On isometries in an affine symmetric space

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In this paper we deal with the group of affine transformations in an affine symmetric space with an indefinite metric and an affine connection. An affine transformation is defined to be a transformation which keeps the connection invariant in the sense of K. Nomizu [2]*. We prove in the paper that every affine transformation is isometric and the group of all isometric transformations has a finite number of components. Our results are applicable to the case of non-compact Riemannian symmetric space, since our case is non-compact and pseudo-Riemannian.

§ 1. We explain some preliminary concepts and prove a lemma necessary for the later. Let $G/H$ be an affine symmetric space i.e. there exists an involutive automorphism $\Sigma$ of $G$ whose fixed point set $H_2$ contains $H$ and its connected component of the identity $(H_2)_0$ is contained in $H$. We always assume that $G$ is a connected semi-simple Lie group and each element of $G$ acts on $G/H$ left and effectively. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of $G$ and $H$ respectively and by $\sigma$ the differential of $\Sigma$. According to the eigenvalues $+1$ and $-1$ of $\sigma$, $\mathfrak{g}$ is decomposed into $\mathfrak{g} = \mathfrak{g} + \mathfrak{m}$, and as usually $\mathfrak{m}$ is identified to the tangent space of $G/H$ at the origin $p_0 = -H$. Let $\mu$ be the Killing form of $\mathfrak{g}$. Then its restriction to $\mathfrak{m}$ gives rise to an invariant indefinite metric of $G/H$. We denote this metric by $\mu$ for the simplicity of the notation.

According to K. Nomizu [2] and A. Lichnerovichz [3], if $\mathfrak{g}/\mathfrak{m}$ is effective, then the connected component of the identity of the affine transformation group of $\mathfrak{g}/\mathfrak{m}$ coincides with $\mathfrak{g}$. Next we prove the following

Lemma. In the decomposition $\mathfrak{g} = \mathfrak{g} + \mathfrak{m}$, the number of involutive automorphisms, such that their restrictions to $\mathfrak{g}$ are identity, is finite.

Proof. Let $A$ be an involutive automorphism as said in the lemma. Since $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{g}$ with respect to the Killing form $\mu$ and $\mathfrak{g}$ is invariant under $A$, $\mathfrak{m}$ is also invariant under $A$. By virtue of the assumption we have a decomposition $\mathfrak{m} = \mathfrak{m}_{+} + \mathfrak{m}_{-}$ according to eigenvalues $+1$ and $-1$ of $A$. We then have

\[[\mathfrak{g}, \mathfrak{m}_{+}] \subset \mathfrak{m}_{+}, \quad [\mathfrak{g}, \mathfrak{m}_{-}] \subset \mathfrak{m}_{-}, \quad [\mathfrak{m}_{+}, \mathfrak{m}_{-}] = 0.\]

From this relations it is easily seen that $[\mathfrak{m}_{+}, \mathfrak{m}_{+}] = \mathfrak{m}_{+}$ and $[\mathfrak{m}_{-}, \mathfrak{m}_{-}] = \mathfrak{m}_{-}$.

*) The numbers of bracets refer to the references at the end of the paper.
$\mathfrak{m}_{-1} + \mathfrak{m}_1$ are ideals of $\mathfrak{g}$ and $([\mathfrak{m}_{-1}, \mathfrak{m}_1] + \mathfrak{m}_1) \cap ([\mathfrak{m}_{-1}, \mathfrak{m}_{-1}] + \mathfrak{m}_{-1})$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. On the other hand, by the assumption $G/H$ is effective. Hence this ideal is zero. As we see in [2], $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{h}$, we thus get

$$\mathfrak{g} = ([\mathfrak{m}_{-1}, \mathfrak{m}_1] + \mathfrak{m}_1) \oplus ([\mathfrak{m}_{-1}, \mathfrak{m}_{-1}] + \mathfrak{m}_{-1}).$$

The direct decomposition of $\mathfrak{g}$ into $\sigma$-invariant ideals is finite in number. Since $\mathfrak{g}$ is semi-simple, each of the above two ideals is expressed as a direct sum of simple ideals and further since the automorphism $A$ is determined by such decomposition, the lemma is proved.

As we stated before the purpose of the paper is to prove the following

**Theorem.** Let $G$ be a connected semi-simple Lie group and $G/H$ an effective affine symmetric space with a pseudo-Riemannian metric and the connection induced from its metric. Then the group of all affine transformations of $G/H$ coincides with that of all isometries. Moreover the number of components of this group is finite.

We conveniently divide the proof into two parts. The first part is proved in §2 and the later in §3.

§2. Let $A (= A(G/H))$ be the group of all affine transformations of $G/H$ and $A_o (= A_o(G/H))$ its connected component of the identity. We define a homomorphism $\theta$ from $A$ into $\text{Aut}(A_o)$, the group of automorphisms, in such a way that $\theta(\alpha)(x) = \alpha x \alpha^{-1}$, $\alpha \in A$, $x \in A_o$. The homomorphism $\theta$ induces a homomorphism $\tilde{\theta}$:

$$A/A_o \longrightarrow \text{Aut}(A_o)/\text{Int}(A_o).$$

As stated in §1, we can assume $A_o = G$. To prove the theorem we study the kernel and the image of $\tilde{\theta}$.

1). The kernel of $\tilde{\theta}$.

At first we put

$$B = \{ \alpha \mid \alpha \in A \text{ and } \alpha x = x \alpha \text{ for any } x \in A_o \}.$$

We further denote $\{ \alpha(p_0) \mid \alpha \in B \}$ by $B(p_0)$ where $p_0 = H$. If $\alpha(p_0) = k \cdot H$, then $\alpha(g \cdot H) = g \cdot k \cdot H$, that is, the transformation $\alpha$ is completely determined by the image point $\alpha(p_0)$. Therefore the set $B(p_0)$ and $B$ is in 1-1 correspondence. Let $A(p_0)$ be the set of fixed points of the isotropy subgroup $H$, then we can easily see $B(p_0) \subset A(p_0)$. Let $K$ be the normalizer of $H$ in $G$. We then see $A(p_0) = K/H$. For any $k \cdot H \in A(p_0)$ we define a diffeomorphism $\alpha$ of $G/H$ by $\alpha(g \cdot H) = g \cdot k \cdot H (g \in G)$. $\alpha$ is commutative with any element of $A_o$, since for any element $x = x(l) \in A_o$, $l \in G$ we obtain

$$\alpha(x \cdot g \cdot H) = \alpha(l \cdot g \cdot H) = l \cdot g \cdot k \cdot H = x(g \cdot H).$$

Next we show that $\alpha$ is an isometry of $G/H$. In fact. from $k \in K$, it follows
that $Ad(k)\hat{\mathcal{G}} = \hat{\mathcal{G}}$ and $\mathfrak{M}$ is the orthogonal complement of $\hat{\mathcal{G}}$ with respect to the Killing form $\mu$, hence we see $Ad(k)\mathfrak{M} = \mathfrak{M}$ and
\[
\mu_{\mathfrak{g}, \mathfrak{h}}(d\alpha X, d\alpha Y) = \mu(d\cdot(k^{-1})d\alpha X, d\cdot(k^{-1})d\alpha Y) \\
= \mu(Ad(k^{-1})X, Ad(k^{-1})Y) \\
= \mu(X, Y) \quad \text{for } X, Y \in \mathfrak{M}
\]
i.e. $\alpha$ is isometric. Since an isometric transformation is affine, $\alpha \in B$ and $k \cdot H \in B(p_o)$ and therefore $A(p_o) = B(p_o)$. From the above correspondence: $k \cdot H \rightarrow \alpha^{-1}$ is an isomorphism of $K/H$ onto $B$. On the other hand we have

$$\theta^{-1}(Int(A_o)) = B \cdot A_o.$$  

In fact, if $\alpha \in \theta^{-1}(Int(A_o))$ there exists an element $\alpha_o$ of $A_o$ such that $\theta(\alpha) = \theta(\alpha_o)$. Then $\alpha_o^{-1} \cdot \alpha \in B$ and $\alpha \in B \cdot A_o$. We thus see that $\theta^{-1}(Int(A_o))$ consists of isometries.

2). The image of $\tilde{\theta}$.

Now consider an automorphism $\zeta$ of $A_o (= G)$ such that $\zeta(H) \subset H$ and define a diffeomorphism $\alpha$ of $G/H$ by $\alpha(g \cdot H) = \zeta(g) \cdot H$, $g \in G$. For any element $x = \tau(l) (l \in G)$ of $A_o$,

$$\alpha x \cdot \tau^{-1}(g \cdot H) = \alpha x \cdot (\zeta^{-1}(g) \cdot H) = \alpha (l \cdot \tau^{-1}(g) \cdot H) = \zeta(l) \cdot g \cdot H = \tau(\zeta(l)) g \cdot H.$$  

It follows from this that $\alpha x \alpha^{-1} \cdot \tau(\zeta(l))$. We further see that $\alpha$ is an isometry and $\theta(\alpha) = \zeta$, in fact

$$\mu(d\alpha X, d\alpha Y) = \mu(d\zeta X, d\zeta Y) = \mu(X, Y) \quad \text{for } X, Y \in \mathfrak{M},$$

because of $d\zeta(\mathfrak{M}) = \mathfrak{M}$.

Next choose an element $\alpha \in A$, $\alpha(p_o) = p_o$ from a class of $A/A_o$, if we put $\theta(\alpha) = \zeta$, $\alpha x \alpha^{-1} = \tau(\zeta(l))$ where $x = \tau(l) \in A_o$, $l \in G$. For any $h \in H$ we see

$$\zeta(h) \cdot H = \alpha \cdot h \alpha^{-1} \cdot H = \alpha \cdot \tau(h) \cdot H = \alpha \cdot H \indent H.$$  

Hence we have $\zeta(H) = H$. From these facts $\tilde{\theta}(A/A_o)$ consists of the classes of $Aut(A_o) / Int(A_o)$ such that each of these classes contains an automorphism $\zeta$, such that $\zeta(H) \supset H$. If we denote by $Aut(G : H)$ the group of automorphisms of $G$ which preserve $H$. We then get

$$\mu = [ Img \, \tilde{\theta} : 1 ] = [ Aut(G : H) : Int_o(K) ],$$

where $Int_o(K)$ means the subgroup of $Int(G)$ by elements of $K$. We can now conclude that the affine transformation group coincides with the isometric transformation group because the kernel of $\theta$ consists of isometries
and an image $\zeta$ by $\theta$ is also an image of an isometry $\alpha$. This proves the first part of the theorem.

§3. We now prove the later part of the theorem. By the assumption $G$ is semi-simple and then $Aut(G)/Int(G)$ is a finite group [4]. This implies $\mu < \infty$. On the other hand we can see from the facts said in §2 that $ker\tilde{\theta} \subseteq \theta^{-1}(Int(A))|A \cong B/B_0$ where $B_0 = B \cap A_0$. We put $\nu = [B : B_0]$. In order to prove that the number of components of $A$ is finite, it is sufficient to show $\nu < \infty$.

Let $Z$ be the center of $G$ and put $\tilde{H} = H \cdot Z$, $\tilde{H}' = H' \cdot Z$ where $H' = K \cap H_2$. On the other hand $\tilde{H}/H \cong B_0$ under the isomorphism $k \cdot H \mapsto \alpha^{-1}$. Thus we have

$$B/B_0 \cong K/H/\tilde{H}/H \cong K/\tilde{H}$$

$$\nu = [K : \tilde{H}'] [\tilde{H}' : \tilde{H}].$$

It is easy to see $[\tilde{H} : \tilde{H}] \leq [H_2 : H]$ and we can see from [1] $[H_2 : H] < \infty$. Hence it is sufficient to prove $[K : \tilde{H}'] < \infty$.

We can assume that $G$ is an adjoint group. In fact, let $\bar{G} = G/Z$, $\bar{H} = H \cdot Z/Z = \tilde{H}/Z$, then $\bar{G}/\bar{H}$ is an effective affine symmetric space. If $\bar{K}$ is the normalizer of $\bar{H}$ in $\bar{G}$, then

$$\nu = [K : \tilde{H}] = [K/Z : \tilde{H}/Z] \leq [K'/Z : \tilde{H}/Z] = \tau,$$

where $K'$ is a suitable subgroup of $G$ such that $K'/Z = \bar{K}$. Under this assumption, we put $k^{-1} \cdot \Sigma(k) = a$ for any $k \in K$. Since $K$ is $\Sigma$-invariant, $a \in K$ and $\Sigma(a) = a^{-1}$ and it is clear that $Ad(a)/\Sigma id$. and $Ad(a)\mathbb{M} = \mathbb{M}$.

Operating $\Sigma$ on the both sides of

$$a(exptx)a^{-1} = exptAd(a)X \quad \text{for} \quad x \in \mathbb{M},$$

we have $a^{-1}(\exp tX)a = \exp tAd(a)X$. Since $a^{-1} \exp tXa = \exp Ad(a^{-1})X$, we get $Ad(a^{-1})X = Ad(a)X$ which implies $Ad(a)^2 = id$. By virtue of the lemma the number of such elements $a$ is finite. We denote these elements by $g_1$, $g_2$, ..., $g_r$. If the element $a$ is identical with some $g_i$, we have $k^{-1} \cdot \Sigma(k) = g_i$, for some element $k' \in K$, then $k' \cdot k^{-1} = \Sigma(k' \cdot k^{-1})$ and hence $k' \cdot k^{-1} \in H_2 \cap K = H'$. Thus we see $k' \cdot H' = k \cdot H'$ and $[K : H'] < l$. The last part of the theorem follows from this.

Example: For $SL(2n, R)/Sp(n, R)$ it is easy to see

if $n = even$, $\nu = 2$ and $\mu = 2$,

if $n = odd$, $\nu = 1$ and $\mu = 4$. 

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REFERENCES


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