Some results on normal bases

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SOME RESULTS ON NORMAL BASES

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Throughout the present paper, A will represent a ring with 1, B a
subring of A containing 1, and G a subgroup of the group \( \hat{G} \) of B-ring
automorphisms of A. Further, the centralizer \( V_d(B) \) of B in A will be
denoted by \( V \), and for any subset \( S \) of A the right multiplication and
the left multiplication by \( S \) (in A) respectively by \( S_r \) and \( S_l \). We set
\( \mathfrak{I} = \text{Hom}(A, A) \), which will be considered on the right side of A. If \( \mathfrak{n}A \) is
completely faithful in the sense of \([1]\), or equivalently, if \( B \) is a direct
summand of \( \mathfrak{n}A \), then \( V_{\mathfrak{n}}(V_{\mathfrak{n}}(B)) = B \) (cf. \([1]\)) In particular, if \( \mathfrak{n}A \) is
free then \( B \) is a direct summand of \( \mathfrak{n}A \) and \( V_{\mathfrak{n}}(V_{\mathfrak{n}}(B)) = B \). Further,
one may remark that if \( B \) is a direct summand of \( \mathfrak{n}A \) and \( V_{\mathfrak{n}}(B) = GA \)
then \( f(G) = \{ x \in A ; x_r = x \text{ for every } \sigma \in G \} \) coincides with B. Those
remarks will be used freely in the sequel.

If \( \mathfrak{n}A \) is finitely generated projective (resp. finitely generated free)
and \( \text{Hom}(\mathfrak{n}A, \mathfrak{n}B) \) contains a left free \( A_\mathfrak{n} \)-basis \( h \) of \( \text{Hom}(\mathfrak{n}A, \mathfrak{n}B) \) then
\( A/B \) is called a Frobenius (resp. free Frobenius) extension. Now, assume
that \( A/B \) is a Frobenius extension and \( h \in \text{Hom}(\mathfrak{n}A, \mathfrak{n}B) \) a left free
\( A_\mathfrak{n} \)-basis of \( \text{Hom}(\mathfrak{n}A, \mathfrak{n}B) \). Then, \( h \) is also a left free \( A_r \)-basis of \( \text{Hom}(A_r, B_r) \) (cf. the proof of \([13; \text{Th. 1}]\)). Moreover, in order that B be a
direct summand of \( \mathfrak{n}A \) (or of \( A_\mathfrak{n} \)), it is necessary and sufficient that \( h \) be
an epimorphism. In case B is a direct summand of \( \mathfrak{n}A \), A is semi-primary
if and only if so is \( B \) ([9; Prop. 7.3]). Finally, let A be an Artinian
simple ring, and G an \( N \)-group of A with \( B = f(G) \). Then \( A/B \) is a free
Frobenius extension with \( V_{\mathfrak{n}}(B) = GA \) ([5; 3. Beispiele] and [13; Prop.
1]) and A is \( GB \)-homomorphic to \( GB \) ([7; Th. 1]).

In the present paper, we shall treat with a kind of free Frobenius
extensions of semi-primary rings and the main theme of our discussion will
concern the normal basis theorems. One of them is an extension of the last
statement (Th. 3.7), and the original of the other will be found in [8] and
[9] (cf. Ths. 2.1 and 2.2). Our theorems obtained in §§2—4 will contain
several results in [7] and [14].

1. A remark on Frobenius extensions

Assume that \( A/B \) is a Frobenius extension and \( h \in \text{Hom}(\mathfrak{n}A, \mathfrak{n}B) \) a
left free \( A_\mathfrak{n} \)-basis of \( \text{Hom}(\mathfrak{n}A, \mathfrak{n}B) \). Then, as was shown in the proof of
\([13; \text{Th. 1}]\), \( \sum x_i h y_{il} \) defines an additive group isomorphism
of \( V_{\mathfrak{n}}(B) \) onto \( V_{\mathfrak{n}}(B) \). We shall expose here the reciprocity between the
conditions \( V_{\mathfrak{n}}(B) = GA \) and \( V_{\mathfrak{n}}(B) = GA \).

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Proposition 1.1. Let \( A/B \) be a Frobenius extension, and \( h \in \text{Hom}(\alpha A, B) \) a left free \( A \)-basis of \( \text{Hom}(\alpha A, B) \). If \( \sigma \) is a \( B \)-ring automorphism of \( A \) then \( \sigma h = v \mu v^{-1} \) with a unit \( v \) in \( V \) (cf. [6; p. 93]), and \( \sigma^* = \sigma^{-1} v r = \sigma^{-1} v u \). In particular, \( 1^* = 1 \).

Proof. We set \( \sigma^* = \sigma^{-1} g \). Since \( x_{\alpha} \sigma = \sigma(x)_{\alpha} \) for every \( x \in A \), we obtain \( \sigma^{-1} g x_{\alpha} = (x \sigma)_{\alpha} = g = u_{\alpha} \) with some \( u \) in \( V \). Hence, \( \sigma^* = \sigma^{-1} u_{\alpha} \). If \( \sigma = \sum y_{\alpha} h x_{\alpha} \) then \( \sigma^* = \sum y_{\alpha} h x_{\alpha} \), so that \( \sigma^* h = \sum y_{\alpha} h x_{\alpha} \). Noting that \( 1 = \sum (x_{\alpha} h) y_{\alpha} \), we readily obtain \( h = 1 \cdot h = \sum y_{\alpha} h (x_{\alpha} h) = \sigma^* h = \sigma^{-1} u_{\alpha} h \), and so \( v_{\alpha} h = \sigma h = \sigma u_{\alpha} h \). Hence, we obtain \( v = u \).

The next is [13; Cor. 1].

Corollary. In order that \( A/B \) be a Frobenius extension, it is necessary and sufficient that there exist some \( h \in \text{Hom}(\alpha A, B) \) and \( x_{\alpha}, \ldots, x_{\alpha}, \ldots, y_{\alpha}, \ldots, y_{\alpha} \in A \) such that \( \sum x_{\alpha} h y_{\alpha} = \sum y_{\alpha} h x_{\alpha} = 1 \).

Proof. It remains only to prove the sufficiency. If \( f \) is in \( \text{Hom}(\alpha A, B) \) then \( f = \sum x_{\alpha} h y_{\alpha} f = \sum x_{\alpha} y_{\alpha} f \). Moreover, if \( a_{\alpha} h = 0 \) (\( a \in A \)) then \( a = a (\sum y_{\alpha} h x_{\alpha}) = \sum y_{\alpha} (a_{\alpha} h x_{\alpha}) = 0 \).

Corollary. Let \( \tilde{V} \) be the set of all \( \tilde{v} \) effected by units \( v \) contained in \( V \). If \( A/B \) is a Frobenius extension and \( V_{\alpha}(B_{\alpha}) = GA_{\alpha} \) then \( V_{\alpha}(B_{\alpha}) = GV_{\alpha} = \tilde{G}A_{\alpha} \).

If \( A/B \) is \( G \)-Galois (cf. for instance [9]) then \( h = \sum_{\sigma \in \sigma} \sigma \) is a left free basis of \( \text{Hom}(\alpha A, B) \) and so \( A/B \) is a Frobenius extension and \( GH \). The next is also an easy consequence of Prop. 1.1.

Corollary. Let \( A/B \) be a Frobenius extension, and \( V_{\alpha}(B) = GA_{\alpha} \). If \( \text{Hom}(\alpha A, B) \) contains a left free \( A \)-basis \( h \) of \( \text{Hom}(\alpha A, B) \) such that \( GH \subset C_{\alpha} \) (\( C = V_{\alpha}(A) \)) then \( V_{\alpha}(B_{\alpha}) = GA_{\alpha} \). In particular, if \( V = C \) then \( V_{\alpha}(B_{\alpha}) = GA_{\alpha} \).

2. Normal basis theorem

Let \( A \) be a semi-primary ring, namely, the residue class ring of \( A \) modulo the (Jacobson) radical \( \mathfrak{R}(A) \) be Artinian. Assume that \( A/B \) is a free Frobenius extension and \( V_{\alpha}(B) = GA_{\alpha} \). Then, the direct sum \( A^{(\alpha)} \) of \( s \) copies of \( A \) is \( GA_{\alpha} \)-isomorphic to \( GA_{\alpha} \) where \( s = [A : B] \). Assume further that \( G \) contains a right free \( A_{\alpha} \)-basis \( H \) of \( GA_{\alpha} \) such that \( GB_{\alpha} = HB_{\alpha} \). Then, to be easily seen, \( GB_{\alpha}^{(\alpha)} \) is \( GB_{\alpha} \)-isomorphic to \( GA_{\alpha} \), and so \( A^{(\alpha)} \) is \( GB_{\alpha} \)-isomorphic to \( (GB_{\alpha})^{(\alpha)} \). Since \( GA_{\alpha} \) is semi-primary, \( GB_{\alpha} \) is also semi-primary by [9; Prop. 7.3]. Hence, as is well known, \( A \) is \( GB_{\alpha} \)-isomorphic to \( GB_{\alpha} \), which proves the following that is fundamental in our subsequent study.

Theorem 2.1. Let \( A/B \) be a free Frobenius extension, and \( V_{\alpha}(B) = \)
\[ \bigoplus_{\sigma \in G} \sigma A \] with a subset \( H \) of \( G \). If \( A \) is semi-primary and \( GB = HB \), then \( A \) is \( GB \)-isomorphic to \( GB \).

**Corollary.** Let \( A/B \) be a free Frobenius extension, and \( V_{\text{hel}}(B) = \bigoplus \sigma_i A \) with some \( \sigma_i \in G \). If \( A \) is semi-primary and \( G \) is abelian then \( A \) is \( GB \)-isomorphic to \( GB \).

**Proof.** Let \( \sigma = \sum \sigma_i y_i \in A \) be an arbitrary element of \( G \). Then, for any \( \tau \in G \) there holds \( \sum \tau_i y_i = \sum \tau_i \sigma_i = \sum \tau_i y_i \). Hence, \( y_i = y_i \) for every \( y_i \). We see therefore every \( y_i \) is in \( J(G) = B \), namely, \( GB = \bigoplus \sigma_i B \). Hence, \( A \) is \( GB \)-isomorphic to \( GB \) by Th. 2.1.

Next, we shall present the following whose proof is quite similar to that of [8; Th. 2.4] (cf. also [8; Example 2.2]).

**Theorem 2.2.** Let \( A/B \) be a Frobenius extension such that \( B \) is a direct summand of \( A \), and \( V_{\text{hel}}(B) = \bigoplus \sigma_i A \) and \( GB = \bigoplus \sigma_i B \) with some \( \sigma_i \in G \). If \( A \) is semi-primary and \( A \) can be generated by a subset of \( s \) elements then \( A \) is \( GB \)-isomorphic to \( GB \).

**Proof.** By the validity of Th. 2.1, it suffices to prove that \( A \) is free. There exists a right \( B \)-epimorphism \( f \) of \( B^{\otimes 1} \) onto \( A \) and a splitting exact sequence

\[ 0 \to \text{Ker} f \to B^{\otimes 1} \to A \to 0. \]

Obviously, the derived sequence

\[ 0 \to \text{Ker} f \otimes_B A \to B^{\otimes 1} \otimes_B A \to A \otimes_B A \to 0 \]

is an exact sequence of right \( A \)-modules. As is well known, \( V_{\text{hel}}(B) = \bigoplus \sigma_i A \) is right \( A \)-isomorphic to \( A \otimes_B A \) (cf. [13; Lemma 1]). Hence, \( B^{\otimes 1} \otimes_B A \otimes_A (A/\mathfrak{N}(A)) \) and \( A \otimes_B A \otimes_A (A/\mathfrak{N}(A)) \) are isomorphic and have the same finite number of irreducible components. Therefore, \( f^{\otimes 1} \otimes 1 \) has to be an isomorphism, and \( f^{\otimes 1} \) is an isomorphism by [8; Lemma 1.7]. Hence, \( \text{Ker} f \otimes_B A = 0 \). Since \( B \) is a direct summand of \( A \), we obtain \( f = 0 \).

If \( A/B \) is \( G \)-Galois and \( A \) is \( GB \)-isomorphic to \( GB \), then it is rather familiar that \( H^i(G, A) = 0 \) for \( i \geq 1 \). By the way, one may remark here that if \( G \) is of finite order and there exists an element \( a \in A \) such that \( T_i(a) = \sum_{\sigma \in G} a \sigma = 1 \) then \( H^i(G, A) = 0 \). In fact, if for every \( \sigma \in G \) there corresponds an element \( x_\sigma \in A \) and there holds \( x_\tau \cdot x_\sigma := x_{\sigma \tau} (\sigma, \tau \in G) \), then for \( x = \sum_{\sigma \in G} x_\sigma a_\sigma \) we have \( x_\sigma = x - x_\sigma a_\sigma \).

**Theorem 2.3.** Let \( A \) be semi-primary, and \( G = \{ \sigma_1, \ldots, \sigma_s \} \) a finite group of ring automorphisms of \( A \) such that \( B = J(G) \) is a direct summand of \( A \). Assume that \( A/B \) is \( G \)-Galois and \( A \) can be generated by a subset of \( s \) elements. If \( a \) is a left \( G \)-normal basis element (abbr. \( G-n.b.e.) of
A/B, namely, if \( \{a_{\sigma_1}, \ldots, a_{\sigma_s} \} \) is a left free B-basis of A, then the matrix \( (a_{\sigma_i} a_j) \) is a unit of \( (A)_n \), and conversely. In particular, if G is abelian then every right G-n. b. e. is a left G-n. b. e.

Proof. By Th. 2.2, A is \( GB \)-isomorphic to \( GB \), so that symmetrically there exists a left G-n. b. e. x of A/B. Since there exist some \( f_j = y_{jv} \cdot \sum \sigma_k \) in Hom \( (bJ, bB) = A_E \cdot \sum \sigma_k \) such that \( \sum x_{\sigma_i} x_{\sigma_j} = x_{\sigma_i} f_j = \delta_{ij} \) \( (i, j = 1, \ldots, s) \), the matrix \( (x_{\sigma_i} x_{\sigma_j}) \) is a unit of \( (A)_n \). Conversely, let a be an arbitrary element of A such that \( (a_{\sigma_i} a_j) \) is a unit of \( (A)_n \). If we set \( a_{\sigma_i} = \sum b_{ij} x_{\sigma_j} (b_{ij} \in B) \) then \( (b_{ij}) = (a_{\sigma_i} a_j) (x_{\sigma_i} x_{\sigma_j})^{-1} \) is a unit of \( (B)_n \), which means that a is a left G-n. b. e.

Finally, assume that A is an Artinian simple ring, and \( G = \{\sigma_1, \ldots, \sigma_s\} \) an F-group of A with \( B = J(G) \). Then, B is an Artinian simple ring and \( [A : B] < s \). If \( [A : B] \) coincides with s, A/B is defined to be strictly Galois with respect to G (cf. [10]). To be easily seen, \( A/B \) is strictly Galois with respect to G if and only if G-Galois. We obtain at once the following that contains [7; Th. 4].

Corollary. Let \( G = \{\sigma_1, \ldots, \sigma_s\} \) be an F-group of an Artinian simple ring A, and \( B = J(G) \). In order that a be a left G-n. b. e. of A/B, it is necessary and sufficient that the matrix \( (a_{\sigma_i} a_j) \) be a unit of \( (A)_n \).

3. Galois extensions of perfect rings

Following [2], a ring A is called left (resp. right) perfect if every left (resp. right) unital A-module possesses a projective cover. As was shown in [2], A is left perfect if and only if any of the following equivalent conditions is satisfied: (1) A is semi-primary and \( \mathfrak{M}(A) \) is left T-nilpotent, and (2) the descending chain condition is satisfied for right principal ideals of A. Now, it will be easy to see that if A is left perfect then every right regular element (i.e. an element that is not a left zero-divisor) of A is a unit. Finally, A is called a local ring if the set of all non-units of A forms an ideal. The following will be found in [2] and [3].

Proposition 3.1. Let A be a left perfect ring.
(a) If M is a left unital A-module then \( \mathfrak{M}(A) \cdot M \neq M \).
(b) Every projective left unital A-module is a direct sum of directly indecomposable direct summands of \( _A A \), and the numbers of isomorphic components are uniquely determined.
(c) If A is primary then A is a complete matrix ring over a local ring, and conversely.
(d) (A), is left perfect. If e is a non-zero idempotent of A then eAe is left perfect.
Now, by the validity of Prop. 3.1 (a) and (b), we can prove the following, whose proof proceeds in the same way as in [12].

**Proposition 3.2.** Let $A$ be a right perfect ring, and let $N$ and $P$ be right unital $A$-modules.

(a) If $P$ is projective and $N^{(n)}$ is $A$-homomorphic to $P^{(n)}$ with positive integers $n > p$ then $N$ is $A$-homomorphic to $P$.

(b) If $N^{(n)}$ is $A$-isomorphic to $A^n$ with a positive integer $n$ and an infinite cardinal number $\omega$ then $N$ is $A$-isomorphic to $A^{(\omega)}$.

(c) If $N^{(n)}$ is $A$-isomorphic to $A^n$ with positive integers $n$, $a$, and $a = nq + r \ (0 < r < n)$, then $N$ is $A$-isomorphic to $A^n \oplus N_0$ for an $A$-homomorphic image $N_0$ of $A$ such that $N_0^{(n)}$ is $A$-isomorphic to $A^n$.

**Proposition 3.3.** Let $B$ be a direct summand of $nA$.

(a) If $A$ is left perfect then so is $B$. In particular, $A$ is left perfect if and only if so is $(A)$. 

(b) If $nA$ is finitely generated projective and $B$ is a right perfect ring then $A$ is right perfect. Particularly, in case $A/B$ is a Frobenius extension, $A$ is a (left and right) perfect ring when and only when so is $B$.

**Proof.** (a) If $\tau$ is a right ideal of $B$ then $\tau A \cap B = \tau$. Hence, the descending chain condition is valid for right principal ideals of $B$. The latter will be obvious by Prop. 3.1 (d).

(b) In virtue of Prop. 3.1 (d), $V_n(B_R)$ is right perfect. Since $A_R$ is a direct summand of the right $A_R$-module $V_n(B_R)$ (cf. [9; Lemma 6.8]), $A_R$ is itself right perfect by (a).

**Proposition 3.4.** Let $A$ be left perfect, and $M_A$ finitely generated free. If $N$ is a finitely generated free submodule of $M_A$ with $[M: A]_F = [N: A]_F$ then $M = N$.

**Proof.** Let $\{m_1, \ldots, m_s\}$ and $\{n_1, \ldots, n_s\}$ be right free $A$-bases of $M$ and $N$, respectively. If $n_j = \sum m_ia_j(a_j \in A)$ then the matrix $(a_{ij})$ is obviously a right regular element of the left perfect ring $(A)_F$ (Prop. 3.1 (d)). Hence, $(a_{ij})$ is a unit of $(A)_F$ and $\{n_1, \ldots, n_s\}$ is a right free $A$-basis of $M$.

As an easy consequence of Props. 3.3 (b) and 3.4, we obtain the next:

**Corollary.** Let $G = \{a_1, \ldots, a_s\}$, $B$ a right perfect ring, and $[A: B]_F = s$. If $(a_{i_1}a_{i_2})$ is a unit of $(A)_F$ then $a$ is a left $G$-b. e. of $A/B$.

Corresponding to Th. 2.1. we obtain the following (cf. [7; Th. 2]):

**Theorem 3.5.** Let $A/B$ be a free Frobenius extension, and $V_n(B_R) = \bigoplus_{a \in H} aA_R$ with a subset $H$ of $G$. If $A$ is left perfect then the following
conditions are equivalent: (1) $A$ is $GB$-isomorphic to $GB_n$, and (2) $GB_n = HB_n$.

Proof. By Th. 2.1, it suffices to prove (1) $\Rightarrow$ (2). By Prop. 3.3 (a), $B$ is left perfect. Since $[GB_n: B_n] = [A : B]_n = [A : B]_k[HA_n : A_n]_n = [HB_n : B_n]_n$, (2) is a consequence of Prop. 3.4.

The next is a generalization of [7; Th. 3].

**Theorem 3.6.** Let $A/B$ be $G$-Galois, and $U$ a $G$-invariant right perfect subring of $A$ such that $A_U$ possesses a free basis $\{y_i : i \in \Lambda\}$.

(a) If $A$ is infinite then there exists a subset $\{x_\lambda : \lambda \in \Lambda\}$ such that $\{x_{\lambda \sigma} : \lambda \in \Lambda, \sigma \in G\}$ is a free basis of $A_U$.

(b) Let $G$ be of order $s$, and $2A = t \leq \infty$. If $t = sq + r$ $(0 < r < s)$ then $A$ contains a subset $X = \{x_1, \ldots, x_t\}$ and a $GU_n$-homomorphic image $M$ of $GU_n$ such that $XG$ is right $U$-free, $M^{(o)}$ is $GU_n$-isomorphic to $(GU_n)^{(o)}$, and $A = (XG)U \oplus M$.

Proof. Since $GU_n = \bigoplus_{\sigma \in G} U_{\sigma}$ is right perfect (Prop. 3.3 (b)) and $A^{(o)}$ is $GU_n$-isomorphic to $(GU_n)^{(o)}$, (a) and (b) are direct consequences of Prop. 3.2 (b) and (c), respectively.

We shall conclude this section with the following that contains [7; Th. 1]:

**Theorem 3.7.** Let $A/B$ be a free Frobenius extension, and $V_{\gamma}(B_\gamma) = GA_n$. If $U$ is a $G$-invariant right Artinian subring of $A$ such that $A_U$ possesses a generating system $\{x_1, \ldots, x_t\}$ with $t \leq [A : B]$, then $A$ is $GU_n$-isomorphic to $GU_n$. In particular, if $A$ is right Artinian then $A$ is $GB_n$-homomorphic to $GB_n$.

Proof. If $s = [A : B]$ then $A^{(o)}$ is $GA_n$-isomorphic to $GA_n$. Since $GA_n$ is right finite over $U_n$, $GU_n$ is right Artinian. Noting that $GA_n = A_nG = \sum x_i U_n G$, we see that $GA_n$ is $GU_n$-homomorphic to $(GU_n)^{(o)}$, and so $A^{(o)}$ is $GU_n$-homomorphic to $(GU_n)^{(o)}$. Hence, by Prop. 3.2 (a), $A$ is $GU_n$-homomorphic to $GU_n$.

4. A special type of Galois extensions of left perfect primary rings

The present section is devoted exclusively to the treaty of a special type of Galois extensions of left perfect primary rings.

**Proposition 4.1.** Let $A$ be a $G$-Galois extension of a left perfect primary ring $B$. If $G'$ is a normal subgroup of $G$, $G = G/G'$ and $B' = J(G')$, then $B'/B$ is free $G$-Galois and $B'$ is left perfect.

Proof. Let $B = \sum B_\gamma e_{i\gamma}$, where $\{e_{i\gamma}\}$ is a system of matrix units and $B_\gamma = V_{\gamma}(\{e_{i\gamma}\})$ a local ring (Prop. 3. (c)). If $A_\gamma = V_{\gamma}(\{e_{i\gamma}\})$ then $A_\gamma/B_\gamma$ is
G-Galois by [9; Th. 5.8]. Hence, By [4; Th. 2], $A_0/B_0$ is free G-Galois, which means that $A/B$ is free G-Galois. Accordingly, $B'/B$ is $\overline{G}$-Galois (cf. for instance [9]), and then free by the above argument. Finally, $B'$ is left perfect by Prop. 3.3 (b).

If $B$ is a left perfect primary ring then, as is well known, the center $Z$ of $B$ is a perfect local ring and the characteristic of $B$ is either 0 or a power of a prime. Now, we shall present a slight generalization of [14; Th. 2].

**Theorem 4.2.** Let $A/B$ be G-Galois, and $G' \neq \{1\}$ a normal subgroup of $G$ with $B' = J(G')$. If $B$ is a left perfect primary ring and $\overline{G} = G/G'$ then the following conditions are equivalent: (1) $a$ is a right G-n. b.e. of $A/B$ whenever $T_\varphi'(a) = \sum_{\sigma \in G'} a \sigma$ is a right $G$-n. b.e. of $B'/B$, and (2) the characteristic of $B$ and the order of $G'$ are powers of a prime $p$.

**Proof.** By Prop. 4.1, $A/B$ is free G-Galois and $A$ is left perfect. Hence, there exists a right $G$-n. b.e. $u$ (Th. 2.1), and $T_\varphi'(u)$ is a right $\overline{G}$-n. b.e. (Props. 3.4 and 4.1). As in the proof of [11; Th.], the mapping $\varphi: \sum_{\sigma \in G} x_{\sigma} \rightarrow \sum_{\sigma \in G} \sigma x_{\sigma}$ is a ring homomorphism of $G B_R$ (isomorphic to the group ring $G B$) onto $\overline{G} B_R$, and one will easily see that $T_\varphi'(u\alpha) = (T_\varphi'(u)) (\alpha \varphi)$ for every $\alpha \in GB_R$. As $GB_R$ is left perfect (Prop. 3.3 (b)), $u\alpha$ is again a right G-n. b.e. when and only when $\alpha$ is a unit of $GB_R$. Similarly, $T_\varphi'(u\alpha)$ is again a right $\overline{G}$-n. b.e. when and only when $\alpha \varphi$ is a unit of $GB_R$. Our equivalence is therefore evident by Th. of [11].

**Corollary.** Let $A/B$ be G-Galois. If $B$ is a left perfect primary proper subring of $A$ then the following conditions are equivalent: (1) $a$ is a right G-n. b.e. of $A/B$ whenever $T_\varphi(a)$ is a unit of $B$, and (2) the characteristic of $B$ and the order of $G$ are powers of a prime $p$. In particular, if $B$ is a perfect primary ring of characteristic $p^e$ and $G$ is of order $p' \times$ then every left G-n. b.e. is a right G-n. b.e.

**References**


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