3 : SZ/2^s \to KO \land X$ satisfying $h_3j_{V} = (\eta \land 1)h_2$, and hence a map $h_4 : SZ/2^r \to \Sigma^1 KU \land X$ satisfying $h_4i_{\eta} = (c \land 1)h_3$ and $h_2i_{V} = (rB_{C}^{-1} \land 1)h_4$. Since the composite map $h_4i : \Sigma^0 \to \Sigma^1 KU \land X$ is trivial, we get a map $h_5 : SZ/2^s \to \Sigma^1 KO \land X$ satisfying $(\eta \land 1)h_5 = h_3$, which admits a coextension $\tilde{h}_5 : \Sigma^0 \to KO \land X$ with $\tilde{h}_5j = h_5$. Consequently we can observe that $(\eta \land 1)h_2 = \tilde{h}_5 \eta jj_{jV} = 0$ as desired.

iii) Note that $KO_1X \oplus KO_5X \cong Z/2 \oplus Z/2$. By a similar argument to the above i) we can easily show that $X$ is quasi $KO_*$-equivalent to the wedge sum $\Sigma^0 \lor U_{r,t,u}$ when $KO_5X = KO_7X = 0$, and to $\Sigma^4 \lor V_{r,t,s}$ when $KO_1X = KO_7X = 0$. So we assume that $KO_1X \cong KO_3X \cong KO_5X \cong Z/2$ and $KO_7X = 0$. According to (3.2) the induced homomorphisms $\tau_* : KT_{2i}X \to KO_{2i+1}X$, whose matrix representations are denoted by $M_{2i+1}(X)$, are divided into the following three types:

1. $M_1(X) = (1 \ 0 \ 0)$, $M_5(X) = (0 \ 0 \ 1)$, $M_3(X) = (0 \ 1)$;
2. $M_1(X) = (0 \ 0 \ 1)$, $M_5(X) = (1 \ 0 \ 0)$, $M_3(X) = (0 \ 1)$;
3. $M_1(X) = (0 \ 0 \ 1)$, $M_5(X) = (1 \ 0 \ 0)$, $M_3(X) = (0 \ 1)$

where $M_1(X), M_5(X) : Z \oplus Z/2^r \oplus Z/2^t \to Z/2$ and $M_3(X) : Z/2^s \oplus Z/2 \to Z/2$. By a similar argument to the above cases we can show that $X$ is quasi $KO_*$-equivalent to the wedge sum $\Sigma^0 \lor V_{r,t,s}$ or $\Sigma^4 \lor U_{r,t,s}$ according as the case (1) or (2). In the case (3) we show that $X$ is quasi $KO_*$-equivalent to the small spectrum $\Sigma^4 R'U_{r,t,s}$. Choose a map $g_{RU} : \Sigma^4 R'U_{r,t,s} \to KT \land X$ such that $(\zeta \land 1)g_{RU} : \Sigma^4 R'U_{r,t,s} \to KU \land X$ is a quasi $KU_*$-equivalence, and then consider the following cofiber sequence

$$\Sigma^3 U_{r,s} \xrightarrow{\tilde{h}_Rj_{jV}} R'_t \xrightarrow{i_{R'_t}} R'U_{r,t,s} \xrightarrow{i_{RU}} \Sigma^4 V_{r,s},$$

obtained from (2.8). Since the induced homomorphism $\tau_* : KT_0X \to KO_*X$ restricted to $Z \subset Z \oplus Z/2^r \oplus Z/2^t$ is trivial, the composite map $(\tau B_{T}^{-1} \land 1)g_{RU}i_{RU}i_{R'} : \Sigma^1 \to KO \land X$ becomes trivial. Hence we obtain a
map $h_0 : \Sigma^5 S\Sigma Z/2^s \to KO \wedge X$ satisfying $h_0 j'_R = (\tau B_T^{-1} \wedge 1) g_{RU} i_{RU}$. By virtue of Lemma 3.2 iii) we see that the composite homomorphism $(\tau B_T^{-1} \wedge 1) g_{RU} : KO_0 R^t U_{t,s} \to KO_5 X$ is trivial, and hence $h_0 : KO_0 S\Sigma Z/2^s \to KO_5 X$ is trivial, too. This implies that the composite map $(\eta \wedge 1)(\tau B_T^{-1} \wedge 1) g_{RU} i_{RU} : \Sigma^5 R'_t \to KO \wedge X$ is trivial. So we obtain a map $h_1 : \Sigma^5 V_{r,s} \to KO \wedge X$ satisfying $h_1 j_{RU} = (\tau B_T^{-1} \wedge 1) g_{RU}$ where the map $g_{RU}$ might be replaced by a new one as in the above i) or ii). Evidently the composite homomorphism $h_{1 \wedge i_V} : KO_0 S\Sigma Z/2^s \to KO_5 X$ is trivial. Therefore there exists a map $h_2 : \Sigma^8 S\Sigma Z/2^s \to KO \wedge X$ satisfying $h_2 j_V = (\eta \wedge 1) h_1$, and hence a map $h_3 : \Sigma^7 S\Sigma Z/2^s \to KU \wedge X$ such that $(c \wedge 1) h_2 = h_3 i_V$ and $(\tau B_T^{-1} \wedge 1) h_3 = h_1 i_V$. Since the composite map $h_3 i : \Sigma^7 \to KU \wedge X$ is trivial, we get a map $h_4 : \Sigma^7 S\Sigma Z/2^s \to KO \wedge X$ satisfying $(\eta \wedge 1) h_4 = h_2$, which admits a coextension $h_4 : \Sigma^8 \to KO \wedge X$ with $h_4 j = h_4$. Consequently we can observe that $(\eta \wedge 1) h_1 = h_4 i h_4 j = h_4 j R'_t i_{RU} j_V$ because $j'_R h_R = \tilde{\eta} : \Sigma^2 \to S\Sigma Z/2^s$, and hence $(\eta \wedge 1) h_1 j_{RU} = 0$ as desired.

Combining Theorem 3.3 with Propositions 2.2, 2.4, 2.5 and 2.7 we obtain

**Corollary 3.4.** i) $U'_{r,t,s}$ and $V'_{r,t,s}(r,t,s \geq 1)$ are quasi $KO_*$-equivalent to $\Sigma^2 U_{t-1,s+1,r}$ and $\Sigma^6 V_{t-1,s+1,r}$ respectively, where $U_{0,s+1,r} = V_{s+1,r}$ and $V_{0,s+1,r} = U'_{s+1,r}$.

ii) $U'M_{t,s}$ and $\Sigma^4 V'M_{r,t,s}(r,t,s \geq 1)$ are quasi $KO_*$-equivalent to $\Sigma^2 M U_{t-1,s+1,r}$ where $M U_{0,s+1,r} = M V_{s+1,r}$.

iii) $U'P'_{r,t,s}$ and $V'P'_{r,t,s}(r,t,s \geq 1)$ are quasi $KO_*$-equivalent to $U'M_{r,t,s-1}$ and $V'M_{r,t,s-1}$ respectively, where $U'M_{r,t,0} = Q V_{r,t}$ and $V'M_{r,t,0} = Q U_{r,t}$.

iv) $U'R'_{r,t,s}$ and $\Sigma^4 V'R'_{r,t,s}(r,t,s \geq 1)$ are quasi $KO_*$-equivalent to $R'U_{t-1,s+1,r}$ where $R'U_{0,s+1,r} = R'V'_{s+1,r}$.

4. The cofibers of certain maps $f : \Sigma^{2i} \to X$

4.1. For any $k(0 \leq k \leq s)$ the cofiber of the map $2^k i : \Sigma^0 \to S\Sigma Z/2^s$ is just the wedge sum $\Sigma^1 \wedge S\Sigma Z/2^k$. Thus we have the following cofiber sequence

$$
\Sigma^0 \xrightarrow{2^k i} S\Sigma Z/2^s \xrightarrow{(j,s)} \Sigma^1 \wedge S\Sigma Z/2^k \xrightarrow{2^{s-k} \wedge (-j)} \Sigma^1
$$

where $\rho = \rho_{s,k} : S\Sigma Z/2^s \to S\Sigma Z/2^k$ is the obvious map. For any $s \geq 2$ we choose a map $h_V : \Sigma^2 \to V_{r,s}$ of order $2^{s-1}$ satisfying the equalities
THE QUASI $K{O}$. TYPES OF THE STUNTED MOD 4 LENS SPACES

$j_V h_V = 2i : \Sigma^0 \to SZ/2^s$ and $h_V j = -i_V i\tilde{\eta} : \Sigma^1 SZ/2 \to V_{r,s}$, whose cofiber is the wedge sum $\Sigma^3 \wedge V_{r+1}$. Note that such a map $h_V$ is uniquely chosen. As is easily calculated, $[\Sigma^2, V_{r,s}] \cong Z/2^{s-1} \oplus Z/2$ whose direct summands are generated by the maps $h_V$ and $i_V \tilde{\eta}$ (cf. (1.4)). For any $k(1 \leq k < s)$ we set

$$f_{V,k} = 2^{k-1} h_V + i_V \tilde{\eta} : \Sigma^2 \to V_{r,s}.$$  

Lemma 4.1. Assume that $1 \leq k < s$. Then the cofiber of the map $f_{V,k} : \Sigma^2 \to V_{r,s}$ is the wedge sum $\Sigma^3 \wedge W_{r,k}$, and the induced homomorphism $f_{V,k*} : KU_0 \Sigma^2 \to KU_0 V_{r,s}$ is given as follows:

$$f_{V,k*}(1) = (2^k, 2^{k-1} + 2^{r-1}) \in Z/2^s \oplus Z/2^r \quad \text{when } r < s;$$
$$f_{V,k*}(1) = (2^r, 2^{k-1}) \in Z/2^{r+1} \oplus Z/2^{s-1} \quad \text{when } r \geq s.$$

Proof. Choose a map $f : \Sigma^2 \to V_{r,s}$ of order $2^{s-k}$ satisfying the equalities $j_V f = 2^k i : \Sigma^0 \to SZ/2^s$ and $-f j = i_V (i\tilde{\eta} + \tilde{\eta} j) : \Sigma^1 SZ/2^k \to V_{r,s}$. Since such a map $f$ is uniquely determined, it is exactly the map $f_{V,k}$ given in (4.2). When $r < s$ the induced homomorphism $h_{V*} : KU_0 \Sigma^2 \to KU_0 V_{r,s}$ is expressed as $h_{V*}(1) = (2, c) \in Z/2^s \oplus Z/2^r$ for some $c$. Using the behavior of the conjugation $\psi_C^{-1}$ on $KU_0 V_{r,s}$ we can immediately show that $c \equiv 1 \mod 2^{r-1}$, thus $c = 1$ or $1 + 2^{r-1}$. Note that the cokernel of $h_{V*}$ is isomorphic to $KU_0 V_{r+1}$ but not to $KU_0 W_{r+1}$ as an abelian group with involution. From this fact it follows that $c = 1$. On the other hand, when $r \geq s$ we may express as $h_{V*}(1) = (a, b) \in Z/2^{r+1} \oplus Z/2^{s-1}$ for some $a, b$ with the relation $-a + 2b \equiv 2 \mod 2^s$. Then it is immediately shown that $a \equiv 0 \mod 2^r$, thus $(a, b) = (0, 1) \text{ or } (2^r, 1)$. We can also verify that $(a, b) = (0, 1)$ by a similar observation to the above case. Evidently the induced homomorphism $i_V \tilde{\eta}_* : KU_0 \Sigma^2 \to KU_0 V_{r,s}$ is given by $i_V \tilde{\eta}_*(1) = (0, 2^{r-1}) \in Z/2^s \oplus Z/2^r$ when $r < s$, and $i_V \tilde{\eta}_*(1) = (2^r, 0) \in Z/2^{r+1} \oplus Z/2^{s-1}$ when $r \geq s$. Therefore our result is now immediate.

Lemma 4.2. Assume that $1 \leq k < s$. Then there exists a map $f_{U,k} : \Sigma^2 \to U_{r,t,s}$ whose cofiber is the wedge sum $P_t \wedge W_{r,k}$ and whose induced homomorphism $f_{U,k*} : KU_0 \Sigma^2 \to KU_0 U_{r,t,s}$ is given as follows:

i) $f_{U,k*}(1) = (-2^{r+s+k-1}, 2^k, 2^{k-1} + 2^{r-1}) \in Z/2^t \oplus Z/2^s \oplus Z/2^r$ when $r < s < t$;

ii) $f_{U,k*}(1) = (2^k, 0, 2^{k-1} + 2^{r-1}) \in Z/2^{s+1} \oplus Z/2^{s-1} \oplus Z/2^r$ when $r < s \geq t$;
iii) \( f_{U,k}(1) = (-2^{t-s+k-1}, 2^r, 2^{k-1}) \in \mathbb{Z}/2^t \oplus \mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^{s-1} \) when \( r \geq s < t \);

iv) \( f_{U,k}(1) = (2^r, 2^{k-1}, 0) \in \mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2^{t-1} \) when \( r \geq s \geq t \).

**Proof.** Consider the cofiber sequence

\[ \Sigma \mathbb{Z}/2^t \xrightarrow{i_U} \Sigma V_{r,t,s} \xrightarrow{j_{U,V}} V_{r,s} \xrightarrow{\bar{n}jj} \Sigma^1 \mathbb{Z}/2^t. \]

Using the map \( f_{U,k} : \Sigma \to V_{r,s} \) we choose a map \( f_{U,k} : \Sigma \to U_{r,t,s} \) satisfying the equalities \( j_{U,V} f_{U,k} = f_{V,k} : \Sigma \to V_{r,s}, \) \( 2^{t-k} f_{U,k} = i_U \tilde{n} : \Sigma \to U_{r,t,s} \) and \( f_{U,k} j jW = 0 : W_{r,k} \to \Sigma U_{r,t,s} \) so that its cofiber is the wedge sum \( P_t \vee W_{r,k} \). Then we can easily check that the induced homomorphism \( f_{U,k} : KU_0 \Sigma \to KU_0 U_{r,t,s} \) is expressed as desired, after replacing the map \( f_{U,k} \) by \( (1 + 2^{s-k}) f_{U,k} = f_{U,k} + i_U \tilde{n} \) if necessary.

Recall that the small spectrum \( MU_{r,t,s} \) is given as the cofiber of the composite map \( i_U \tilde{n} : \Sigma \to U_{r,t,s} \) where \( i_U : \mathbb{Z}/2^t \to U_{r,t,s} \) is the canonical inclusion. Using the map \( f_{U,k} : \Sigma \to U_{r,t,s} \) obtained in Lemma 4.2 we consider the composite map

\[ f_{MU,k} = i_{U,MU} f_{U,k} : \Sigma \to MU_{r,t,s} \]

for any \( k(1 \leq k < s) \), where \( i_{U,MU} : U_{r,t,s} \to MU_{r,t,s} \) is the canonical inclusion.

**Lemma 4.3.** Assume that \( 1 \leq k < s \). Then the cofiber of the map \( f_{MU,k} : \Sigma \to MU_{r,t,s} \) is the wedge sum \( MP_t \vee W_{r,k} \) and the induced homomorphism \( f_{MU,k} : KU_0 \Sigma \to KU_0 MU_{r,t,s} \) is given by \( f_{MU,k}(1) = (0, f_{U,k}(1)) \in \mathbb{Z} \oplus KU_0 U_{r,t,s} \) where \( f_{U,k}(1) \) is precisely expressed in Lemma 4.2.

**Proof.** The cofiber of the map \( f_{MU,k} \) coincides with that of the map

\( (i_UP, 0) : \Sigma \to P_t \vee W_{r,k} \) where \( i_P : \mathbb{Z}/2^t \to P_t \) is the canonical inclusion. Thus it is exactly the wedge sum \( MP_t \vee W_{r,k} \) as desired. The latter part of our result is obvious.

**4.2.** Recall that the small spectrum \( U'_{r,t,s} \) is obtained as the cofiber of the map \( i_V \tilde{n} j : \Sigma^1 \mathbb{Z}/2^s \to V_{r,t} \). Using the map \( f_{V,k} : \Sigma \to V_{r,t} \) given in (4.2) we consider the composite map

\[ f'_{U,k} = i_{V,U'} f_{V,k} : \Sigma \to U'_{r,t,s}. \]
for any $k(1 \leq k < t)$, where $i_{V,U^t} : V_{r,t} \to U_{r,t,s}'$ is the canonical inclusion.

**Lemma 4.4.** Assume that $s < k < t$. Then the cofiber of the map $f_{U,k} : \Sigma^2 \to U_{r,t,s}'$ is the wedge sum $\Sigma^3 \vee W_{r,k} \vee \Sigma^2 SZ/2^s$, and the induced homomorphism $f_{U,k*} : KU_0 \Sigma^2 \to KU_0 U_{r,s}'$ is given as follows:

i) $f_{U,k*}(1) = (2^k, 2^{k-1} + 2^{r-1}, 0) \in Z/2^r \oplus Z/2^r \oplus Z/2^s$ when $s < r < t$;

ii) $f_{U,k*}(1) = (2^r, 2^{k-1}, 0) \in Z/2^{r+1} \oplus Z/2^r \oplus Z/2^s$ when $s < r \geq t$;

iii) $f_{U,k*}(1) = (2^k, 2^s - 2^{s-r+k}, 2^{k-1}) \in Z/2^t \oplus Z/2^{s+1} \oplus Z/2^{r-1}$ when $s \geq r < t$.

**Proof.** Under the assumption that $k > s$ we observe that $f_{V,k,j} = f_{V,k,j} \rho_s,k = i_{\Sigma^1} i(\tilde{\eta}j) \rho_s,k = -i_{\Sigma^1} \tilde{\eta}j : \Sigma^1 SZ/2^s \to V_{r,t}$. Therefore the cofiber of the composite map $f_{U,k} = f_{V,U} \cdot f_{V,k}$ coincides with the wedge sum $\Sigma^3 \vee W_{r,k} \vee \Sigma^2 SZ/2^s$ because Lemma 4.1 says that the cofiber of the map $f_{V,k}$ is just the wedge sum $\Sigma^3 \vee W_{r,k}$ when $1 \leq k < t$. By a routine computation we can easily show the latter part of our result.

For the map $h'_{V^t} : \Sigma^2 \to V_{r,s}'$ chosen in (2.5) its induced homomorphism $h'_{V^t*} : KU_0 \Sigma^2 \to KU_0 V_{r,s}'$ is expressed as follows:

\[ h'_{V^t*}(1) = (-2^{r-s-1}, 1) \in Z/2^r \oplus Z/2^s \quad \text{when } r > s; \]
\[ h'_{V^t*}(1) = (1, 0) \in Z/2^{s+1} \oplus Z/2^{r-1} \quad \text{when } r \leq s. \]

Here the map $h'_{V^t}$ might be replaced by $(1 + 2^s)h'_{V^t} = h'_{V^t} + i_{\Sigma^1} \tilde{\eta}$ if necessary. Recall that the small spectrum $MV_{r,s}'$ is given as the cofiber of the composite map $i_{\Sigma^1} \eta : \Sigma^1 \to V_{r,s}'$. For any $k(0 \leq k \leq s)$ we set

\[ h'_{V,k} = 2^k h'_{V^t} : \Sigma^2 \to V_{r,s}' \quad \text{and} \quad h'_{MV,k} = i_{\Sigma^1} \eta \cdot h'_{V,k} : \Sigma^2 \to MV_{r,s}' \]

where $i_{\Sigma^1} \eta : V_{r,s}' \to MV_{r,s}'$ is the canonical inclusion.

**Lemma 4.5.** Assume that $0 \leq k \leq s$. Then the cofibers of the maps $h'_{V,k} : \Sigma^2 \to V_{r,s}'$ and $h'_{MV,k} : \Sigma^2 \to MV_{r,s}'$ are the wedge sums $P_r \vee \Sigma^2 SZ/2^k$ and $MP_r \vee \Sigma^2 SZ/2^k$ respectively, and the induced homomorphisms $h'_{V,k*} : KU_0 \Sigma^2 \to KU_0 V_{r,s}'$ and $h'_{MV,k*} : KU_0 \Sigma^2 \to KU_0 MV_{r,s}'$ are given by $h'_{V,k*}(1) = 2^k h'_{V^t*}(1)$ and $h'_{MV,k*}(1) = (0, 2^k h'_{V^t*}(1)) \in Z \oplus KU_0 V_{r,s}'$ where $h'_{V^t*}(1)$ is precisely expressed in (4.5).

**Proof.** Choose a map $f_k : \Sigma^2 \to V_{r,s}'$ satisfying the equalities $j_i f_k = 2^k i : \Sigma^0 \to SZ/2^s$, $2^{s-k} f_k = i_{\Sigma^1} \tilde{\eta} : \Sigma^2 \to V_{r,s}'$ and $f_k j = 0 : \Sigma^1 SZ/2^k \to V_{r,s}'$, so that its cofiber is the wedge sum $P_r \vee \Sigma^2 SZ/2^k$. Evidently $f_s =
\[ i_V j = 2^s h_V + a i_V j \] for some \( a \) because \( i_V j \) does not vanish when \( r \geq 2 \). Note that the cofiber of the maps \( h_{V,k} \) and \( f_k = (1 + 2^{s-k}a)h_{V,k} \) coincide in the \( k < s \) case. Now the first of our result is easily shown. The latter part is obvious.

4.3. The cofiber of the map \( 2^\ell i_j : \Sigma^{-1} S/2^k \to S/2^s \) is just the wedge sum \( S/2^{s+k-\ell} \vee S/2^\ell \) for any \( \ell \leq \text{Min}(s,k) \). Thus there exists a cofiber sequence

\[ \Sigma^{-1} S/2^k \xrightarrow{2^\ell i_j} S/2^{s+k-\ell} \vee S/2^\ell \xrightarrow{\rho(-\rho)} S/2^k \]

in which each of \( \rho \) is the obvious map. As is easily seen, under the assumption that \( \ell \leq \text{Min}(s,k-1) \) we get a map \( g_{V,\ell} : \Sigma S/2^k \to V_{r,s} \) satisfying the equalities \( j_{\ell} g_{V,\ell} = 2^\ell i_j : S/2^k \to \Sigma S/2^s \), \( g_{V,\ell} \rho = i_V(i\eta + \tilde{\eta}j) : \Sigma S/2^{s+k-\ell} \to V_{r,s} \) and \( g_{V,\ell}\rho = 0 : \Sigma S/2^\ell \to V_{r,s} \) so that its cofiber is the wedge sum \( W_{r,s+k-\ell} \vee \Sigma^2 S/2^\ell \). Thus we have the following cofiber sequence

\[ \Sigma S/2^k \xrightarrow{g_{V,\ell}} V_{r,s} \xrightarrow{(\rho_{\Sigma^2, W}, \rho j_{\ell})} W_{r,s+k-\ell} \vee \Sigma^2 S/2^\ell \xrightarrow{\rho_{\Sigma^2, W}(-\rho)} \Sigma^2 S/2^k. \]

Assume that \( 0 \leq \ell < k \leq s + k - \ell < t \). Using the above two maps \( h_{V,k} : \Sigma^2 \to V'_{r,s} \) and \( g_{V,\ell} : \Sigma S/2^k \to V_{r,s} \), we then obtain a map \( f_{V,k,\ell} : \Sigma^2 \to V'_{r,t,s} \) making the following diagram with four cofiber sequences commutative

\[
\begin{array}{ccc}
\Sigma^2 & = & \Sigma^2 \\
\downarrow f'_{V,k,\ell} & & \downarrow h_{V,k} \\
V_{r,s}' & \to & V_{r,t,s}' \\
\downarrow \varphi & & \downarrow \varphi \\
V_{r,s}' & \to & V_{r,t,s}' \\
\downarrow \psi & & \downarrow \psi \\
\Sigma^3 & = & \Sigma^3 \\
(4.7) & & (4.7)
\end{array}
\]

because \( g_{V,\ell}\rho = i_V(i\eta + \tilde{\eta}j)\rho = i_V i\eta j_{\ell} = i_V i\eta j_{\ell} \).

Lemma 4.6. Assume that \( 0 \leq \ell < s < r < s + k - \ell < t \). Then there exists a map \( f'_{V,k,\ell} : \Sigma^2 \to V'_{r,t,s} \) whose cofiber is the wedge sum \( P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2 S/2^\ell \) and whose induced homomorphism \( f'_{V,k,\ell} : KU_0 \Sigma^2 \to \).
THE QUASI \(KO\)-TYPES OF THE STUNTED MOD 4 LENS SPACES

\(KU_0V'_{r,t,s}\) is given by \(f'_{V,k,t,s}(1) = (2^k, 2^{r-s+\ell-1}, -2^\ell) \in \mathbb{Z}/2^t \oplus \mathbb{Z}/2^r \oplus \mathbb{Z}/2^s\).

**Proof.** Use the commutative diagram (4.7). The induced homomorphism \(f'_{V,k,t,s} : KU_0\Sigma^2 \to KU_0V'_{r,t,s}\) is expressed as \(f'_{V,k,t,s}(1) = (2^k, b, c) \in \mathbb{Z}/2^t \oplus \mathbb{Z}/2^r \oplus \mathbb{Z}/2^s\) with the relation \(b + 2^{r-s-1}c \equiv 0 \mod 2^{r-1}\). On the other hand, the induced homomorphism \(\varphi_* : KU_0V'_{r,t,s} \to KU_0W_{r,s+k-\ell} \oplus KU_0\Sigma^2SZ/2^0\) is represented by a certain matrix

\[
\begin{pmatrix}
x & 2m+1 & u \\
y & -1 & v \\
z & 0 & 1
\end{pmatrix} : \mathbb{Z}/2^t \oplus \mathbb{Z}/2^r \oplus \mathbb{Z}/2^s \to \mathbb{Z}/2^{s+k-\ell+1} \oplus \mathbb{Z}/2^{r-1} \oplus \mathbb{Z}/2^\ell
\]

with the relations \(x + 2^{s+k-\ell}y - 2^{r-\ell}z \equiv 1 \mod 2^k\) and \(u + 2^{m+1}v \equiv 2^{k-\ell} \mod 2^{s+k-\ell}\) where \(m = s - r + k - \ell \geq 1\). Using the behavior of the conjugation \(\psi_C^{-1}\) on \(KU_0W_{r,s+k-\ell} \oplus KU_0\Sigma^2SZ/2^0\) we see that \(u \equiv 2(2^{m-1})v \equiv 2^{r-s} \mod 2^r\). Hence it follows that \(u \equiv 2^{k-\ell+1}(1 - 2^{m-1}) \mod 2^{s+k-\ell}\) and \(v \equiv 2^{r-s-1}(2^{m-1} - 1) \mod 2^{r-1}\), thus \(u = 2^{k-\ell+1}u'\) for some \(u'\) and \(v = 2^{r-s-1}v'\) for some odd \(v'\) whenever \(s \geq 1\). Since \(\varphi f'_{V,k,t,s} \Sigma^2 \to P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2SZ/2^\ell\) is trivial, it is shown that \(2^kx + 2^{m+1}b + 2c' \equiv 0 \mod 2^{s+k-\ell+1}\), \(-b + c' \equiv 0 \mod 2^{r-1}\) and \(c \equiv 0 \mod 2^\ell\), thus \(b = 2^{r-s+\ell-1}b'\) and \(c = 2^\ell c'\) for some \(b', c'\), and in addition \(x + b' + 2c' \equiv 0 \mod 2^{s+k-\ell+1}\). Moreover we notice that \(b' + c' \equiv 0 \mod 2^{r-\ell}\) because \(b + 2^{r-s-1}c \equiv 0 \mod 2^{r-1}\). Consequently \(f'_{V,k,t,s}(1) = (2^k, 2^{r-s+\ell-1}b', -2^\ell b') \in \mathbb{Z}/2^t \oplus \mathbb{Z}/2^r \oplus \mathbb{Z}/2^s\) for some odd \(b'\). In this case we may take \(b' = 1\) by replacing suitably the direct sum decomposition of \(KU_0V'_{r,t,s} \cong KU_0V'_{t} \oplus KU_0V'_{r,s}\) if necessary.

Since the map \(\bar{\eta} : \Sigma^1SZ/2 \to \Sigma^0\) has order 4 we can choose a map \(k'_{P_1} : P_1 \to \Sigma^0\) satisfying \(k'_{P_2} = 4 : \Sigma^0 \to \Sigma^0\) and \(ik'_{P_1} = \bar{\eta}_{1,2}j'_{P_1} : P_1 \to SZ/4\), where cofiber is the small spectrum \(U_1\) constructed as the cofiber of the map \(\bar{\eta}_{1,2} : \Sigma^2SZ/2 \to SZ/4\) with \(j\bar{\eta}_{1,2} = \bar{\eta}\) (see [14, 1.1]). Composing this map \(k'_{P_1} : P_1 \to \Sigma^0\) before the map \(f'_{V,k,t} : \Sigma^2 \to V'_{r,t,s}\) obtained in (4.7) we get a map

\[
g'_{V,k,t,s} = f'_{V,k,t,s}k'_{P_1} : \Sigma^2P_1 \to V'_{r,t,s}
\]

when \(0 \leq \ell < k \leq s + k - \ell < t\).

**Lemma 4.7.** Assume that \(0 \leq \ell < s < r < s + k - \ell < t\). Then the cofiber of the map \(g'_{V,k,t,s} : \Sigma^2P_1 \to V'_{r,t,s}\) is quasi \(KO\)-equivalent to the wedge sum \(\Sigma^7 \vee W_{r,s+k-\ell} \vee \Sigma^6V_{t+1}\), and the induced homomorphism \(g'_{V,k,t,s} : KU_0\Sigma^2P_1 \to KU_0V'_{r,t,s}\) is given by \(g'_{V,k,t,s}(1) = (2^k+1, 2^{r-s+\ell}, -2^\ell+1) \in\)
\[ Z/2^t \oplus Z/2^r \oplus Z/2^s. \]

**Proof.** The cofiber of the composite map \( g'_{v,k,\ell} = f'_{v,k,\ell} k'_{P} \) coincides with the fiber of the map \( i_U \psi = \psi_1 \vee \psi_2 \vee \psi_3 : P_1 \vee W_{\tau, s+k-\ell} \vee \Sigma^2 S^2 / 2^t \to \Sigma^3 U_1 \) where \( \psi = 2^{t-k-1} j_P \vee (-2^{s-\ell} j_{W}) \vee j : P_1 \vee W_{\tau, s+k-\ell} \vee \Sigma^2 S^2 / 2^t \to \Sigma^3. \)

Since \( P_1 \) and \( U_1 \) have the same quasi \( KO_* \)-types as \( \Sigma^7 \) and \( \Sigma^6 S^2 / 2 \) respectively, it follows that \( [P_1, \Sigma^3 KO \wedge U_1] \cong KO_6 S^2 / 2 = 0 \) and \( [\Sigma^0, KO \wedge U_1] \cong [\Sigma^2, KO \wedge S^2 / 2] \cong Z / 4. \) Obviously the composite map \((i_U \wedge 1)\psi_1 : P_1 \to \Sigma^3 KO \wedge U_1\) is trivial where \( i_U : S \to KO \) denotes the unit of \( KO \). Since \( 2n j_{W} = \eta^2 j_{W} : W_{\tau, s+k-\ell} \to \Sigma^1 S^2 / 2 \) is trivial, the composite map \((i_U \wedge 1)\psi_2 : W_{\tau, s+k-\ell} \to \Sigma^3 KO \wedge U_1\) becomes trivial under the assumption that \( \ell < s \). On the other hand, the cofiber of the map \( \psi_3 = i_U j : \Sigma^1 S^2 / 2 \to U_1\) coincides with the small spectrum \( U_{t+1} \) obtained as the cofiber of the map \( 2^t k'_{P} : P_1' \to \Sigma^0\), which has the same quasi \( KO_* \)-type as \( \Sigma^4 V_{t+1}\) (see [14, (1.4)]). Using these facts we observe that the cofiber of the map \( g'_{v,k,\ell} \) is quasi \( KO_* \)-equivalent to the wedge sum \( P_1 \vee W_{\tau, s+k-\ell} \vee \Sigma^2 U_{t+1} \) and hence it is quasi \( KO_* \)-equivalent to the wedge sum \( \Sigma^7 \vee W_{\tau, s+k-\ell} \vee \Sigma^6 V_{t+1} \) as desired. Since the induced homomorphism \( k'_{P*} : KU_0 P_1' \to KU_0 \Sigma^0 \) is the multiplication by \( 2 \) on \( Z \), the latter part of our result is immediate from Lemma 4.6.

**4.4.** For any \( s \geq 1 \) we choose a map \( h_W : \Sigma^2 \to W_{\tau, s} \) of order \( 2^s \) satisfying the equalities \( j_W h_W = 2i : \Sigma^0 \to S^2 / 2^s, 2^{s-1} h_W = i_W \tilde{\eta} : \Sigma^2 \to W_{\tau, s} \) and \( h_W j = -i_W \tilde{\eta} : \Sigma^1 S^2 / 2 \to W_{\tau, s} \) so that its cofiber is the small spectrum \( PV_{t,1} \) constructed in (2.7). Evidently \( [\Sigma^2, W_{\tau, s}] \cong Z / 2^s \) which is generated by the map \( h_W \). After the map \( h_W \) is replaced by \( (1 + 2^{s-1}) h_W = h_W + i_W \tilde{\eta} \) if necessary, the induced homomorphism \( h_{W*} : KU_0 \Sigma^2 \to KU_0 W_{\tau, s} \) is expressed as follows:

\[
(4.9) \quad \begin{align*}
\text{(i)} \quad & h_{W*}(1) = (-2^{r-s+1}, 1) \in Z / 2^{r+1} \oplus Z / 2^{s-1} \quad \text{when } r > s; \\
\text{(ii)} \quad & h_{W*}(1) = (2, 1) \in Z / 2^s \oplus Z / 2^r \quad \text{when } r = s; \\
\text{(iii)} \quad & h_{W*}(1) = (2 - 2^{s-r+1}, 1) \in Z / 2^{s+1} \oplus Z / 2^{r-1} \quad \text{when } r < s.
\end{align*}
\]

For any \( k(1 \leq k \leq s) \) we set

\[
(4.10) \quad h_{W, k} = 2^{k-1} h_W : \Sigma^2 \to W_{\tau, s}.
\]

**Lemma 4.8.** Assume that \( 1 \leq k \leq s \). Then the cofiber of the map \( h_{W, k} : \Sigma^2 \to W_{\tau, s} \) is the small spectrum \( PV_{r,k} \) and the induced homomorphism \( h_{W, k*} : KU_0 \Sigma^2 \to KU_0 W_{r, s} \) is given by \( h_{W, k*}(1) = 2^{k-1} h_{W*}(1) \) where \( h_{W*}(1) \) is precisely expressed in (4.9).
THE QUASI $K^O_*$-TYPES OF THE STUNTED MOD $4$ LENS SPACES

Proof. Similarly to the proof of Lemma 4.5 we choose a map $f_k : \Sigma^2 \to W_{r,s}$ satisfying the equalities $j_W f_k = 2^k i : \Sigma^0 \to S^2/Z/2^s \to S^{2^s - k}$, $i_W \bar{\eta} \to W_{r,s}$ and $f_k \bar{\eta} = i_W \bar{\eta} : \Sigma^1 S^2/Z/2^k \to W_{r,s}$ so that its cofiber is the small spectrum $P^V_{r,k}$. Evidently $f_s = i_W \bar{\eta} = h_{W,s}$ and $f_k = h_{W,k}$ or $h_{W,k} + i_W \bar{\eta}$ when $k < s$. Since the cofibers of the maps $h_{W,k}$ and $h_{W,k} + i_W \bar{\eta} = (1 + 2^{s-k}) h_{W,k}$ coincide under the assumption that $k < s$, our result is immediate.

Denote by $\tilde{\eta}_V : \Sigma^2 \to V_t$ the composite map $i_V \tilde{\eta} : \Sigma^2 \to S^2/Z/2^{t-1} \to V_t$ when $t \geq 2$, and the bottom cell inclusion $i : \Sigma^2 \to S^2 Z/2$ when $t = 1$. Using the maps $h_{W,k} : \Sigma^2 \to W_{r,s}$, $\bar{\eta} : \Sigma^2 \to S^2/Z/2^t$ and $\tilde{\eta}_V : \Sigma^2 \to V_t$ we consider the two maps

$$
(4.11) \quad f_{W,k} = (h_{W,k}, \bar{\eta}) : \Sigma^2 \to W_{r,s} \vee S^2/Z/2^t \quad \text{and} \quad f_{WV,k} = (h_{W,k}, \tilde{\eta}_V) : \Sigma^2 \to W_{r,s} \vee V_t
$$

for any $k (1 \leq k \leq s)$.

**Lemma 4.9.** Assume that $1 \leq k \leq s$. The cofibers of the maps $f_{W,k} : \Sigma^2 \to W_{r,s} \vee S^2/Z/2^t$ and $f_{WV,k} : \Sigma^2 \to W_{r,s} \vee V_t$ are the small spectra $P^U_{r,t,k}$ and $P^V_{r,t,k}$ respectively, and the induced homomorphisms $f_{W,k}^* : KU^0 \Sigma^2 \to KU^0 W_{r,s} \oplus KU^0 S^2/Z/2^t$ and $f_{WV,k}^* : KU^0 \Sigma^2 \to KU^0 W_{r,s} \oplus KU^0 V_t$ are given by $f_{W,k}^*(1) = f_{WV,k}^*(1) = (h_{W,k}^*(1), 2^{t-1}) \in KU^0 W_{r,s} \oplus Z/2^t$ where $h_{W,k}^*(1)$ is expressed in Lemma 4.8.

Proof. The cofiber of the map $f_{W,k}$ coincides with that of the composite map $\tilde{\eta} (2^{s-k} \vee (-j)) f_{PV} : PV_{r,k} \to \Sigma^3 \vee \Sigma^2 S^2/Z/2^k \to \Sigma^3 \to \Sigma^1 S^2/Z/2^t$. Note that $(i_W^2 \vee 0) f_{PV} : PV_{r,k} \to \Sigma^1 S^2/Z/2^t$ is trivial because $(\tilde{\eta} \vee i \tilde{\eta}) f_{PV} : PV_{r,k} \to \Sigma^1 S^2/Z/2^t$ is trivial. Hence the above composite map $\tilde{\eta} (2^{s-k} \vee (-j)) f_{PV}$ is rewritten to be $(0 \vee (-\tilde{\eta})) f_{PV}$. Therefore the cofiber of the map $f_{W,k}$ coincides with that of the map $\tilde{\eta} \vee i \tilde{\eta} f_{PV} : \Sigma^2 \vee \Sigma^{-1} V'_t \to S^2/Z/2^t$. Thus it is the small spectrum $P^U_{r,t,s}$ given in (2.7). By a similar argument we can also observe that the cofiber of the map $f_{WV,k}$ is the small spectrum $P^V_{r,t,s}$. The latter part of our result is obvious.

**4.5.** For any $k (0 \leq k \leq r)$ and $\ell (0 \leq \ell \leq t)$ we set

$$
(4.12) \quad g_{M,k} = 2^k i_M i : \Sigma^0 \to M_r \quad \text{and} \quad g_{W,\ell} = 2^\ell i_W i : \Sigma^0 \to W_{t,s}.
$$

The cofiber of the map $g_{M,k}$ is the wedge sum $\Sigma^1 \vee M_k$ where $M_0 = \Sigma^2$. 

Produced by The Berkeley Electronic Press, 1993
Thus we have the following cofiber sequence

\[ \Sigma^0 \xrightarrow{g_{M,k}} M_r \xrightarrow{(k_M, g_M)} \Sigma^1 \vee M_k \xrightarrow{2^t - k \vee (-k_M)} \Sigma^1 \]

in which the map \(k_M\) is appearing in (2.3). On the other hand, the cofiber of the map \(g_{W,t}\) is the wedge sum \(\Sigma^1 \vee W_t, s\), and when \(t < s\) the cofiber of the map \(g_{W,\ell}\) coincides with the small spectrum \(V M_{\ell,s}'\), constructed as the cofiber of the map \((\eta \bar{j}, i \bar{n}) : \Sigma^1 S \Sigma/2^s \to \Sigma^1 \vee S \Sigma/2^t\). Note that \(V M_{0,s}' = \Sigma^1 M_s'\) and \(V M_{\ell,s}'\) is the S-dual of \(MV_{s, \ell}'\) given in (2.4), thus \(V M_{\ell,s}' = \Sigma^3 DM_{V_{s, \ell}}'\). We see immediately that the induced homomorphism \(g_{M,k*} : KU_0 \Sigma^0 \to KU_0 M_r\) and \(g_{W,\ell*} : KU_0 \Sigma^0 \to KU_0 W_t, s\) are expressed as follows:

\[(4.13) \quad \begin{align*}
   &i) \quad g_{M,k*}(1) = (0, 2^k) \in Z \oplus Z/2^r; \\
   &ii) \quad g_{W,\ell*}(1) = (2^t + s - t + 1, 2^\ell) \in Z/2^{s+1} \oplus Z/2^{t-1} \quad \text{when} \quad t < s; \\
   &iii) \quad g_{W,\ell*}(1) = (0, 2^\ell) \in Z/2^s \oplus Z/2^t \quad \text{when} \quad t = s; \\
   &iv) \quad g_{W,\ell*}(1) = (2^t + 1 - 2^{t-s} + t + 1, 2^\ell) \in Z/2^{t+1} \oplus Z/2^{s-1} \quad \text{when} \quad t > s.
\end{align*}\]

Using these two maps \(g_{M,k} : \Sigma^0 \to M_r\) and \(g_{W,\ell} : \Sigma^0 \to W_t, s\) we consider the map

\[(4.14) \quad g_{MW, k, \ell} = (g_{M,k}, g_{W,\ell}) : \Sigma^0 \to M_r \vee W_t, s\]

for any \(k(0 \leq k \leq r)\) and \(\ell(0 \leq \ell \leq t)\).

**Lemma 4.10.** Assume that \(0 \leq k \leq r\) and \(0 \leq \ell \leq t \leq r - k + \ell\). Then the cofiber of the map \(g_{MW, k, \ell} : \Sigma^0 \to M_r \vee W_t, s\) is the wedge sum \(\Sigma^1 \vee M_{U_{\ell,t-k-\ell}} \vee (k, k_M - k_M)\) or \(\Sigma^1 \vee M_k \vee W_{t, s}\) according as \(k > \ell < t\) or otherwise, and the induced homomorphism \(g_{MW, k, \ell*} : KU_0 \Sigma^0 \to KU_0 M_r \oplus KU_0 W_{t, s}\) is given by \(g_{MW, k, \ell*}(1) = (0, 2^k, g_{W,\ell*}(1)) \in Z \oplus Z/2^s \oplus KU_0 W_{t, s}\) where \(g_{W,\ell*}(1)\) is precisely expressed in (4.13).

**Proof.** The cofiber of the map \(g_{MW, k, \ell}\) is obtained as the cofiber of the composite map \(g_{W,\ell}(2^t - k \vee (-k_M)) : \Sigma^0 \vee \Sigma^{-1} M_k \to W_t, s\). Evidently the latter map is rewritten to be \(0 \vee (-2^\ell i_{W, k_M})\) under the assumption that \(r - k + \ell \geq t\). Set \(g_{MW,\ell} = 2^\ell i_{W, k_M} : \Sigma^{-1} M_k \to W_t, s\). Then the cofiber of the map \(g_{MW, k, \ell}\) is just the wedge sum of \(\Sigma^1\) and the cofiber of the map \(g_{MW,\ell}\). Since \(2^k i_{k_M} = i_{\eta, k_M} : \Sigma^{-1} M_k \to S \Sigma/2^t\), it is easily seen that the map \(g_{MW,\ell} : \Sigma^{-1} M_k \to W_{t, s}\) is trivial if \(k \leq \ell\) or \(t = \ell\). Therefore
the cofiber of the map \( g_{MW} \) is exactly the wedge sum \( M_k \vee W_{t,s} \) in the \( k \leq t \) or \( t = t \) case. Assume that \( k > t < t \). Then the cofiber of the map
\[ 2k M : \Sigma^{-1} M_k \to \Sigma^0 \]
is just the wedge sum \( P \vee \Sigma^2/2^t \) because the cofiber of the map \( M : \Sigma^{-1} M_k \to \Sigma^0 \) is the elementary spectrum \( P \). Evidently the cofiber of the composite map \( 2iM : \Sigma^{-1} M_k \to \Sigma^2/2^t \) is the wedge sum \( M_{t+k+\ell} \vee \Sigma^2/2^t \). Thus there exists a cofiber sequence
\[ \Sigma^{-1} M_k \to \Sigma^2/2^t \to M_{t+k+\ell} \vee \Sigma^2/2^t \to M_k. \]

Here \( \varphi_1 = 2k-t \) and \( \varphi_2 = 2k \), \( i : \Sigma^0 \to M_{t+k+\ell} \) and \( i : \Sigma^2/2^t \to \Sigma^1 \) and \( i : \Sigma^2/2^t \to \Sigma^2/2^t \) in which the map \( i : \Sigma^0 \to M_{t+k+\ell} \) is appearing in (2.3). As is easily seen, \( \varphi_1 = i : \Sigma^0 \to M_{t+k+\ell} \) and \( \varphi_2 = \rho_{t+\ell} + \alpha \eta j : \Sigma^2/2^t \to \Sigma^2/2^t \) for some \( a \) where \( \rho_{t+\ell} \) is the obvious map. Since \( 2iM : \Sigma^2/2^t \to \Sigma^3/2^t \) is trivial, it follows that \( \varphi_1(i + \eta j) = i_M \eta j : \Sigma^3/2^t \to \Sigma^3/2^t \) and \( \varphi_2(i + \eta j) = \eta(1 + a \eta j) : \Sigma^3/2^t \to \Sigma^3/2^t \). Hence we can observe that when \( k > t < t \) the cofiber of the map \( g_{MW} = 2iM \) coincides with that of the map \( i_M : \Sigma^0 \to M_{t+k+\ell} \vee \Sigma^2/2^t \), which is exactly the desired spectrum \( M_{U(t+k+\ell) \vee t} \).

5. The stunted mod 4 lens spaces

5.1. Let \( L^k(4) \) be the \((2k+1)\)-dimensional standard mod 4 lens space and \( L_0^k(4) \) its \( 2k \)-skeleton. For simplicity we set \( L^{2k+1} = L^k(4) \) and \( L^k = L_0^k(4) \). Recall the structure of \( KU \)-cohomology \( K^* U^n \) (see [5] or [7]). The inclusion \( i : \Sigma^0 \to L^{2k+1} \) induces an isomorphism \( i^* : KU^0 L^{2k+1} \cong KU^0 L^k \), and \( KU^1 L^{2k+1} \cong Z \) and \( KU^1 L^k = 0 \). The ring \( KU^0 L^{2k+1} \cong KU^0 L^k \) is generated by \( \sigma = \gamma - 1 \), whose multiplicative structure is given by the two relations \((\sigma + 1)^4 = 1 \) and \( \sigma^{k+1} = 0 \). Here \( \gamma \) denotes the canonical complex line bundle over \( L^{2k+1} = L^k(4) \) or its restriction to \( L^k = L_0^k(4) \). According to [5, Theorem 4.6] the \( KU \)-cohomology \( KU^0 L^{2k+1} \cong KU^0 L^k(2m = 2m + 1) \) is explicitly given as follows:

\[ KU^0 L^{4m+1} \cong KU^0 L^{4m} \cong Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1} \]
\[ KU^0 L^{4m+3} \cong KU^0 L^{4m+2} \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m \]

whose direct summands are generated by the elements \( \sigma, \sigma(1) \) and \( \sigma(1) + 2^{m+1} \) in the former case, and \( \sigma, \sigma(1) + 2^{m+1} \) and \( \sigma(1) \) in the latter case, where \( \sigma = \gamma - 1 \) and \( \sigma(1) = \gamma^2 - 1 = \gamma^2 + 2 \).
We next study the behavior of the complex Adams operation $\psi^r_C$ on $KU^0L^{2k+1} \cong KU^0L^{2k}$ after changing the above direct summands slightly as follows:

\[(5.1) \quad \begin{array}{l}
\text{i) } KU^0L^{4m+1} \cong KU^0L^{4m} \cong Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1} \\
\text{with generators } \sigma, \sigma(1) + \sigma(1) \text{ and } \sigma(1) + 2^{m+1}\sigma, \text{ and} \\
\text{ii) } KU^0L^{4m+3} \cong KU^0L^{4m+2} \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m \\
\text{with generators } \sigma, \sigma(1) + \sigma(1) \text{ and } 2^{m+1}\sigma + \sigma(1) + 2^{m+1}\sigma.
\end{array}\]

Since $\psi^{r+4}_C = \psi^r_C \sigma$ and $\psi^{r+2}_C(1) = \psi^r_C(1)$, it is evident that $\psi^{r+4}_C = \psi^r_C$ on $KU^0L^{2k+1} \cong KU^0L^{2k}$. As is easily calculated, the complex Adams operation $\psi^r_C$ on $KU^0L^{2k+1} \cong KU^0L^{2k}(k = 2m \text{ or } 2m + 1)$ is given as follows:

\[(5.2) \quad \begin{array}{l}
\text{i) } \psi^{4s}_C = 0 \text{ and } \psi^{4s+1}_C = 1; \\
\text{ii) } \psi^{4s+2}_C = \begin{pmatrix} 2^{m+1} & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2^{m+1} & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\
\text{iii) } \psi^{4s+3}_C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 - 2^{m+1} & 2^{m+2} & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\]

which operate respectively on $KU^0L^{4m+1} \cong KU^0L^{4m} \cong Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$ and $KU^0L^{4m+3} \cong KU^0L^{4m+2} \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$ whose direct summands are given as in (5.1) i) and ii). Here the matrices behave always as left action.

Dualizing (5.1) and (5.2) we can study the behavior of the complex Adams operation $\psi^r_C$ on $KU_1L^n \otimes Z[1/\tau]$, and in particular the conjugation $\psi^{-1}_C$ on $KU_1L^n$. Note that $KU_1L^{2k} \cong KU^0L^{2k}$, $KU_1L^{2k+1} \cong KU_1\Sigma^{2k+1} \otimes KU_1L^{2k}$ and $KU_0L^{2k} = KU_0L^{2k+1} = 0$. By virtue of (5.1) the induced homomorphism $i^* : KU^0L^{2k+2} \to KU^0L^{2k+1}$ is actually represented by the following matrix $A_k(k = 2m \text{ or } 2m + 1)$:

\[(5.3) \quad A_{2m} = \begin{pmatrix} 1 & 2^{m+1} & 2^{m+2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{2m+1} = \begin{pmatrix} 1 & -2^{m+1} & 2^{m+2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\]

where $A_{2m} : Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m \to Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$ and $A_{2m+1} : Z/2^{2m+3} \oplus Z/2^{m+1} \oplus Z/2^m \to Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$. Therefore the induced homomorphism $i_\ast : KU_1L^{2k+1} \to KU_1L^{2k+2}$ is given by the following matrix $B_k(k = 2m \text{ or } 2m + 1)$:
THE QUASI $K\text{O}$-TYPES OF THE STUNTED MOD 4 LENS SPACES

(5.4) $B_{2m} = \begin{pmatrix} x & 2 & 0 & 0 \\ y & 1 & 1 & 0 \\ z & 2 & 1 & -1 \end{pmatrix}$ and $B_{2m+1} = \begin{pmatrix} u & 2 & 0 & 0 \\ v & -1 & 2 & 0 \\ w & 1 & 1 & -1 \end{pmatrix}$

where $B_{2m} : Z \oplus Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1} \to Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$ and $B_{2m+1} : Z \oplus Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m \to Z/2^{2m+3} \oplus Z/2^{m+1} \oplus Z/2^m$.

Since the above induced homomorphism $i_*$ is an epimorphism in any case, it follows that $x$ and $u$ must be odd. Using this fact we show

**Proposition 5.1.** The suspended mod 4 lens space $\Sigma^1 L^n(n \geq 2)$ has the same $\varepsilon$-type as the small spectrum $U_{m-1,2,m+1,n}, MU_{m-1,2,m+1,n}$, $SZ/2^m \vee W_{2m+1,m+1}$ or $\Sigma^0 \vee SZ/2^m \vee W_{2m+1,m+1}$ according as $n = 4m, 4m + 1, 4m + 2$ or $4m + 3$, where $W_{1,1}$ should be replaced by $\Sigma^2 SZ/4$ in the $n = 2$ and 3 cases.

**Proof.** The $n = 2k$ case is just shown as the dual of (5.2). On the other hand, the conjugation $\psi_C^{-1}$ on $KU_{-1}L^{2k+1}$ is represented by the following matrix:

$$
\psi_C^{-1} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
1 & 2^{m+1} & 0 & 0 \\
a & 0 & -1 & 0 \\
b & 0 & -1 & 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
1 & 0 & 0 & 0 \\
d & 1 - 2^{m+1} & 2^{m+2} & 0 \\
ea & 1 & -1 & 0 \\
f & 0 & 0 & 1
\end{pmatrix}
$$

according as $k = 2m$ or $2m + 1$. Using the equality $\psi_C^{-1}i_* = i_*\psi_C^{-1} : KU_{-1}L^{2k+1} \to KU_{-1}L^{2k+2}$ we get immediately that $a = x + 2^ma', b = 0, c = x - z, d = 2^{m+1}d', e = -d'$ and $f = 0$. As is easily verified, we may take $x = -1, a' = c = 0$ and $d' = 0$ after changing the direct sum decomposition of $KU_{-1}L^{2k+1} \cong Z \oplus KU_{-1}L^{2k}$ suitably if necessary. Now our result is immediate from Propositions 2.1 and 2.3.

**5.2.** The stunted mod 4 lens space $L^n/L^m(n > m \geq 0)$ is simply written to be $L^n_{m+1}$ as usual. We here study the behavior of the conjugations $\psi_C^{-1}$ on $KU^n_{-1}L^m_{m+1}$ and $KU^n_{-1}L^m_{m+1}$. Similarly to (5.3) the induced homomorphism $i^* : KU^0L^{2\ell} \to KU^0L^{2k}(\ell > k)$ is represented by the following matrix $A_{\ell,k}$.
Z. YOSIMURA

\[
A_{2n,2m} = \begin{pmatrix}
1 & 0 & 2^{n+1} - 2^{m+1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
A_{2n+1,2m} = \begin{pmatrix}
1 & 2^{n+1} & 2^n + 1 + 2^{m+1} \\
0 & 1 & 1 \\
0 & 0 & -1
\end{pmatrix},
\]

\[
A_{2n+1,2m+1} = \begin{pmatrix}
1 & 2^{n+1} - 2^{m+1} & 2^n + 1 - 2^{m+1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
A_{2n+2,2m+1} = \begin{pmatrix}
1 & -2^{m+1} & 2^{n+2} \\
0 & 1 & 1 \\
0 & 0 & -1
\end{pmatrix}
\]

where \(A_{2n,2m} : \mathbb{Z}/2^{2n+1} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^{n-1} \rightarrow \mathbb{Z}/2^{2m+1} \oplus \mathbb{Z}/2^m \oplus \mathbb{Z}/2^{m-1}\),
\(A_{2n+1,2m} : \mathbb{Z}/2^{2n+2} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^{n-1} \rightarrow \mathbb{Z}/2^{2m+1} \oplus \mathbb{Z}/2^m \oplus \mathbb{Z}/2^{m-1}\),
\(A_{2n+1,2m+1} : \mathbb{Z}/2^{2n+2} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^{n-1} \rightarrow \mathbb{Z}/2^{2m+2} \oplus \mathbb{Z}/2^m \oplus \mathbb{Z}/2^{m-1}\) and
\(A_{2n+2,2m+1} : \mathbb{Z}/2^{2n+3} \oplus \mathbb{Z}/2^{n+1} \oplus \mathbb{Z}/2^n \rightarrow \mathbb{Z}/2^{2m+2} \oplus \mathbb{Z}/2^m \oplus \mathbb{Z}/2^{m-1}\).

The projection \(p : L_{2k+1}^{2\ell} \rightarrow L_{2k+1}^{2\ell}\) induces a monomorphism \(p^* : KU^0 L_{2k+1}^{2\ell} \rightarrow KU^0 L_{2k+1}^{2\ell}\), which is represented by the following matrix \(C_{k,\ell}\):

\[
C_{2m,2n} = \begin{pmatrix}
2^{2m} & 0 & 0 \\
0 & 2^m & 0 \\
2^{2m-1} - 2^{m-1} & 0 & 2^{m-1}
\end{pmatrix},
\]

\[
C_{2m,2n+1} = \begin{pmatrix}
2^{2m-1} - 2^{m-1} & 2^m & 0 \\
0 & 2^{2m+1} & 0 \\
2^{m-1} - 2^{2m-1} & 0 & 2^{m-1}
\end{pmatrix},
\]

\[
C_{2m+1,2n+1} = \begin{pmatrix}
2^{2m} & 2^m & 0 \\
0 & 2^{m+1} & 0 \\
2^m - 2^{2m} & -2^m & 2^{m+1}
\end{pmatrix},
\]

\[
C_{2m+1,2n+2} = \begin{pmatrix}
2^m & 2^{m+1} & 0 \\
0 & 2^{m+1} & 0 \\
2^{2m} - 2^m & 0 & 2^m
\end{pmatrix}
\]

where \(C_{2m,2n} : \mathbb{Z}/2^{2n-2m+1} \oplus \mathbb{Z}/2^{n-m} \oplus \mathbb{Z}/2^{n-m-1} \rightarrow \mathbb{Z}/2^{2n+1} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^{n-1}\),
\(C_{2m,2n+1} : \mathbb{Z}/2^{2n-2m+2} \oplus \mathbb{Z}/2^{n-m} \oplus \mathbb{Z}/2^{n-m} \rightarrow \mathbb{Z}/2^{2n+2} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^{n-1}\),
\(C_{2m+1,2n+1} : \mathbb{Z}/2^{2n-2m-1} \oplus \mathbb{Z}/2^{n-m} \oplus \mathbb{Z}/2^{n-m-1} \rightarrow \mathbb{Z}/2^{2n+2} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n\) and \(C_{2m+1,2n+2} : \mathbb{Z}/2^{2n-2m+2} \oplus \mathbb{Z}/2^{n-m} \oplus \mathbb{Z}/2^{n-m} \rightarrow \mathbb{Z}/2^{2n+3} \oplus \mathbb{Z}/2^{n+1} \oplus \mathbb{Z}/2^n\).

Using the equality \(p^* \psi_C^{-1} = \psi_C^{-1} p^* : KU^0 L_{2k+1}^{2\ell} \rightarrow KU^0 L_{2k+1}^{2\ell}\) we can easily show the following result by virtue of Proposition 2.1.
THE QUASI $KO_*$-TYPES OF THE STUNTED MOD 4 LENS SPACES 221

Proposition 5.2. The $S$-dual $DL^{2\ell+2k}_{2k+1}$ of the stunted mod 4 lens space $L^{2\ell+2k}_{2k+1}(\ell \geq 1)$ has the same $\varepsilon$-type as the small spectrum $U_{2n,n,n}$, $SZ/2^n \vee W_{2n+1,n+1}$, $U_{n-1,2n+1,n}$ or $SZ/2^{2n+2} \vee W_{n,n}$ according as $(k,\ell) = (2m,2n)$, $(2m,2n+1)$, $(2m+1,2n)$ or $(2m+1,2n+1)$, where $W_{1,1}$ should be replaced by $\Sigma^2SZ/4$ in the $(k,\ell) = (2m,1)$ case.

Dualizing Proposition 5.2 we can immediately obtain

Corollary 5.3. The suspended stunted mod 4 lens space $\Sigma^1L^{2\ell+2k}_{2k+1}$ ($\ell \geq 1$) has the same $\varepsilon$-type as the small spectrum $U_{n-1,2n+1,n}$, $SZ/2^n \vee W_{2n+1,n+1}$, $U_{2n,n,n}$ or $SZ/2^{2n+2} \vee W_{n,n}$ according as $(k,\ell) = (2m,2n)$, $(2m,2n+1)$, $(2m+1,2n)$ or $(2m+1,2n+1)$, where $W_{1,1}$ should be replaced by $\Sigma^2SZ/4$ in the $(k,\ell) = (2m,1)$ case.

The induced homomorphism $p_* : KU_{-1}L^{2\ell}_2 \rightarrow KU_{-1}L^{2\ell}_{2k+1}$ is represented by the following matrix $C'_{\ell,k}$ dual to (5.6):

$$C'_{2n,2m} = \begin{pmatrix} 1 & 0 & 2^n \frac{1}{2} - 2^{n-m+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C'_{2n+1,2m} = \begin{pmatrix} 1 & 2^n \frac{1}{2} - 2^{n-m+1} - 2^{n-m+1} - 2^{n+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C'_{2n+1,2m+1} = \begin{pmatrix} 1 & 2^n \frac{1}{2} - 2^{n-m+1} - 2^{n+1} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C'_{2n+2,2m+1} = \begin{pmatrix} 1 & 2^n \frac{1}{2} - 2^{n+2} - 2^{n-m+2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $C'_{2n,2m} : Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^n \rightarrow Z/2^{n-2m+1} \oplus Z/2^{n-m} \oplus Z/2^{n-m}$, $C'_{2n+1,2m} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \rightarrow Z/2^{n-2m+2} \oplus Z/2^{n-m} \oplus Z/2^{n-m}$, $C'_{2n+1,2m+1} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \rightarrow Z/2^{n-2m+1} \oplus Z/2^{n-m} \oplus Z/2^{n-m}$ and $C'_{2n+2,2m+1} : Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n \rightarrow Z/2^{n-2m+2} \oplus Z/2^{n-m} \oplus Z/2^{n-m}$.

Notice that $KU_{-1}L^{2\ell+1}_{2k+1} \cong KU_{-1}2^{2\ell+1} \oplus KU_{-1}L^{2\ell}_{2k+1}$ and $KU_{0}L^{2\ell+1}_{2k+1} = 0$. The induced homomorphism $p_* : KU_{-1}L^{2\ell}_2 \rightarrow KU_{-1}L^{2\ell}_{2k+1}$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & C'_{\ell,k} \end{pmatrix} : Z \oplus KU_{-1}L^{2\ell}_2 \rightarrow Z \oplus KU_{-1}L^{2\ell}_{2k+1}$$
in which the matrix $C_{i,k}$ is explicitly expressed in (5.7). Since the induced homomorphism $p_* : KU_{-1}L^{2\ell+1}_k \rightarrow KU_{-1}L^{2\ell+1}_{2k+1}$ is an epimorphism, we can easily show the following result by means of Proposition 5.1.

**Proposition 5.4.** The suspended stunted mod 4 lens space $\Sigma^1 L^{2\ell+2k+1}_{2k+1} (\ell \geq 1)$ has the same $\aleph$-type as the small spectrum $MU_{n-1,2n+1,n}, \Sigma^0 \vee SZ/2^n \vee W_{2n+1,2n+1}, \Sigma^0 \vee U_{2n,n,n}$ or $M_{2n+2} \vee W_{n,n}$ according as $(k,\ell) = (2m,2n), (2m,2n+1), (2m+1,2n)$ or $(2m+1,2n+1)$, where $W_{1,1}$ should be replaced by $\Sigma^2 SZ/4$ in the $(k,\ell) = (2m,1)$ case.

5.3. By means of (5.6) we can easily give the matrix representation of the induced homomorphism $j^* : KU^0 L^{2\ell+2k}_{2k+1} \rightarrow KU^0 L^{2\ell+2k}_{2k-1}$ where $j : L^{2\ell+2k}_{2k-1} \rightarrow L^{2\ell+2k}_{2k+1}$ denotes the canonical projection. Note that $KU^0 L^{2\ell+2k}_{2k} \cong KU^0 \Sigma^{2k} \oplus KU^0 L^{2\ell+2k}_{2k+1}$ and $KU^1 L^{2\ell+2k}_{2k} = 0$. Then the bottom cell collapsing $j : L^{2\ell+2k}_{2k-1} \rightarrow L^{2\ell+2k}_{2k}$ induces an epimorphism $j^* : KU^0 L^{2\ell+2k}_{2k-1} \rightarrow KU^0 L^{2\ell+2k}_{2k-1}$, which is represented by the following matrix $B_{k,\ell}$ similarly to (5.4):

$$B_{2m,2n} = \begin{pmatrix} x_1 & 2 & 0 & 0 \\ y_1 & 1 & 1 & 0 \\ z_1 & 1 & 0 & 2 \end{pmatrix}, \quad B_{2m,2n+1} = \begin{pmatrix} x_2 & 2 & 0 & 0 \\ y_2 & 1 & 1 & 2 \\ z_2 & 1 & 1 & 0 \end{pmatrix}$$

(5.9)

$$B_{2m+1,2n} = \begin{pmatrix} x_3 & 2 & 0 & 0 \\ y_3 & 1 & 1 & 0 \\ z_3 & 0 & 1 & 2 \end{pmatrix}, \quad B_{2m+1,2n+1} = \begin{pmatrix} x_4 & 2 & 0 & 0 \\ y_4 & 1 & 1 & 2 \\ z_4 & 0 & 0 & 1 \end{pmatrix}$$

where $B_{2m,2n} : Z \oplus Z/2^{n+1} \oplus Z/2^n \oplus Z/2^{n-1} \rightarrow Z/2^{n+2} \oplus Z/2^n \oplus Z/2^n$, $B_{2m,2n+1} : Z \oplus Z/2^{n+2} \oplus Z/2^n \oplus Z/2^n \rightarrow Z/2^{n+3} \oplus Z/2^{n+1} \oplus Z/2^n$, $B_{2m+1,2n} : Z \oplus Z/2^{n+1} \oplus Z/2^n \oplus Z/2^{n-1} \rightarrow Z/2^{n+2} \oplus Z/2^n \oplus Z/2^n$, and $B_{2m+1,2n+1} : Z \oplus Z/2^{n+2} \oplus Z/2^n \oplus Z/2^n \rightarrow Z/2^{n+3} \oplus Z/2^{n+1} \oplus Z/2^n$. Notice that all of $x_i (1 \leq i \leq 4)$ must be odd. By a quite similar argument to Proposition 5.1 we show

**Proposition 5.5.** The S-dual $DL^{2\ell+2k}_{2k}$ of the stunted mod 4 lens space $L^{2\ell+2}_{2k} (\ell \geq 1)$ has the same $\aleph$-type as the small spectrum $\Sigma^0 \vee U_{2n,n,n}$, $\Sigma^0 \vee SZ/2^n \vee W_{2n+1,2n+1}, MU_{n-1,2n+1,n}$ or $M_{2n+2} \vee W_{n,n}$ according as $(k,\ell) = (2m,2n), (2m,2n+1), (2m+1,2n)$ or $(2m+1,2n+1)$, where $W_{1,1}$ should be replaced by $\Sigma^2 SZ/4$ in the $(k,\ell) = (2m,1)$ case.
THE QUASI KO-TYPES OF THE STUNTED MOD 4 LENS SPACES

Proof. By virtue of Proposition 5.2 the conjugation \( \psi_C^{-1} \) on \( KU^0 L_{2k}^{2\ell+2k} \) is expressed by the following matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

according as \((k, \ell) = (2m, 2n), (2m, 2n+1), (2m+1, 2n)\) or \((2m+1, 2n+1)\). Using the equality \( j^* \psi_C^{-1} = \psi_C^{-1} j^* \) : \( KU^0 L_{2k}^{2\ell+2k} \to KU^0 L_{2k}^{2\ell+2k} \) we see that

i) \( a_1 = c_1 = 0, b_1 = -y_1 - z_1 \); ii) \( a_2 = 2^{\ell+1} a_2', b_2 = -a_2', c_2 = 0 \);

iii) \( a_3 = x_3 + 2^n a_3', b_3 = 0, c_3 = z_3 \); and iv) \( a_4 = x_4, b_4 = -z_4, c_4 = -2z_4 \).

As in the proof of Proposition 5.2 we may take \( a_2' = a_3' = 0, x_3 = x_4 = -1 \) and \( b_1 = c_3 = b_4 = c_4 = 0 \). Thus \( a_i, b_i \) and \( c_i (1 \leq i \leq 4) \) are taken to be zero except \( a_3 \) and \( a_4 \), while \( a_3 = a_4 = 1 \) as desired. Now our result is immediate from Propositions 2.1 and 2.3.

By means of (5.7) we can represent the induced homomorphism \( j_* : KU_{-1} L_{2k}^{2\ell+2k} \to KU_{-1} L_{2k}^{2\ell+2k} \) by the following matrix \( D_{\ell,k} \):

\[
D_{2n,2m} = \begin{pmatrix}
1 & -2^{n+1} & 2^n & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix},
\]

(5.10)

\[
D_{2n,2m+1} = \begin{pmatrix}
1 & 2^n & 0 \\
0 & 1 & -1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( D_{2n,2m} : \mathbb{Z}/2^{2n+2} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \to \mathbb{Z}/2^{2n+1} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^{n-1} \),

\( D_{2n,2m+1} : \mathbb{Z}/2^{2n+3} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \to \mathbb{Z}/2^{2n+2} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^{n-1} \) and \( D_{2n+1,2m+1} : \mathbb{Z}/2^{2n+3} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \to \mathbb{Z}/2^{2n+2} \oplus \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \).

Evidently the induced homomorphism \( j_* : KU_{-1} L_{2k}^{2\ell+2k+1} \to KU_{-1} L_{2k+1}^{2\ell+2k+1} \) is represented by the matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & D_{\ell,k}
\end{pmatrix} : \mathbb{Z} \oplus KU_{-1} L_{2k-1}^{2\ell+2k} \to \mathbb{Z} \oplus KU_{-1} L_{2k+1}^{2\ell+2k}
\]

(5.11)
in which the matrix $D_{\ell,k}$ is explicitly expressed in (5.10).

Consider the exact sequence

$$0 \to KU_1 L_{2k}^n \to KU_1 L_{2k+1}^n \to KU_0 \Sigma_{2k} \to KU_0 L_{2k}^n \to KU_0 L_{2k+1}^n \to 0$$

induced by the cofiber sequence $\Sigma_{2k} \to L_{2k}^n \to L_{2k+1}^n \to \Sigma_{2k+1}^k (n > 2k)$

where $KU_0 L_{2k+1}^n = 0$. Assume that $KU_0 L_{2k}^n = 0$, and then $KU_1 L_{2k+1}^n \cong Z \oplus KU_1 L_{2k}^n$. When $n = 2\ell$ this is evidently a contradiction because $KU_1 L_{2k+1}^{2\ell} \oplus Q = 0$. In the $n = 2\ell + 1$ case our assumption implies that $KU_1 L_{2k}^{2\ell+1} \cong KU_1 L_{2k+1}^{2\ell}$ because $KU_1 L_{2k+1}^{2\ell+1} \cong Z \oplus KU_1 L_{2k+1}^{2\ell}$. In $KU_1 L_{2k-1}^{2\ell+1}$ there exists an element of order $2^{\ell-k+2}$, but in $KU_1 L_{2k+1}^{2\ell+1}$ there exist no elements of order $2^{\ell-k+2}$ under our assumption. As is easily checked, this is a contradiction, too. Therefore it is verified that $KU_0 L_{2k}^n \cong Z$, and hence there exist isomorphisms

$$(5.12) \quad i_* : KU_0 \Sigma_{2k} \cong KU_0 L_{2k}^n \quad \text{and} \quad j_* : KU_1 L_{2k}^n \cong KU_1 L_{2k+1}^n$$

for any $n > 2k$ where $i : \Sigma_{2k} \to L_{2k}^n$ and $j : L_{2k}^n \to L_{2k+1}^n$ denote the bottom cell inclusion and collapsing respectively.

Using (5.11) and (5.12) we can immediately show

**Lemma 5.6.** The induced homomorphism $i_* : KU_{-1} \Sigma_{2k+1} \to KU_{-1} L_{2k+1}^{2\ell+2k+1} (\ell \geq 1)$ is identified with the homomorphism $\varphi_{k,\ell}$ defined as follows:

$$\varphi_{2m,2n}(1) = (0, 2^{2n-1}, 2^{n-1}, 0) \in Z \oplus Z / 2^{2n+1} \oplus Z / 2^n \oplus Z / 2^{n-1};$$

$$\varphi_{2m,2n+1}(1) = (0, 2^{2n}, 2^{n-1}, 0) \in Z \oplus Z / 2^{2n+2} \oplus Z / 2^n \oplus Z / 2^{n-1};$$

$$\varphi_{2m+1,2n}(1) = (0, 2^{2n-1}, 2^n, 0) \in Z \oplus Z / 2^{2n+1} \oplus Z / 2^n \oplus Z / 2^{n-1};$$

$$\varphi_{2m+1,2n+1}(1) = (0, 2^{2n}, 0, 2^{n-1}) \in Z \oplus Z / 2^{2n+2} \oplus Z / 2^n \oplus Z / 2^{n-1}.$$

**5.4.** In order to determine the quasi $KO_\ast$-types of the stunted mod 4 lens spaces $L_{m+1}^n = L^n / L^m$ we only need the following part (cf. [15, Lemma 2.2]), although $KO_\ast L_{m+1}^n$ and hence $KO_\ast L_{m+1}^n$ are completely calculated in [6, Theorem 2] and [8, Theorem 2].

**Lemma 5.7.** i) $KO_{4m} L_{4m+1}^{4m+n} = 0 = KO_{4m} L_{4m-1}^{4m+n}$ if $n \equiv 1, 2, 3, 4, 5 \text{mod } 8$.

ii) $KO_{4m+4} L_{4m+1}^{4m+n} = 0 = KO_{4m+4} L_{4m-1}^{4m+n}$ if $n \equiv 0, 1, 5, 6, 7 \text{mod } 8$.

iii) $KO_{4m+6} L_{4m+1}^{4m+n} = 0 = KO_{4m+6} L_{4m-1}^{4m+n}$ for all $n$.

iv) $KO_{4m-3} L_{4m+1}^{4m+2\ell} = 0 = KO_{4m-3} L_{4m-2}^{4m+2\ell}$ if $\ell \equiv 1, 2 \text{mod } 4$. 

http://escholarship.lib.okayama-u.ac.jp/mjou/vol35/iss1/15
THE QUASI $KO_*$-TYPES OF THE STUNTED MOD 4 LENS SPACES

v) $KO^{4m-7}L_{4m}^{4m+2\ell} = 0 = KO^{4m-7}L_{4m-2}^{4m+2\ell}$ if $\ell \equiv 0, 3 \mod 4$.

vi) $KO^{4m-5}L_{4m}^{4m+2\ell} = 0 = KO^{4m-5}L_{4m-2}^{4m+2\ell}$ for all $\ell$.

Using Corollary 5.3, Propositions 5.4 and 5.5 and Lemma 5.7 and then applying Proposition 3.1 and Theorem 3.3, we can first determine the quasi $KO_*$-types of $L_{2k+1}^{2k}$ and $DL_{2k}^{2k+2\ell}$.

**Theorem 5.8.** i) $\Sigma^{-4m+1}L_{4m+1}^{4m+n}(n \geq 2)$ is quasi $KO_*$-equivalent to the following small spectrum: $U_{2r-1,4r+1,2r}, MU_{2r-1,4r+1,2r}, V_{2r} \lor W_{4r+1,2r+1}, \Sigma^4 \lor V_{2r} \lor W_{4r+1,2r+1}, V_{2r,4r+3,2r+1}, MU_{2r,4r+3,2r+1}, Sz/2^{2r+1} \lor W_{4r+3,2r+2}, \Sigma^0 \lor Sz/2^{2r+1} \lor W_{4r+3,2r+2}$ according as $n = 8r, 8r + 1, \ldots, 8r + 7$. Here $V_0 \lor W_{1,1}$ should be replaced by $\Sigma^2 Sz/4$ in the $n = 2$ and $3$ cases.

ii) $\Sigma^{-4m+1}L_{4m-1}^{4m+n-2}(n \geq 2)$ is quasi $KO_*$-equivalent to the following small spectrum: $U_{4r,2r,2r}, \Sigma^0 \lor U_{4r,2r,2r}, Sz/2^{4r+2} \lor W_{2r,2r}, M_{4r+2} \lor W_{2r,2r}, V_{4r+2,2r+1,2r+1}, \Sigma^4 \lor V_{4r+2,2r+1,2r+1}, V_{4r+4} \lor W_{4r+1,2r+1}, M_{4r+4} \lor W_{2r+1,2r+1}$ according as $n = 8r, 8r + 1, \ldots, 8r + 7$.

iii) $\Sigma^4 DL_{4m}^{4m+2\ell}(\ell \geq 1)$ is quasi $KO_*$-equivalent to the following small spectrum: $\Sigma^0 \lor U_{4r,2r,2r}, \Sigma^0 \lor \Sigma^4 V_{2r} \lor W_{4r+1,2r+1}, \Sigma^0 \lor \Sigma^4 V_{4r+2,2r+1,2r+1}, \Sigma^0 \lor Sz/2^{2r+1} \lor W_{4r+3,2r+2}$ according as $\ell = 4r, 4r + 1, 4r + 2, 4r + 3$. Here $\Sigma^4 V_0 \lor W_{1,1}$ should be replaced by $\Sigma^{-2} Sz/4$ in the $\ell = 1$ case.

iv) $\Sigma^4 DL_{4m-2}^{4m+2\ell-2}(\ell \geq 1)$ is quasi $KO_*$-equivalent to the following small spectrum: $MU_{2r-1,4r+1,2r}, M_{4r+2} \lor W_{2r,2r}, \Sigma^4 MU_{2r,4r+3,2r+1}, \Sigma^4 M_{4r+4} \lor W_{2r+1,2r+1}$ according as $\ell = 4r, 4r + 1, 4r + 2, 4r + 3$.

From [10, Corollary 1.8] we recall that (5.13) two finite spectra $X$ and $Y$ have the same quasi $KO_*$-type if and only if their $S$-duals $DX$ and $DY$ have the same quasi $KO_*$-type.

By virtue of Theorem 5.8 iii) and iv) and (5.13) we can next determine the quasi $KO_*$-types of $L_{2k}^{2k+2\ell}$ with the aid of Corollary 3.4.

**Theorem 5.9.** i) $\Sigma^{-4m+1}L_{4m}^{4m+2\ell}(\ell \geq 1)$ is quasi $KO_*$-equivalent to the following small spectrum: $\Sigma^1 \lor U_{2r-1,4r+1,2r}, \Sigma^1 \lor V_{2r} \lor W_{4r+1,2r+1}, \Sigma^1 \lor V_{2r,4r+3,2r+1}, \Sigma^1 \lor Sz/2^{2r+1} \lor W_{4r+3,2r+2}$ according as $\ell = 4r, 4r + 1, 4r + 2, 4r + 3$. Here $V_0 \lor W_{1,1}$ should be replaced by $\Sigma^2 Sz/4$ in the $\ell = 1$ case.
ii) \( \Sigma^{-4m+1}L_{4m-2}^{4m+2\ell-2}(\ell \geq 1) \) is quasi \( KO_* \)-equivalent to the following small spectrum: \( PU_{4r+1,2r,2r}, P_{4r+3} \vee W_{2r,2r}, \Sigma^4PU_{4r+3,2r+1,2r+1}, \Sigma^4P_{4r+5} \vee W_{2r+1,2r+1} \) according as \( \ell = 4r, 4r+1, 4r+2, 4r+3 \).

5.5. Using the maps appearing in Lemmas 4.3, 4.4, 4.5, 4.7, 4.9 and 4.10 we here consider the following maps \( f_{k,\ell} : Y_{k,\ell} \to X_{k,\ell} \) modelled on the bottom cell inclusions \( i : \Sigma^{2k-4m+2} \to \Sigma^{-4m+1}L_{2k+1}^{2k+2\ell+1} \) with \( k = 2m \) or \( 2m-1 \):

\[
\begin{align*}
(5.14) & \quad (1) \ f_{2m,1} = (0, i) : \Sigma^2 \to \Sigma^4 \vee \Sigma^2SZ/4; \\
& \quad (2) \ f_{2m,2} = h_{MV,0}' : \Sigma^2 \to MV_{3,1}; \\
& \quad (3) \ f_{2m,2n+2} = f_{MV,n} : \Sigma^2 \to MU_{n,2n+3,n+1}; \\
& \quad (4) \ f_{2m,4r-1} = (0, f_{W,2r-1}) : \Sigma^2 \to \Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}; \\
& \quad (5) \ f_{2m,4r+1} = (0, f_{W,2r}) : \Sigma^2 \to \Sigma^4 \vee W_{4r+1,2r+1} \vee V_{2r}; \\
& \quad (6) \ f_{2m,4r-1} = i_Mi : \Sigma^0 \to M_2; \\
& \quad (7) \ f_{2m-1,2} = (0, iv_i) : \Sigma^0 \to \Sigma^4 \vee V_{2,2}; \\
& \quad (8) \ f_{2m-1,4r} = (0, f_{V,2r}) : \Sigma^0 \to \Sigma^0 \vee \Sigma^{-2}U_{2r,4r+1,2r-1}; \\
& \quad (9) \ f_{2m-1,4r+2} = (0, g_{W,2r}) : \Sigma^4P_1' \to \Sigma^4 \vee \Sigma^2V_{2r+1,4r+3,3r}; \\
& \quad (10) \ f_{2m-1,2n+1} = g_{MW,2n,n-1} : \Sigma^0 \to M_{2n+2} \vee W_{n,n} \\
\end{align*}
\]

\( (n, r \geq 1) \) where the small spectrum \( P_1' \) has the same quasi \( KO_* \)-type as \( \Sigma^4 \). According to Theorem 5.8 i) and ii) combined with Corollary 3.4, the small spectrum \( X_{k,\ell} \) has the same quasi \( KO_* \)-type as \( \Sigma^{-4m+1}L_{2k+1}^{2k+2\ell+1} \) where \( k = 2m \) or \( 2m-1 \). Using Lemmas 4.3, 4.4, 4.5, 4.7, 4.9 and 4.10 with the aid of (5.13) and Corollary 3.4, we can observe that

\[
(5.15) \quad i) \quad \text{the cofiber of the map } f_{k,\ell} \text{ has the same quasi } KO_*-\text{type as the following small spectrum } Z_{k,\ell} : \Sigma^4 \vee \Sigma^3, MP_3, MP_{2n+3} \vee W_{n,n}, \\
\Sigma^0 \vee PU_{4r-1,2r-1,2r-1}, \Sigma^4 \vee \Sigma^4PU_{4r+1,2r,2r}, \Sigma^1 \vee \Sigma^2, \Sigma^1 \vee \Sigma^4 \vee \Sigma^2SZ/4, \\
\Sigma^1 \vee \Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}, \Sigma^1 \vee \Sigma^4 \vee W_{4r+1,2r+1} \vee V_{2r}, \Sigma^1 \vee MU_{n-1,2n-1,n} \text{ corresponding to the case } (1), (2), \cdots, (10) \text{ of } (5.14), \text{ and moreover}
\]

ii) \( \text{the induced homomorphism } f_{k,\ell} : KU_0Y_{k,\ell} \to KU_0X_{k,\ell} \) is identified (up to signs) with the homomorphism \( \varphi_{k,\ell} \) defined in Lemma 5.6.

**Proposition 5.10.** Let \( X \) and \( Y \) be CW-spectra having the same quasi \( KO_* \)-types as \( X_{k,\ell} \) and \( Y_{k,\ell} \) given in (5.14) respectively. Let \( f : Y \to X \) be a map whose induced homomorphism \( f_* : KU_0Y \to KU_0X \) is
THE QUASI $KO_\ast$-TYPES OF THE STUNTED MOD 4 LENS SPACES

identified with the homomorphism $\varphi_{k,\ell}$ defined in Lemma 5.6. Then the cofiber of the map $f$ is quasi $KO_\ast$-equivalent to the small spectrum $Z_{k,\ell}$ appearing in (5.15) i).

Proof. Choose quasi $KO_\ast$-equivalences $h_0 : Y \rightarrow KO \wedge Y_{k,\ell}$ and $h_1 : X \rightarrow KO \wedge X_{k,\ell}$ satisfying $(c \wedge f_{k,\ell})h_0 = (c \wedge 1)h_1 f : Y \rightarrow KU \wedge X_{k,\ell}$ where $c : KO \rightarrow KU$ denotes the complexification. It is sufficient to show that the equality $(1 \wedge f_{k,\ell})h_0 = h_1 f : Y \rightarrow KO \wedge X_{k,\ell}$ holds in any case. By means of [10, Propositions 4.2 and 4.5] and Propositions 2.2 and 2.4 it is immediate that $[Y, \Sigma^1 KO \wedge X_{k,\ell}] \cong [Y_{k,\ell}, \Sigma^1 KO \wedge X_{k,\ell}] = 0$ except $(k, \ell) = (2m, 4r - 1)$. Therefore our assertion that the equality $(1 \wedge f_{k,\ell})h_0 = h_1 f$ holds is valid unless $(k, \ell) = (2m, 4r - 1)$. In the $(k, \ell) = (2m, 4r - 1)$ case we next show that our assertion is also valid after changing the quasi $KO_\ast$-equivalence $h_1 : X \rightarrow KO \wedge X_{2m,4r-1}$ suitably if necessary. As is easily seen, we can choose a certain map $g = (a\eta^2, 2^{2r-2}h_W, \tilde{\eta} + b\eta^2) : \Sigma^2 \rightarrow \Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}$ with $a, b \in Z/2$ satisfying $(1 \wedge g)h_0 = h_1 f : Y \rightarrow KO \wedge (\Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1})$. Consider the involution $\alpha$ on $\Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}$ represented by the matrix $\begin{pmatrix} 1 & 0 & a\eta^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + b\eta^2 \end{pmatrix}$, and replace the quasi $KO_\ast$-equivalence $h_1$ by the composite map $h'_1 = (1 \wedge \alpha)h_1$. Then we get the equalities $(c \wedge 1)h'_1 = (c \wedge 1)h_1$ and $(1 \wedge f_{k,\ell})h_0 = h'_1 f$ for the new quasi $KO_\ast$-equivalence $h'_1$. Hence our assertion is valid even if $(k, \ell) = (2m, 4r - 1)$.

Combining Proposition 5.10 with Lemma 5.6 we can finally determine the quasi $KO_\ast$-types of $L_{2k}^{2k+2\ell+1}$.

Theorem 5.11. i) $\Sigma^{-4m+1}L_{4m+2\ell+1}^4(\ell \geq 0)$ is quasi $KO_\ast$-equivalent to the following small spectrum: $\Sigma^1 \vee MU_{2r-1,4r+1,2r}, \Sigma^1 \vee \Sigma^4 \vee V_{2r} \vee W_{4r-1,2r+1}, \Sigma^1 \vee MU_{2r,4r+1,2r+1}, \Sigma^1 \vee \Sigma^0 \vee SZ/2^{2r+1} \vee W_{4r+3,2r+2}$ according as $\ell = 4r$, $4r + 1$, $4r + 2$, $4r + 3$. Here $MU_{1,1,0} = \Sigma^2$ and $V_0 \vee W_{1,1} should be replaced by $\Sigma^2 S Z/4 in the $\ell = 1$ case.

ii) $\Sigma^{-4m+1}L_{4m-2}^{4m+2\ell-1}(\ell \geq 0)$ is quasi $KO_\ast$-equivalent to the following small spectrum: $\Sigma^0 \vee PU_{4r+1,2r,2r}, \Sigma^4 MP_{4r+3} \vee W_{2r,2r}, \Sigma^4 \vee \Sigma^4 PU_{4r+3,2r+1,2r+1}, \Sigma^4 MP_{4r+5} \vee W_{2r+1,2r+1}$ according as $\ell = 4r$, $4r + 1$, $4r + 2$, $4r + 3$ where $PU_{1,0,0} = \Sigma^{-1}$. 

Produced by The Berkeley Electronic Press, 1993
Z. YOSIMURA

REFERENCES


DEPARTMENT OF MATHEMATICS
OSACA CITY UNIVERSITY
SUGIMOTO, SUMIYOSHI, OSAKA, 558 JAPAN

(Received September 9, 1992)

CURRENT ADDRESS:
DEPARTMENT OF MATHEMATICS
NAGOYA INSTITUTE OF TECHNOLOGY
GOKISO, SHOWA, NAGOYA 466, JAPAN