ExtA(Z2[y]/Z2,Z2), A being the mod 2 Steenrod algebra

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Aikawa: ExtA(Z2\[y]/Z2, Z2), A being the mod 2 Steenrod algebra

§ 1. Introduction.

Let A be the mod 2 Steenrod algebra, Z be a ring of integers, Zm = Z/mZ, (m: a positive integer). Let M^k be a polynomial algebra with one generator y of degree 2^k, (k: a non-negative integer) over Z. Let x be the generator of M^0. Let M^i be a A-submodule of M^j generated by y^j, q\geq i, and M^i/p be a quotient module of M^i by an A-submodule generated by y^j, p<q. Particularly we denote M^x=M^0, M_1=M^1, M_i=x=M^i/p.

Let RP^i, CP^i, HP^i be i-dimensional real (complex, quaternion) projective space, respectively. Then reduced cohomology groups of them with coefficient group Z are

\[ H^k(RP^n) = M^0, \quad H^k(CP^n) = M^1, \quad H^k(HP^n) = M^2. \]

There is no space such that H^k(X) = M^k, for k\geq 3. (see [10] Chapter 1, Theorem 4.5; [3] Theorem 4.6.1.) But we can naturally make M^k (and M^i) a left A-module algebraically such that the axiom [10] by Steenrod hold on M^0. This definition has no contradiction since in the case k=0 there is a space X (=RP^n) such that H^k(X) = M^0 and since we have Proposition 2.3 in this paper.

The determination of Ext_i(M^k, Z), k=0, 1, 2, is used to the determination of 2-primary components of stable homotopy of S^o to RP^n/PP^{n-1}, CP^n/CP^{n-1}, HP^n/HP^{n-1}, respectively by [1] and [2].

After the author determined Ext^4_i(M^0, Z), t-s\leq 27, the author fined that M. Mahowald [6] had determined Ext^4_i(M^0, Z), t-s\leq 29, by his own method different to my method. Since his representation of generators is different to mine and the relationship between generators of Ext_i(M^0, Z) and those of Ext_i(Z_2, Z_2) is more clear at a glance by my method, we will offer the table of Ext_i(M^0, Z) in the last of this parer, for reference. So the main purpose of this paper is the determination of generators and relations of Ext^4_i(M^0, Z) in general (without restriction on t-s).

We conjecture by our table of Ext_i(M^0, Z), t-s\leq 27, that Ext_i(Z_2, Z_2)[h_0] is isomorphic to a direct summand of Ext_i(M^0, Z) by an appropriate correspondence. But we have not found the effective method to prove this.
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§ 2. \( \text{Ext}^0_A(M_i^k, Z_2) \).

If 2-adic expansion of a positive integer \( i \) is
\[
(2.1) \quad i = 2^{i_1} + \cdots + 2^{i_m}, \quad i_1 > \cdots > i_m \geq 0,
\]
then we define \( 2\text{-th set} \([i]\) \) and \( 2\text{-th number} \#[i] \) of \( i \) in the following;
\[
[i] = \{i_1, \ldots, i_m\}, \quad \#[i] = m.
\]

If 2-adic expansion of another positive integer \( j \) is
\[
(2.2) \quad j = 2^{j_1} + \cdots + 2^{j_n}, \quad j_1 > \cdots > j_n \geq 0,
\]
then \([i] \supseteq [j]\) means the condition that
\[
m \geq n, \quad i_1 \geq j_{m-n+1}, \ldots, i_m \geq j_n:
\]
or
\[
m \leq n, \quad i_\ell \geq j_{m+n-1}, \ldots, i_m \geq j_n.
\]

The following lemma on binomial coefficients is an alternative representation of Lemma 2.6 Chapter 1 in [10] and plays an essential part of proving many propositions and lemmas in this paper.

**Lemma 2.1**
\[
\binom{i}{j} \equiv \begin{cases} 1, & [j] \subseteq [i] \\ 0, & [j] \not\subseteq [i] \end{cases} \pmod{2}
\]

**Remark.** \([j] \subseteq [i]\) means that the set \([j]\) is contained in the set \([i]\).

**Proposition 2.2**

If \( k \geq 0, \ a < 2b \), then as operations on \( M_i^k \),
\[
\text{Sq}^a \text{Sq}^b = \sum_{t=0}^{\frac{1}{2}\binom{a}{b}} (2^t - 1) \text{Sq}^{a+t} \text{Sq}^t
\]
where \( \binom{a}{b} \) stands for the maximal integer which does not exceed \( i \) only in this proposition.

**Proof.** By the following equality and congruence:
\[
\text{Sq}^a \text{Sq}^b = \sum_{t=0}^{\frac{1}{2}\binom{a}{b}} (2^t - 1) \text{Sq}^{a+t} \text{Sq}^t
\]
\[
\left(2^b - 2^t - 1\right) = \left(2^t \left(b-t-1\right) + 2^s - 1\right) \equiv \left(2^t \left(b-t-1\right)\right) \equiv (b-t-1) \pmod{2}
\]

**Proposition 2.3**
\[
\text{Sq}^r y' = \begin{cases} y'^{r+t}, & j = 2^k y' \text{ and } [j'] \subseteq [i] \\ 0, & \text{otherwise} \end{cases}
\]
Proof. By Cartan formula. This proposition is a generalization of Lemma 2.4 Chapter 1 in [10].

Theorem 2.4

\[ A \cdot y^j = Z_2[y^j; i \geq j, [i] \geq [j], \#[i] \leq \#[j]], \text{ where } Z_2\{a : C(a)\} \text{ means } \]

a \text{ Z}_2\text{-module generated by } a \text{ satisfying the condition } C(a).

Proof. We denote by \( C \) the right hand side of the equality to prove. By Proposition 2.2 and 2.3, it is sufficient to prove this Theorem in the case \( k = 0 \). Since \( A \cdot x' \subseteq C \) easily follows from Proposition 2.3, we will only prove \( A \cdot x' \supset C \).

If \( e \) is a positive integer such that

\[ (2.3) \quad e > j, [e] \geq [j], \#[e] \leq \#[i], \]

then we denote

\[ B = Z_2[x^i \in C; e > i]. \]

We will show by induction on \( e \) that if \( B \subseteq A \cdot x' \), then

\[ x' = a \cdot x', \text{ for some } a \in A, x' \in B, a \neq 1. \]

If 2-adic expansion of \( e \) is

\[ e = 2^a_11 \cdot \cdots \cdot 2^a_n, e \geq 0 \]

and that of \( i, j \) is the same as in (2.1) and (2.2), then by \( j < e \), there is an integer \( a \) satisfying conditions (2.4) and, either (2.5), (2.6), (2.7) or (2.8):

\[ (2.4) \quad j_i = e_1, \cdots, j_{a-1} = e_{a-1}, j_a < e_a, 0 \leq a \leq q; \]

\[ (2.5) \quad e_a = e_{a+1} = 1, a \leq b < q, e_b > e_{a+1} + 1, \text{ for some } b; \]

\[ (2.6) \quad e_a = e_{a+1} = 0, a \leq b < q; \]

\[ (2.7) \quad a = q. \]

In all cases, \( [e] \geq [j] \) implies \( [e] \geq [j_{a-\varphi+b}] \). We have \( e_a > j_{a-q+b} \).

(If \( e_a = j_{a-q+b} \), then by \( e_a > j_a \)

\[ \# \left[ \sum_{i=a}^{2^b} 2^i \right] > \# \left[ \sum_{i=a+q+b}^{2^b} 2^i \right]. \]

Therefore \( \#[e] > \#[j] \). This is contrary to (2.3).)

In the case (2.5), set \( e' = 2^{a+1} \), then (2.4) implies \( e - e' > j, e > j_{a-\varphi+b} \)

implies \( [e - e'] \geq [j] \), and we have

\[ \# [e - e'] = \# [e] \leq \# [j]. \]

Therefore by the inductive hypothesis,

\[ x' = \text{Sq}^{e-e'} x'^{e-e'}, x'^{e-e'} \in B. \]

In the cases (2.6), (2.8), we have \( e_a > 0 \).
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(If otherwise, then we have not both \( e > j \), and \( \#[e] \leq \#[j] \).) The proof in the case is the same as in the case (2.5), after replacing \( b \) with \( q \).

In the case (2.7) or (2.8), if \( e_0 > j_{n+1} \), then the proof is similar to that in the case (2.5), after replacing \( b \) with \( a \).

In the case (2.5), set \( e' = 2^a - 1 \) then (2.4) implies \( e - e' > j \), \( e_a > j_{n+1} \) implies \( [e - e'] \geq [j] \), and we have

\[
\# [e - e'] = \# [e] \leq \# [j].
\]

Therefore by the inductive hypothesis,

\[ x^e = \text{Sq}^{e - e'} \circ x^{e - e'} \in B. \]

In the case (2.7), if

\[
e_0 = j_a + 1; j_u = j_{a+1} + 1, a \leq u < n \quad \text{or} \quad a = e = q,
\]

then \( \# [e] \leq \# [j] \) implies \( [j] - [e] \neq \emptyset \) and we take

\[ c = \min ([j] - [e]), \]

where signature "-" means a subtraction of two sets \( [j] \) and \( [e] \). Take

\[ e' = 2^e + \sum_{u=a}^{n} 2^u. \]

Clearly \( [e'] \subset [j] \), and \( x^e = \text{Sq}^{e} \circ x', x^e \in B \).

In the case (2.7), let

\[ e_a = j_a + 1; j_u = j_{a+1} + 1, a \leq u < b, j_b > j_{b+1} + 1, \quad \text{or} \quad j_a > j_{a+1} + 1
\]

(If \( j_a > j_{a+1} + 1 \), take \( b = a \).)

If \( e_{a+1} = j_b - 1 \), then we take

\[
c = \min \{ j_a, \ldots, j_b, j_b - 1 - [e] \},
\]

\[
d = \min \{ u; e_u \geq j_b - 1 \},
\]

\[ e' = 2^e + \sum_{u=a+1}^{b} 2^u, \]

\[ j' = j + 2^{b-1} - 2^{b+1}. \]

\[ j' > j, \quad [j'] \geq [j], \quad \# [j'] = \# [j]
\]

implies by the inductive hypothesis

(2.9) \[ x' = g \circ x^{e'} \in C, \quad \text{for some} \quad g \in A, \quad x^e \in B. \]

Clearly \( [e'] \subset [j'] \), and

\[ x^e = \text{Sq}^{e} \circ x'. \]

If \( e_{a+1} < j_b - 1 \), then we have (2.9). Clearly \( j_b - 1 \in [j'] \), and

\[ x^e = \text{Sq}^{j_b - 1} \circ x'. \]

In the case (2.8), if

\[
e_a = j_a + 1; j_u = j_{a+1} + 1, a \leq u < n \quad \text{or} \quad a = q = n,
\]

then \( j_u \in [j] \) implies
\[ x' = \text{Sq}' x^j, \quad x' \in B. \]

(We write \( j' = 2^i u \). In this case \( j = e - j' \).

In the case (2.8), if, for some \( b \),
\[ e_b = j_a + 1, \quad j_u = j_{u+1} + 1, \quad a \leq u < b < n, \quad j_b > j_{b+1} + 1, \]
and we take \( j' = 2^b - 1 \), then
\[ j_b - 1 \in [e - j'] \]

implies
\[ x' = \text{Sq}' x^{e-j'}, \quad x' \leq j', \in B \]

[Q. E. D. of Theorem 2.4]

For the next theorem we give the following notation:

If \( i \) is such an integer as (2.1), and \( u \) is such an integer that \( i > \)
\( u > i + 1 \), or \( i > u \geq 0 \), we define (in the last case, we set \( v = m \))
\[ (i, u) = 2^{i+1} + \cdots + 2^{i+2} + 2^{u+1}. \]

**Theorem 2.5**

(1) If \( 2^{i-1} < i \leq 2^{i+1} - 1 \), then
\[ \text{Ext}^0_0(M_i^b, Z_2) = Z_1 \left\{ h_u, \ max[p, \ u \geq j+k, \ h_u, \ j \geq 0, \ u \notin [i], \ p \geq (i, u) \right\} \]
\[ \text{Ext}^0_0(M_i^b, Z_2) = Z_1 \left\{ h_u, \ u \geq j+k, \ h_u, \ j \geq 0, \ u \notin [i] \right\} \]

(2) In the particular case \( i = 2^i - 1 \),
\[ \text{Ext}^0_0(M_i^b, Z_2) = Z_1 \left\{ h_u, \ u \geq j+k \right\}. \]

**Remark.** where \( Z_1 \{ a ; C(a) \} \) stand for a \( Z_2 \)-free module
genrated by \( a \) satisfying the condition \( C(a) ; h_u, h_u \) stands for the cohomology
classes of \( [\ ] y_{p-k}, [\ ] y_{(i, u)} \) of degrees \( 2^u - 2^k \), \( (i, u) \) in the coher construction \( \overline{F}(A^*, M_i^b) \) of \( M_i^b \) over \( A^* \), and \( y_u \) stands for the element in \( M_i^b \) dual to \( y_u \) in \( M_i^b \).

**Proof.** By Proposition 2.3, it is sufficient to prove the proposition in the case \( k = 0 \) and \( p = \infty \).

First we show that
\[ x^{u-1} > j; \ x^{(i, u)}, \ j > u \geq 0, \ u \notin [i], \]
genenerate \( M_i \) as a left \( A \)-module. If \( 2^{j-1} < e < 2^{j+1} - 1 \), then
\[ #[e] = j = #\left[ 2^j - 1 \right], \quad [e] \geq \left[ 2^j - 1 \right], \]
so by Theorem 2.4,
\[ x' = a \cdot x^{j-1}, \text{ for some } a \in A. \]

If \( i < e < 2^{j+1} - 1 \), and \( e \) cannot be expressed in the form of \( (i, u) \), we denote
\[ u = \max([e] - [i]). \]

Clearly \( u \notin [i] \), \( e > i, u \), and \( e \neq (i, u) \) implies \( \{u - 1, \ldots, 1, 0\} - [e] \neq \emptyset \).

Therefore
\[ [e] \geq [(i, u)], \quad \#(e) \leq \#((i, u)). \]

Then by Theorem 2.4,
\[ x = a \cdot x^{(i, u)} \text{ for some } a \in A. \]

Secondly we show that (2.10) is a minimal generating set of \( M \) as a left \( A \)-module. If \( u > v > j \), then
\[ \#([2^n - 1] = u > v = \#([2^n - 1]), 2^n - 1 > 2^n - 1 \]

implies by Theorem 2.4 that \( x^{2^n - 1} \) and \( x^{2^v - 1} \) are linearly independent. If \( j > u > v \geq 0, u \notin [i], v \notin [i], \)

then
\[ \#((i, u)) - \#((i, v)) = \#([2^n - 1]) - \#([2^v - 1]) = v - v > 0, \]
\[ (i, u) - (i, v) = (2^n - 1) - (2^v - 1) > 0, \]

implies by Theorem 2.4 that \( x^{(i, u)} \) and \( x^{(i, v)} \) are linearly independent. If \( u > j > v \geq 0, v \notin [i], \)

then
\[ 2^n - 1 > 2^j + 1, \]
\[ \#([2^n - 1]) = u > j \geq \#((i, v)) \]

implies by Theorem 2.4 that \( x^{2^n - 1} \) and \( x^{(i, v)} \) are linearly independent. Thus the proof is completed.

For the next alternative Proof of Theorem 2.5 (2), we give the following definition.

**Definition 2.6**

We define \( A \)-maps
\[ f_k : \overline{A} \rightarrow M^k, \quad \overline{A} = A/Z_n, \]
\[ f_k : \overline{A} \rightarrow M^k \]

for an admissible monomial \( S^1 S^2 \cdots S^n \) in the following:

\[ f_k (S^1 S^2 \cdots S^n) = \begin{cases} y^{i-1}, i = 2^k, i \geq 1, n = 1, \\ 0, \text{ otherwise}. \end{cases} \]

\[ f_k (S^1 S^2 \cdots S^n) = \begin{cases} y^{i-1}, i = 2^k, i \geq 1, n = 1, \\ 0, \text{ otherwise}. \end{cases} \]

We denote
\[ f = f_{\emptyset}, \quad f = f_{\emptyset}, \quad L^k = \ker f_k, \quad K = \ker f. \]

**Remark.** Adem relations ensure that \( f_k \) and \( f_k \) are \( A \)-maps in the following: If \( 0 < i < 2^j \), then
(We show only in the case \( k=0 \) by Proposition 2.2 and 2.3.)

\[
\begin{align*}
\mathcal{J}(\text{Sq}^i \text{Sq}^j) &= \sum_{t} \binom{i-t-1}{2t} \mathcal{J}(\text{Sq}^{i+j-t} \text{Sq}^j) \\
&= \binom{i-1}{i} \mathcal{J}(\text{Sq}^{i+j}) = \binom{i-1}{i} x^{i+j-1},
\end{align*}
\]

\[
\text{Sq}^i \mathcal{J}(\text{Sq}^j) = \text{Sq}^i x^{j-1} = \binom{j-1}{i} x^{i+j-1}.
\]

[Alternative proof of (2) in Proposition 2.5]

Since \( f_k \) is an \( A \)-map, and by Lemma 4.2, Chapter 1 in [10], and \( \text{Sq}^i \) is indecomposable if and only if \( i \) is a power of 2, if \( y' \notin A \cdot M^k \), then \( i=2^u-1 \), for some \( u \geq k \).

If \( y^{i-1} = a \cdot y^j \), for some \( j \) and \( a \in A \), then

\[
\mathcal{J}(\text{Sq}^{u+k}) = y^{i-1} = a \cdot y^j = a \cdot \mathcal{J}(\text{Sq}^{(j+k)}) = \mathcal{J}(a \cdot \text{Sq}^{(j+k)})
\]

Therefore

\[
\text{Sq}^{u+k} + a \cdot \text{Sq}^{(j+k)} = b, \text{ for some } b \in \ker f_k
\]

This is contrary to the fact that \( \text{Sq}^{u+k} \) is indecomposable.

\section{3. Relations in \( \text{Ext}_d(M^*_t, Z_2) \).}

We determine some typical relations in \( \text{Ext}_d(M^*_t, Z_2) \) by using the cobar construction \( F(A^*, M^*_t) \) of \( M^*_t \) over \( A^* \) in this section.

We denote by \( \alpha \beta \) the image of \( \alpha \otimes \beta \) by the composition map

\[
\text{Ext}_d(M^*_t, Z_2) \otimes \text{Ext}_d^*(Z_2, Z_2) \longrightarrow \text{Ext}^{*+*+*}_d(M^*_t, Z_2)
\]

Let \( h_n \) be the generator in \( \text{Ext}_d^{*+*}(Z_2, Z_2) \) corresponding to \( \text{Sq}^{n} \).

\textbf{Theorem 3.1}

If \( n \geq 0, i \) is such as (2.1), then in \( \text{Ext}_d(M^*_t, Z_2) \),

\[
\begin{align*}
h_{n+1}h_{n} &= 0, \quad n+1 \geq i_1 + k, \\
h_{n+2}h_{n} &= h_{n+1}h_{n+1}, \quad n+1 \geq i_1 + k, \\
h_{n+2}h_{n+2}h_{n} &= 0, \quad n+2 \geq i_1 + k.
\end{align*}
\]

\textbf{Remark.} Similar relations holds in \( \text{Ext}_d(Z_2, Z_2) \), but the following relations are not true:

\[
h_nh_{n+1} = 0, \quad h_nh_{n+2} = 0.
\]

The remainder of this section is devoted to the proof of this theorem. The direct proof is remained in the last of this section.

\textbf{Lemma 3.2}

\[
a < 2b, \quad c \geq 2d,
\]

\[
\text{Sq}^a \text{Sq}^b = \text{Sq}^a \text{Sq}^b + \cdots (\text{Adem relation})
\]
implies \( a, b \equiv 0 \pmod{2^n} \).

**Proof.** If \( a = a_2 2^n + a_1 \), \( 0 < a_1 < 2^n \), then
\[
\left( 2^n - a_1 - 1 \right) a_2 \equiv 0 \pmod{2}
\]
Therefore
\[
\left( b - 2^d d - 1 \right) a_2 = \left( 2^n (a_1 - 2d) + a_2 - 1 \right) = \left( c - a_1 - 2d \right) \left( 2^n - a_2 - 1 \right) \equiv 0 \pmod{2}
\]

**Proposition 3.3**

\( r \geq 2s \),
\[
\{(a, b) : a < 2b, \text{Sq}^a \text{Sq}^b = \text{Sq}^b \text{Sq}^a + \cdots (\text{Adem relation})\}
\]
\( g \rightarrow \{(c, d) : c < 2d, \text{Sq}^c \text{Sq}^d = \text{Sq}^c \text{Sq}^d + \cdots (\text{Adem relation})\}
\]
This map is a bijection by defining
\( g(a, b) = (2^n a, 2^n b) \)

**Proof.** The latter equality in the proof of Proposition 2.2 implies that \( g \) is a map and the definition of \( g \) implies that \( g \) is a monomorphism. If the latter Adem relation in this proposition holds, then by Lemma 3.2
\( c = 2^n c', d = 2^n d', c' < 2d'. \)

Therefore by the latter equality in the proof of Proposition 2.2, \( g \) is an epimorphism.

**Proposition 3.4**

Let \( B \) be a module over a field \( R \), \( \{b_u, u \in U\} \) be a basis for \( B \), \( b^u \) be the element in the dual \( R \)-module \( B^* \) dual to \( b_u \).

1. If \( B \) is an algebra with product \( \varphi \), and
\[
\varphi(b_u \otimes b_v) = \sum c_{u,v}^w b_w, \quad c_{u,v}^w \in R
\]
then \( B^* \) is a coalgebra with coproduct \( \varphi^* \) such that
\[
\varphi^* (b^v) = \sum (-1)^e c_{u,v}^w b^u \otimes b^v, \quad e = \deg b_u \times \deg b_v
\]
2. If \( B \) is a coalgebra with coproduct \( \psi \) and
\[
\psi(b_u) = \sum c_{u,v}^w b_u \otimes b_v, \quad c_{u,v}^w \in R,
\]
then \( B^* \) is an algebra with product \( \psi^* \) such that
\[
\psi^* (b^u \otimes b^v) = \sum (-1)^e c_{u,v}^w b^w.
\]

**Proof.** standard.

**Lemma 3.5**

If \( \text{Sq}^a \neq \text{Sq}^b \), then \( (\text{Sq}^a)^* \) has not \( \xi_1^a \) as a summand.

The proof is left to my paper to appear. This lemma is used only to prove Proposition 3.6 in the case of \( \deg (f) = 2^n \), \( q \geq 0 \), and this is not

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necessary to prove Theorem 3.1.

**Proposition 3.6**

\[(\text{Sq}^j)^* = \sum \xi^j\]

implies

\[(\text{Sq}^{n+j})^* = \sum \xi^{n+j},\]

where \(J\) runs over the same set in the two summations above and if \(I = (i_1, \ldots, i_n)\), then we denote

\[2^n \cdot I = (2^n i_1, \ldots, 2^n i_n).\]

**Proof.** By induction on \(\text{deg}(I)\).

\[\varphi^*(\sum \xi^{n+j}) = \varphi^*[(\text{Sq}^{n})^*] = \sum (\text{Sq}^{j})^* \otimes (\text{Sq}^{j})^*,\]

where the last summation runs over all pairs \(I_1, I_2\) such that

\[\text{Sq}^{I_1} \text{Sq}^{I_2} = \text{Sq}^j + \ldots.\]

By inductive hypothesis

\[\varphi^*[(\sum \xi^{n+j})] = \varphi^*[(\sum \xi^{j})]^n = \sum (\text{Sq}^{j_1})^* \otimes (\text{Sq}^{j_2})^* + (\text{Sq}^{j})^* \otimes 1 + 1 \otimes (\text{Sq}^{j})^*;\]

On the other hand, by Proposition 3.4 (1),

\[\varphi^*[(\text{Sq}^{n+j})^*] = \sum (\text{Sq}^{n+j})^* \otimes (\text{Sq}^{n+j})^*.\]

Using Lemma 3.5, we have the conclusion.

For the next proposition we denote \(\text{Sq}^j = \text{Sq}(i_1, i_2, \ldots, i_n)\), for convenience’ sake, if \(I = (i_1, i_2, \ldots, i_n)\) is complicated.

**Proposition 3.8**

1. \(\text{Sq}(2^n + \ldots, 2^{n+1}, 2^n)^* = \xi_j^{2^n+1}, j \geq 0\)
2. \(\text{Sq}(2^n(2^j + 2^{m+n}), \ldots, 2^n(2^j + 2^{m+n}), 2^{n+j-q-1}, \ldots, 2^{n+1}, 2^n)^* = \xi_j^{2^n+1}, 0 \leq m \leq j, 0 \leq q \leq j.\)
3. \(\text{Sq}(2^n(2^{n+q} - j^1 + 2^j), \ldots, 2^n(2^{n+q} - j^1 + 2^j), 2^n(2^n - j^1 + 1), 2^n - m + q - j^1, \ldotsj, 2^{n+1}, 2^n)^* = \xi_j^{2^n+1}, 0 \leq m \leq j \leq q.\)
4. \(\text{Sq}(2^n + j^1, \ldots, 2^n + j^1, 2^{j^1}, \ldots, 2^{n+1}, 2^n)^* = \xi_j^{2^n+1}, m \geq 2, j \geq 0.\)
5. \(\text{Sq}(2^n(2^m + 1), \ldots, 2^{n+j-m+1}(2^m + 1), 2^n - m, \ldots, 2^{n+1}, 2^n)^* = \xi_j^{2^n+1}, j \geq m - 1 \geq 0.\)
(6) \(\text{Sq}(2^{n+1} + 1, \ldots, 2^n, 2^{n+1} + 1, 2^n, 2^{n+m-1}, \ldots, 2^n, 2^{n+j}, \ldots, 2^n) = \xi_m^{n+j} + \xi_{j+1}^n + \xi_{j+m+1}^n, m \geq j \geq 0.\)

(7) \(\text{Sq}(2^{n+j+m} + 2^n, \ldots, 2^n, 2^{n+j+m} + 2^n, \ldots, 2^n, 2^n, \ldots) = \xi_m^{n+j} + \xi_{j+m+1}^n + \xi_{j+m}^n, 2 \leq m \leq j + 2.\)

(8) \(\text{Sq}(2^{n+j+m} + 2^n, \ldots, 2^n, 2^n, \ldots, 2^n) = \xi_m^{n+j} + \xi_{j+1}^n + \xi_{j+m+1}^n, m \geq j + 2.\)

**Proof.** It is sufficient by Proposition 3.6 to prove this proposition in the case \(n=0.\)

Proof of (5); If
\[
\psi(\text{Sq}^i) = \text{Sq}(2^{i+m}, \ldots, 2^{i+j}, 2^{i+1}) \otimes \text{Sq}(2^i, \ldots, 2, 1) + \ldots,
\]
then \(I\) is either
\[
I_1 = (2^{i+m}, \ldots, 2, 1)
\]
or
\[
I_2 = (2^{i+m} + 2^i, \ldots, 2^{i+j} + 2^{i-1} + 2^{i-m+1}, 2^{i-m}, \ldots, 2, 1)
\]

Applying Proposition 3.4 (2),
\[
\xi_m^{j+1} \xi_{j+1} = \text{Sq}(2^{i+m}, \ldots, 2^{i+j}) \otimes \text{Sq}(2^i, \ldots, 2, 1) + \ldots
\]
then (4) is a special case of (2).

Proof of (7); If
\[
\psi(\text{Sq}^i) = \text{Sq}(2^{i+m}, \ldots, 2^{i+j} \otimes \text{Sq}(2^i, \ldots, 2, 1) + \ldots
\]
then \(I\) is either
\[
I_1 = (2^{i+m}, \ldots, 2^{i+j}, 2^i, \ldots, 2, 1)
\]
or
\[
I_2 = (2^{i+m} + 2^i, \ldots, 2^{i+j} + 2^{i-m+1}, 2^{i-m}, \ldots, 2, 1)
\]

Applying Proposition 3.4 (2) and the formula (4),
\[
\xi_m^{j+1} \xi_{j+1} = \text{Sq}(2^{i+m}, \ldots, 2^{i+j}) \otimes \text{Sq}(2^i, \ldots, 2, 1) + \ldots
\]

Thus the proof is completed.

The proofs of other formulas are similar.

**Remark.** The following formula is expected to be true:

http://escholarship.lib.okayama-u.ac.jp/mjou/vol13/iss2/7
\((\text{Sq})^* = \sum b(J) \xi^J, \ \text{deg}(J) = j,\)

where if \(J = (j_1, \ldots, j_n)\), then we denote
\[
b(J) = \frac{(j_1 + \ldots + j_n)!}{j_1! \ldots j_n!}.
\]

In particular
\((\text{Sq}^{n-1})^* = \sum \xi^I, \ \text{deg}(I) = 2^n - 1.\)

For the proof of Theorem 3.1, we use only the following special cases
of the formulas above:
\[
\begin{align*}
\text{Sq}(2^{n+1}, 2^n)^* &= \xi_1^{n+1} \xi_2^n \\
\text{Sq}(2^{n+1} + 2^k)^* &= \xi_1^{n+1} \xi_2^n + \xi_2^1.
\end{align*}
\]

We denote the \(A^\ast\)-comodule map of \(M_i^*\) by
\(\Delta : M_i^* \to A^* \otimes M_i^*\)

There are some properties of this map.

**Proposition 3.9**

\(\Delta x_{j} = \sum \xi^J \otimes x_{n}\)

implies
\(\Delta y_{j} = \sum \xi^{J+1} \otimes y_{n}.\)

**Proof.** If \(\text{Sq}^j x^u = x^i, j \geq u \geq i\), in \(M_i\), then by Proposition 2.3, \(\text{Sq}^j y^u = y^i\) in \(M_i\). If \(\text{Sq}^i y^u = y^i, j \geq u \geq i\), then by Proposition 2.3, \(J = 2^n \cdot I\)
for some \(I\), and \(\text{Sq}^i x^u = x^i\). Therefore by Proposition 3.6, we have the proposition.

**Lemma 3.10**

\[
\begin{align*}
\{(I, q); \text{Sq}^j y^u = y^m\} &\xrightarrow{g} \{(J, j); \text{Sq}^j y^l = y^{m+1} - 1\}
\end{align*}
\]

This map \(g\) is a bijection by defining
\(g(I, q) = (2^n \cdot I, 2^n q + 2^n - 1)\)

**Proof.** By Proposition 2.3, \(g\) is a monomorphism. If
\(\text{Sq}^j y^l = y^m + 1\),
then by Theorem 2.4, \([j] \leq [2^n m + 2^n - 1]\), that is, \([j] \leq [2^n - 1]\).
Therefore \(j = 2^n q + 2^n - 1\) for some \(q\) and \(\text{Sq}^j\) is such that
\(\text{Sq}^j y^m = y^{m+1}\).

By Proposition 2.3, \(J = 2^n \cdot I\), for some \(I\). Thus \(g\) is an epimorphism
and a bijection.
Proposition 3.11
\[ \Delta y_m = \sum \xi^j \otimes y_j \]
implies
\[ \Delta y_{n+m+n-1} = \sum \xi^j \otimes y_{n+j+n-1} \cdot \]

*Proof.* By Lemma 3.10.

Proposition 3.12
\[ \Delta x_j = 1 \otimes x_j \]
\[ \Delta x_{n+1} = \sum x_{n+1} \]
\[ \Delta x_{2n-1} = \sum 2^{n-1} x_{2n-1} \]
\[ \Delta x_{3n-1} = \sum 3^{n-1} x_{3n-1} \]
\[ \Delta y_{3n-1} = \sum y_{3n-1} \]
\[ \Delta y_{4n-1} = \sum y_{4n-1} \]

The formulas replaced \( x_j \) with \( y_j \) and \( \xi^j \) with \( \xi^{j+i} \) above are true.

*Proof.* It is sufficient by Proposition 3.9 to prove the formulas in the case \( k=0 \). We will prove only the second, for example. The proof of the second is reduced by Proposition 3.11 to that of
\[ \Delta x_j = 1 \otimes x_j + \xi^j \otimes x_j \]
which is clearly true.

[The proof of Theorem 3.1]

Let \( \delta \) be the coboundary map of the cobar construction \( \overline{F}(A^*, M^{t*}) \).

Then it is sufficient to calculate
\[ \delta \left( \sum x_{3n-1} \right) \]
\[ \delta \left( \sum x_{3n-1} + \sum x_{3n-1} \right) \]
\[ \delta \left( \sum x_{3n-1} + \sum x_{3n-1} \right) \]


For the next proposition we denote by \( K, L, K, \overline{L} \) a \( Z \)-module generated by the following admissible monomials, respectively:

- \( K \): \( n \geq 2 \)
- \( L \): \( n \geq 2 \)
- \( K \): \( a > b \geq 0 \)
- \( L \): \( a > b \geq 0 \).

Since \( K = \ker f, L = \ker f \), and \( f \) and \( f \) are \( A \)-maps, \( K \) and \( L \) are left \( A \)-modules.
We finally prove in Proposition 5.3 that
\[ K = \overline{K} + \overline{A} \cdot K \quad \text{(direct sum)} \]

**Proposition 4.1**

\[ K = \overline{K} + \overline{A} \cdot K \quad \text{(not direct sum)} \]

**Proof.** It is sufficient to prove that
\[ \text{Sq}^a \text{Sq}^b \in \overline{A} \cdot K, \text{ if } a \geq 2b, b > 0 \text{ and unless } a = 2^a, b = 2^b, \text{ for any } a', b'. \]

Let 2-adic expansions of \( a \) and \( b \) are
\[ a = 2^{a_1} + \cdots + 2^{a_r}, \quad a_r > \cdots > a_s \geq 0, \]
\[ b = 2^{b_1} + \cdots + 2^{b_r}, \quad b_r > \cdots > b_s \geq 0. \]

The set of all cases not satisfying
\[ a = 2^a, b = 2^b, \text{ for any } a' \text{ and } b' \]
are classified into following four cases (with no intersection to each other):

1. \[ r \geq 2, \quad a_r \geq b_r + 2, \quad (4.1) \]
2. \[ r \geq 2, \quad a_r = b_r + 1, \quad q \geq 2, \quad (4.2) \]
3. \[ a_r \leq b_r, \quad q \geq 2, \quad (4.3) \]
4. \[ r = 1, \quad a_r > b_r, \quad q \geq 2, \quad (4.4) \]

**Proof of the case (4.1):** Let
\[ a = a' 2^{n+2}, \quad b = b' 2^{n+1} + 2^n, \quad a' > b' > 0. \]

Then
\[ \text{Sq}(2^{n+1}, a - 2^n, b' 2^{n+1}) = (\text{Sq}^a \text{Sq}^b + \text{Sq}^{a+b}) \text{Sq}^{n+1} \]
\[ = \text{Sq}^a \text{Sq}^b + \sum_{i=0}^{n-1} \text{Sq}(a, b - 2^i, 2^i) + \text{Sq}(a + 2^n, b' 2^{n+1}). \]

The last summand is reduced to the case (4.3).

**Proof of the case (4.2):** Let
\[ a = a' 2^{n+2} + 2^{n+1}, \quad b = b' 2^{n+1} + 2^n, \quad a' \geq b' > 0. \]

Then
\[ \text{Sq}(2^{n+1}, a' 2^{n+2} + 2^n, b' 2^{n+1}) = \text{Sq}(a, 2^n, b' 2^{n+1}) \]
\[ = \text{Sq}^a \text{Sq}^b + \sum_{i=0}^{n-1} \text{Sq}(a, b - 2^i, 2^i). \]

**Proof of the case (4.3):** Let
\[ a = a' 2^{n+1} + 2^n, \quad b = b' 2^n, \quad a' \geq b' > 0. \]

Then we prove it by induction on \( n \). If \( n = 0 \), then
\[ \text{Sq}^a \text{Sq}^b = \text{Sq}^b \text{Sq}^{a-1} \text{Sq}^a. \]
Therefore \( \text{Sq}^a \text{Sq}^b \in A \cdot K \). If \( n > 0 \), then
\[(4.5) \quad \text{Sq}(2^n, a^{2n+1}, b) = \text{Sq}^a \cdot \text{Sq}^b + \sum_{i=0}^{n-1} \text{Sq}(a-2^i, b+2^i) + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \text{Sq}(a-2^i, b+2^j - 2^i, 2^i) \]

\(\text{Sq}(a-2^i, b+2^i)\) is not admissible only in the case \(a' = b', \ t = n-1, \) but if \(n \geq 2, \) then
\[\text{Sq}(a-2^{n-1}, b+2^{n-1}) = \text{Sq}(a'2^{n+1}+2^{n-1}, a'2^n+2^{n-1}) = \sum_{i=0}^{n-1} \text{Sq}(2^{n+1}a'+2^{n-2}, 2^n a'+2^i).\]

Transform the summands of \(n-2 \geq t \geq 0\) in the form as \((4.5)\), and we know that they are contained in \(\overline{A} \cdot K\) by inductive hypothesis. The summand of \(t = n-2\) is
\[\text{Sq}(2^{n-1}c+2^{n-3}, 2^{n-1}c), \ c = 4a'+1.\]

Apply the same method above to this summand, and we know that there remains only one summand
\[\text{Sq}(2^{n-3}d+2^{n-4}, 2^{n-4}d), \ d = 4c+1,\]

which is unknown to be contained in \(\overline{A} \cdot K\), if \(n \geq 4.\)

But by applying this method repeatedly the problem is reduced to either \(\text{Sq}(4e+1, 2e+1)\) or \(\text{Sq}(8e+2, 4e+2)\) according that \(n\) is odd or even. We have
\[\text{Sq}(4e+1, 2e+1) = 0\]
\[\text{Sq}(8e+2, 4e+2) = \text{Sq}(8e+3, 4e+1) = \text{Sq}(1, 8e+2, 4e+1)\]

Then in the cases \((4.1), (4.2), \) and \((4.3), \) by inductive hypothesis, \(\text{Sq}^2 \in \overline{A} \cdot K.\)

Proof of the case \((4.4): \) Let \(b = 2^n.\) Using Proposition 3.2 we can decompose \(\text{Sq}^a\) in the form of
\[\text{Sq}^a = \sum_{u>0} c_u \text{Sq}^{u}, \ c_u \in A.\]

Therefore \(\text{Sq}^a \text{Sq}^u \in \overline{A} \cdot K.\)

Thus the proposition has been proved.

In this proof, we use the following formulas.

**Lemma 4.2**
\[
\text{Sq}(2^n, 2^{n+1}a+2^i) = \sum_{i=0}^{n-1} \text{Sq}(2^{n+1}a+2^{n+1}-2^i, 2^n)
\]
\[
\text{Sq}(2^n, 2^{n+1}a) = \sum_{i=0}^{n-1} \text{Sq}(2^{n+1}a+2^i-2^i, 2^n) + \text{Sq}(2^{n+1}a+2^n)
\]
\[
\text{Sq}(2^{n+1}, 2^{n+1}a+2^n) = \text{Sq}(2^{n+1}a+2^{n+1}, 2^n)
\]
\[
\text{Sq}(a2^{n+2}+2^i, 2^{n+1}a+2^n) = \sum_{i=0}^{n-1} \text{Sq}(2^{n+2}a+2^{n+1}-2^i, 2^{n+1}a+2^n).
\]

We define an \(A\)-map
Aikawa: $\text{Ext}(\mathbb{Z}_2[y]/\mathbb{Z}_2, \mathbb{Z}_2)$, $A$ being the mod 2 Steenrod algebra

$\overline{\mathcal{f}}: \overline{A}^3 \to N$

$N = \overline{A} \cdot M^0 = \mathbb{Z}_2 \{x^j : j > 0, j \neq 2^n - 1, \text{ for any } n\}$

to be the restriction of $f: A \to M^0$. then

$0 \to K \to \overline{A}^3 \xrightarrow{\mathcal{f}} N \to 0$

is an exact sequence of left $A$-modules.

Lemma 4.3

$N = \overline{A} \cdot N + \mathbb{Z}_2 \{x^{n+1}y^{n-1}, n \geq 0\}$. (direct sum)

Proof. If

$a = 2^{m+1}a' + 2^m + 2^n - 1, m \geq n + 2, a' \geq 0,$

or $m = n + 1, a' > 0$,

then

$\text{Sq}^{n+1}x^{n-2}y^{n-1} = x^n, x^{n-2}y^{n-1} \in N.$

If $m > n$, set

$m' = 2^{m+1} + 2^m - 1, n' = 2^{m+1} + 2^n - 1,$

then

$m' > n', \#[m'] = m + 1 > n + 1 = \#[n'].$

Therefore $x^{n'}$ and $x^{n'}$ are linearly independent by Theorem 2.4. Thus

the proof is completed.

Proposition 4.4

$L^0 = \overline{L}^0 + \overline{A} \cdot L^0$ (not direct sum).

Proof. By Proposition 4.1,

$L^0 = K + \mathbb{Z}_2 \{\text{Sq}^1\}$

$= \overline{K} + \overline{A} \cdot K + \mathbb{Z}_2 \{\text{Sq}^1\}$

$= \overline{L}^0 + \overline{A} \cdot L^0 + \mathbb{Z}_2 \{\text{Sq}^1, j > 0\} + \mathbb{Z}_2 \{\text{Sq}^1\}$

$= \overline{L}^0 + \overline{A} \cdot L^0$ (not direct sum).

§ 5. Exact sequences for Ext.

The author imagines that somebody has ever proved the following proposition.

Proposition 5.1

Let $R$ be a commutative ring with unit and $B$ be an algebra over $R$.

(1) Then an short exact sequence of left $B$-modules

$0 \to L \xrightarrow{i} N \xrightarrow{f} M \to 0$
and a left $B$-module $G$ induce an exact sequence of right $\text{Ext}_i^s(G, G)$-modules, $r \geq 0$,

\[
\cdots \rightarrow \text{Ext}_i^s(M, G) \rightarrow \text{Ext}_i^s(N, G) \rightarrow \text{Ext}_i^s(L, G) \\
\rightarrow \text{Ext}_i^{s+n}(M, G) \rightarrow \cdots
\]

(2) $F_s$, $I_s$ and $\partial_s$ are compatible with Massey products; in detail, if $m \in \text{Ext}_i(M, G)$, $l \in \text{Ext}_i(L, G)$, $n \in \text{Ext}_i(N, G)$, $a, b \in \text{Ext}_i(G, G)$, then

- $F_s<m, a, b> \subset <F(m), a, b>$, if $ma = 0 = ab$,
- $I_s<n, a, b> \subset <I(n), a, b>$, if $na = 0 = ab$,
- $\partial_s<l, a, b> \subset <\partial(l), a, b>$, if $la = 0 = ab$.

These properties hold for iterated Massey products. For example,

- $F_s<m, a, b>, a', b'> \subset <F(m), a, b>, a', b'>$,

if $ma = 0 = ab$, $<m, a, b>a' \equiv 0$, and $a'b' = 0$, where $F_s$, $I_s$ and $\partial_s$ stand for $F_s$, $I_s$ and $\partial_s$ for an appropriate $s$.

We apply this proposition to the following short exact sequence of left $A$-modules:

\[
0 \rightarrow L^k \rightarrow \overline{A} \rightarrow M^k \rightarrow 0
\]

Then the following exact sequence is induced:

\[
\cdots \rightarrow \text{Ext}_i^{s-1}(L^k, Z_2) \rightarrow \text{Ext}_i^{s-2}(M^k, Z_2) \\
\rightarrow \text{Ext}_i^s(\overline{A}, Z_2) \rightarrow \text{Ext}_i^s(L^k, Z_2) \rightarrow \cdots
\]

(5.1)

By comparing the dimensions of generators,

$F_s(h_n) = h_n$, $n > k$.

**Proposition 5.2**

\[
\text{Ext}_i^s(M^n, Z_2) = Z_2 \{h_n h_b \; : \; a \not\equiv b := 1, a > 0, b \geq 0 \}.
\]

$L^n = \overline{A} \cdot L^n + \overline{L}^b$. (direct sum)

**Remark.** By Theorem 3.1, we have

$h_n h_{a - 1} = 0$, $a > 0$.

**Proof.**

\[
\text{Sq}^{a^b} x^{b-1} = 0, \quad a \geq b.
\]

\[
\text{Sq}^{a^b} x^{b-1} = \text{Sq}^{a^b-1} \text{Sq}^{a^b-1}, \quad a + 2 \leq b,
\]

implies
Aikawa: \( \text{Ext}A(\mathbb{Z}_2[y]/\mathbb{Z}_2, \mathbb{Z}_2) \), \( A \) being the mod 2 Steenrod algebra

\[
\begin{align*}
\text{Ext}_* & (\mathbb{Z}_2[y]/\mathbb{Z}_2, \mathbb{Z}_2) \\
& 167 \\
\text{Ext}_*(\mathbb{Z}_2[y]/\mathbb{Z}_2, \mathbb{Z}_2) \\
\end{align*}
\]

\[h_0h_a \neq 0, \ b > 0, \ a \geq 0, \ b \neq a + 1,
\]
\[h_0h_a \neq h_0h_b, \ a \geq b - 2, \ b > 0.
\]

Since
\[F_i(h_0h_{i+1}) = h_0h_{i+1} = 0, \ b > 0 \]
\[F_i(h_0h_a) = h_0h_a = h_a h_a = F_i(h_a h_b), \ a - 2 \geq b > 0 \]
\[I_0(h_0) = h_0
\]

(where \( h'_0 \) is the cohomology class of \( [ ] \xi_1 \) in the cobar construction \( \overline{F}(A^*, L^{**}) \)), we have
\[Z_2 \{h_0h_{i+1}, b > 0; h_0h_a + h_0h_b, a - 2 \geq b > 0\}
\subset \ker F_i = \text{im} \ \partial_a = \text{coker} \ I_0.
\]

coker \( I_0 \) is a \( \mathbb{Z}_2 \)-module generated by \( g_{a,b} \), which is the cohomology class of \( [ ](\text{Sq}^{a}\text{Sq}^{-b})^* \) in the cobar construction \( \overline{F}(A^*, L^{**}) \), for \( a, b \) such that \( a > b > 0 \) and \( \text{Sq}^{a}\text{Sq}^{-b} \notin \overline{A} \cdot L^0 \). (Therefore \( g_{a,b} \neq 0 \), if exists.) By comparing the dimensions,
\[
coker I_0 = Z_2 \{g_{a,b}, a > b > 0\},
\]
\[
\partial_a(g_{a,b}) = h_0h_a + h_0h_b, \ a - 2 \geq b > 0
\]
\[
\partial_a(g_{a+1,b}) = h_a h_{a+1}, \ a > 0
\]

and two sets of generators in the left hand side and right hand side correspond bijectively to each other. Thus the proof is completed.

**Proposition 5.3**

\[K = \overline{A} \cdot K + \overline{K} \ (\text{direct sum}).
\]

**Proof.** Let \( S = \mathbb{Z}_2[\text{Sq}^1] \). Then the short exact sequence of left \( A \)-modules:
\[0 \longrightarrow K \longrightarrow L^0 \longrightarrow S \longrightarrow 0
\]
induces the long exact sequence:
\[\cdots \longrightarrow \text{Ext}_i^*(L^0, \mathbb{Z}_2) \overset{I_*}{\longrightarrow} \text{Ext}_i^*(K, \mathbb{Z}_2) \overset{\partial_*}{\longrightarrow} \text{Ext}_{i+1}^{*,1}(S, \mathbb{Z}_2) \longrightarrow \text{Ext}_{i+1}^{*,1}(L^0, \mathbb{Z}_2) \longrightarrow \cdots
\]

By Proposition 5.2 and 4.4, \( \text{Ext}_0^*(K, \mathbb{Z}_2) \) is a \( \mathbb{Z}_2 \)-free module generated by \( g_{a,b} \), which is the cohomology class in the cobar construction \( \overline{F}(A^*, K) \) for \( a, b \), such that \( a > b \geq 0 \) and \( \text{Sq}^{a}\text{Sq}^{-b} \notin \overline{A} \cdot K \). In \( \text{Ext}_*(L^0, \mathbb{Z}_2) \), \( h'_0h_0 \neq 0, \ h'_0h_0 = 0, \ u > 0 \).

By comparing the dimensions of generators, \( g_{u,v}, \ u > v \geq 0 \), are generators in \( \text{Ext}_*(K, \mathbb{Z}_2) \) and \( F_0(1) = h'_0, \ F_1(h_0) = h'_0h_0 \),

Produced by The Berkeley Electronic Press, 1967
\[ I_0(g_{u,v}) = g_{u,v}, \quad u > v > 0, \]
\[ \partial_0(g_{u,v}) = h_u, \quad u > 0. \]

**Corollary 5.4**
\[ \overline{A} = \overline{A}^0 + Z_2 \{ \text{Sq}^a \text{Sq}^b ; a > b \geq 0, a + 1 = b > 0 \}. \] (direct sum)

**Proof.** Apply Proposition 5.3 to the exact sequence (4.6) and we have \( I_0(g_{u,v}) = g_{u,v}, \quad a > b \geq 0, \) and \( g_{u,v}, \quad a > b \geq 0, \) are generators of \( \text{Ekt}_0(A^*, Z_2). \)

Let \( N^k \) be a \( Z_2 \)-module generated by \( x^n; n \geq 0 \) and \( n \equiv -1 \pmod{2^k} \) or \( n = 2^k - 1 \)

**Lemma 5.5**
\[ N^k = \overline{A} \cdot N^k + Z_2 \{ x^{j-1}, 0 \leq j \leq k : x^{j-1-2^k-1}, j \geq k + 2 \}. \] (direct sum)

**Proof.** By Theorem 2.4. We denote by \( h_j, b_{i,j} \) the cohomology classes of \( [x_{j-1}] \) and \( [x_{j-1}^{j-2^k-1}] \) in \( \overline{F}(A^*, N^k) \), respectively. \( \deg h_j = 2^j - 1, \deg b_{i,j} = 2^j - 2^{k-1}. \)

**Proposition 5.6**
\[ \text{Ext}_1^i(L^k, Z_2) = Z_2 \{ h_u, u \leq k ; g_{u,v}, u > v > k ; b_{i,j}, j \geq k + 2 \}. \]
\[ \deg h_u = 2^u, \quad \deg b_{i,j} = 2^i - 2^{k-1}, \quad \deg g_{u,v} = 2^u + 2^v. \]

**Proof.** There is a morphism of short exact sequences of left \( A \)-modules:

\[
\begin{array}{cccc}
0 & \rightarrow & L^k & \rightarrow & A & \rightarrow & M^0 & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & K & \rightarrow & L^k & \rightarrow & N^k & \rightarrow & 0 \\
\end{array}
\]

This induces a morphism of long exact sequences for \( \text{Ext} \):

\[
\begin{array}{cccc}
\text{Ext}^{i+1}_1(Z_2, Z_2) & \xrightarrow{r_x} & \text{Ext}^i_1(M^0, Z_2) & \xrightarrow{r_x^1} & \text{Ext}^i_1(A, Z_2) & \xrightarrow{r_x} & \text{Ext}^i_1(L^k, Z_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Ext}^{i+1}_1(N^k, Z_2) & \xrightarrow{r_x} & \text{Ext}^i_1(L^k, Z_2) & \xrightarrow{r_x} & \text{Ext}^i_1(K, Z_2) \\
\end{array}
\]

We denote \( F_0'(h_k) = h_j, \quad k \geq j \geq 0, \quad F_0'(b_{i,j}) = b_{i,j}. \) Then \( h_j = F_0'(h_j) = q_j F_0'(h_j) = q_j(h_j). \)
Aikawa: Ext\(_A(Z_2[y]/Z_2,Z_2)\), \(A\) being the mod 2 Steenrod algebra

\[
\partial_\alpha(g_{u,v}) = \begin{cases} 
0, & u > v > k \\
h_0h_v, & u - 2 \geq v, \ u > k \geq v \\
h_0h_v + h_0h_{u+1}, & k \geq u \geq v + 2 \\
h_0h_{u+1}, & k \geq v = u-1 \geq 0.
\end{cases}
\]

Therefore \(\ker \partial_\alpha = Z_2\{g_{u,v}, u > v > k\}\). Thus the proof is completed.

**Theorem 5.7**

If \(k > 0\), then

\[\text{Ext}_4(M^k, Z_2) = Z_2\{h_0h_v, k < u \neq v + 1; b_{k,j}, j \geq k + 2\}.
\]

\[\deg (h_0h_v) = 2^u - 2^v + 2^k, \ \deg b_{k,j} = 2^j - 3 \cdot 2^{k-1}.
\]

**Proof.** By comparing the dimensions in the exact sequence (5.1),

\[\partial_\alpha(g_{u+1,v}) = h_0h_v + h_0h_{u+1}, \ u > k; \ \partial_\alpha(b_{k,j}) = b_{k,j}, \ j \geq k + 2.
\]

\[\partial_\alpha(g_{u,v}) = h_0h_v, \ u > v > k,
\]

and \(F_0(h_0h_v) = h_0h_v = F_1(h_0h_v)\) implies \(h_0h_v \neq h_0h_v\).

(Also we can show this directly by the method similar to the proof of Proposition 5.2.) Thus the proof is completed.

**Proposition 5.8**

\(\text{In Ext}_A(L^2, Z_2), g_{a,b}h_a \neq 0, g_{a,b}h_{a-1} = 0, g_{a,b}h_{b-1} = 0,
\)

\(g_{a,b}h_{a+1} = g_{a+1,b}h_a, g_{a,b}h_{b+1} = g_{a+1,b}h_a, a > b > k:
\)

\[g_{a,b}h_a + g_{a,b}h_b + g_{b,b}h_a = 0, a - 4 \geq b - 2 \geq c > k.
\]

\(\text{In Ext}_A(L^0, Z_2), a \geq 0,
\)

\[g_{a+3,a+3}h_a^2 = g_{a+3,a+3}h_{a+1}^2 = g_{a+3,a+3}h_{a+1}^2.
\]

**Theorem 5.9**

1. If \(\alpha\) and \(\beta\) are non-zero elements of \(\text{Ext}_A(Z_2, Z_2)\), and \(\alpha \beta \neq 0\), then \(\alpha \beta \neq 0\). In particular \(h_0 \beta \neq 0, u > k, \text{ in Ext}_4(M^k, Z_2),
\)

2. If \(\alpha, \beta, \text{ and } \gamma\) are in \(\text{Ext}_A(Z_2, Z_2), \text{ then we denote an iterated Massey product by}
\)

\[M(\alpha) = \langle \langle \cdots \langle \langle \alpha, \beta, \gamma\rangle, \beta, \gamma\rangle, \cdots \rangle, \beta, \gamma\rangle.
\]

If \(M(\alpha)\) and \(M(\alpha)\) are defined and \(M(\alpha) \neq 0\) in \(\text{Ext}_4(M^k, Z_2)\).

**Proof.** By Proposition 5.1.
Corollary 5.10

(1) \( h_{\alpha}h_{\alpha}h_{\alpha} > k, w \neq v \pm 1, v \neq w \pm 1, u \neq w \pm 1; \)
\( h_{\alpha}h_{\alpha}h_{\alpha} > k \)
are non-zero in \( \text{Ext}_{\alpha}(M^k, Z_2) \).

(2) \( c_{\alpha}, h_{\alpha}c_{\alpha} = c_{\alpha}h_{\alpha}, d_{\alpha}, P'_{\alpha}h_{\alpha}, P'_{\alpha}h_{\alpha}, P'_{\alpha}c_{\alpha}, P'_{\alpha}d_{\alpha} \)
are non-zero in \( \text{Ext}_{\alpha}(M^0, Z_2) \).

Remark. In theorem 5.8 and Corollary 5.9, if \( \alpha \) is an element in \( \text{Ext}_{\alpha}(M^k, Z_2) \) which is mapped to \( \alpha \) by \( F_n \), for an appropriate \( n \), then we denote this element by \( \gamma \). The representation of generators of \( \text{Ext}_{\alpha}(Z_2, Z_2) \) is due to [9] and [7].

§ 6. Tables.

We offer the tables of \( \text{Ext}_{\alpha}^s(L^0, Z_2), t - s \leq 29 \), and \( \text{Ext}_{\alpha}^s(M^0, Z_2), t - s \leq 27 \), in this section.

We first determine the former by determining the partial minimal resolution of \( L^0 \) over \( A \). Secondly we determine the latter by the former and the table of \( \text{Ext}_{\alpha}(Z_2, Z_2) \) in [9], [7] and the exact sequence (5.1) in the case \( k = 0 \). We only remark the fact that \( I_s \) is trivial for all generators in \( \text{Ext}_{\alpha}^s(Z_2, Z_2) \), except for \( h_0^{\alpha + 1} \), when \( F_s(h_0^{\alpha + 1}) = h_0^{\alpha}h_0^{s} \), \( s \geq 0 \), in that range of \( s \), \( t \).

Since \( \alpha_2 = <g_2, h_1, h_2^2>, \alpha_3 = <g_3, h_0, h_2^2>, \)
\( F(\alpha_2) = <F(g_2, i), h_0, h_2> = h_1 <h_3, h_1, h_2^2> = h_1 c_1 \)
\( F(\alpha_3) = <F(g_3, i), h_0, h_2^2> = h_1 h_3 + h_3 h_1 h_0, h_2^2> \)
\( = h_1 h_0, h_2^2> + <h_1, h_0, h_2^2> h_4 = h_1 c_0 + c_0 h_4. \)

By exactness \( h_4 c_0 = c_0 h_4 \). By constructing a minimal resolution \( h_4 c_0 = c_0 h_4 \neq 0 \) and by Theorem 5.8 \( h_4 c_0 h_4 \neq 0 \). By \( F(\alpha_2 h_1) = h_4 c_0 h_1 + c_0 h_4 h_1, h_4 c_0 h_1 \)
\( = c_0 h_4 h_1 = h_4 c_0 h_1. \)

As above \( h_4 c_0 \neq 0 \), but by constructing a minimal resolution \( c_0 h_4 = 0. \)

In Table 6.1 and 6.2, “bar” means “multiplied by \( h_0, h_1 \) or \( h_2 \).”

We imagine that \( bideg (h_0 h_3^2 d_3) = (6, 23), bideg (i) = (7, 23), \)
\( d_s(h_0 h_3^2 d_3) = s P_1 d_3, \pi_s(P_3) = Z_2 \oplus Z_1 \) in Table 8.3, and \( (h_0 h_3^2 h_3) h_3 \neq 0 \) in Table 8.2 in [6]
are misprints and we think that they must be corrected in the following:
\( bideg (h_0 h_3^2 d_3) = (7, 23), bideg (i) = (6, 23), \)
\( d_s(i) = s P_1 d_3, \pi_s(P_3) = Z_2 \oplus Z_1, \)
and \( (h_0 h_3^2 h_3) h_3 = 0, (c_0 h_4 h_3) h_4 \neq 0. \)

In Table 8.3, \( (s P_1 h_3 c_0) h_1 = (s P_1 c_0) h_3^2, \)
\( (s P_1 h_1 c_0) h_2 = (s P_1 d_3) h_3, \)
\( (s P_1 h_3) h_3 = (s P_1 c_0) h_3^2. \)

In Table 8.4, \( (s P_1 h_3) h_3 = (s P_1 c_0) h_3^2. \)
Aikawa: ExtA(Z2[y]/Z2, Z2), A being the mod 2 Steenrod algebra

ExtA(Z2[y]/Z2, Z2)

REFERENCES


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(Received July 9, 1968)
### Table 6.1

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\[
\alpha_2 = \langle g_{s,1}, h_1, h_3^2 \rangle
\]

\[
\alpha_4 = \langle g_{s,1}, h_0, h_3^2 \rangle
\]

\[
\alpha_3 h_3 = \alpha_1 h_3
\]
Aikawa: $\text{Ext}(\mathbb{Z}_2[y]/\mathbb{Z}_2, \mathbb{Z}_2)$, $A$ being the mod 2 Steenrod algebra

Table 6.2

$\text{Ext}(\mathcal{M}^0, \mathbb{Z}_2)$

Ext of $\mathcal{M}^0$ with coefficients in $\mathbb{Z}_2$.