Hadamard matrices of bush type

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HADAMARD MATRICES OF BUSH TYPE

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In [1] Bush suggested a method for constructing a Hadamard matrix of order $n$ using a Hadamard matrix of order $\frac{1}{2}n-2$ and a skew Hadamard matrix of $\frac{1}{4}n+1$, where $n \equiv 12 \pmod{16}$. A Hadamard matrix of order $n$ constructed by the method of Bush will be called a Hadamard matrix of Bush type of order $n$.

The purpose of this note is to prove two propositions on Hadamard matrices of Bush type of order $n$.

For basic facts on Hadamard matrices see [2].

1. Introduction. We want to construct a Hadamard matrix of order $n = 16u + 12$ under certain "inductive" assumptions, where $u$ is a non-negative integer. Obviously it suffices to construct a symmetric $2 - (16u + 11, 8u + 5, 4u + 2)$ design $D = (P, B)$, where $P = \{1, 2, \ldots, 16u + 11\}$ and $B$ denote the sets of points and blocks of $D$ respectively.

We make the following "inductive" assumptions: (1) There exists a Hadamard matrix $L$ of order $8u + 4$, and (2) there exists a skew Hadamard matrix $R$ of order $4u + 4$. Put $L = (\lambda(i), 1 \leq i \leq 8u + 4$, where $\lambda(i)$ denotes the $i$-th row vector of $L$ and we may assume that $\lambda(1)$ is the all one vector. Let $L(\lambda(1)) = (P(\ell), B(\ell))$ be the Hadamard 3-design associated with $L$ at $\lambda(1)$. We put $P(\ell) = \{1, 2, \ldots, 8u + 4\}$ so that the block $\sigma(i)$ of $L(\lambda(1))$ corresponding to $\lambda(i)$ contains the point $j$ if and only if the $j$-th component of $\lambda(i)$ equals 1, where $2 \leq i \leq 8u + 4$ and $\sigma(i)^* = P(\ell) - \sigma(i)$ is also a block of $L(\lambda(1))$, for $2 \leq i \leq 8u + 4$. Clearly we have that $\sigma(i) \cap \sigma(i)^* = \emptyset$ and $|\sigma(i) \cap \sigma(j)| = |\sigma(i) \cap \sigma(j)^*| = 2u + 1$ for $i \neq j$.

We pick up any $4u + 3$ distinct disjoint block pairs from the $1\sigma(i)$, $\sigma(i)^*$, $2 \leq i \leq 8u + 4$. For simplicity of notation we denote them by $1\sigma(i)$, $\sigma(i)^*$, $2 \leq i \leq 4u + 4$. This configuration $\Sigma$ consists of $8u + 6$ blocks of size $4u + 2$.

Next we may assume that $R$ is in a skew-normalized skew form:

$$R = \begin{bmatrix}
-1 & 1 & \cdots & 1 \\
-1 & -1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & -1
\end{bmatrix} = (\rho(i)),$$

where $\rho(i)$ denotes
the $i$-th row vector of $R$, $1 \leq i \leq 4u+4$.  we label the $j$-th column of $R$ by $8u+2j+2$, for $2 \leq j \leq 4u+4$, and notice that the first column is still labelled 1.

Let $D(r) = (P(r), B(r))$ be a symmetric $2-(4u+3, 2u+2, u+1)$ design which is the complement of the symmetric $2-(4u+3, 2u+1, u)$ design associated with $R$ at $\rho(1)$ punctured at 1. We put $P(r) = 8u+6, 8u+8, ..., 16u+10$ so that the block $\tau(i)$ of $D(r)$ corresponding to $\rho(i)$ contains the point $8u+2j+2$ if and only if the $j$-th component of $\rho(i)$ equals $-1$ ($2 \leq i, j \leq 4u+4$). Let us define a mapping $T$ from $B(r)$ to $P(r)$ by $\tau(i)T = 8u+i2+2$, for $2 \leq i \leq 4u+4$. Then by the skew property of $R$ we have that $\tau(i)T \in \tau(i)$ and that $\tau(i)T \in \tau(j)$ if and only if $\tau(j)T \notin \tau(i)$ for $i \neq j$.

Now we are going to double points and blocks of $D(r)$ as follows. The block $\tau(i)$ will be developed into two blocks $\tau(i1)$ and $\tau(i2)$, $2 \leq i \leq 4u+4$. If $8u+2j+2 \in \tau(i)$ and $i \neq j$, then both $\tau(i1)$ and $\tau(i2)$ contain both $8u+2j+2$ and $8u+2j+3$. If $i = j$, then $\tau(i)$ contains only $8u+i2+2$ and $\tau(i2)$ contains only $8u+i2+3$. Then clearly we have that $|\tau(i1) \cap \tau(i2)| = 4u+2$, for $2 \leq i \leq 4u+4$. Moreover, since $|\tau(i) \cap \tau(j)| = u+1$ and since $\tau(i)T \in \tau(j)$ if and only if $\tau(j)T \notin \tau(i)$ for $i \neq j$, we have that $|\tau(ik) \cap \tau(j\ell)| = 2u+1$ for $i \neq j$ and $1 \leq k, \ell \leq 2$. In this way we get a configuration $\mathfrak{M}$ consisting of $8u+6$ blocks of size $4u+3 = 1+2(2u+1)$.

Finally we match $\mathcal{Q}$ and $\mathfrak{M}$ together in any possible way under the condition that $|\sigma(i), \sigma(i)^*|$, and $|\tau(j1), \tau(j2)|$ should be matched if a member of $|\sigma(i), \sigma(i)^*|$ is matched together with a member of $|\tau(j1), \tau(j2)|$. For simplicity of notation we assume that $\sigma(i)$ and $\tau(i)$, and hence $\sigma(i)^*$ and $\tau(i2)$, are matched together, $2 \leq i \leq 4u+4$.

Put $\sigma(1) = P(\mathcal{Q} \cup |8u+5|$, $\sigma(2i-2) = \sigma(i) \cup \tau(i1)$ and $\sigma(2i-1) = \sigma(i)^* \cup \tau(i2)$, for $2 \leq i \leq 4u+4$. Then it is easy to see that $|\sigma(i)| = 8u+5, 1 \leq i \leq 8u+7$ and $|\sigma(i) \cap \sigma(j)| = 4u+2$ for $i \neq j$.

So the configuration $\mathfrak{Q} = (\mathcal{Q}, |\sigma(i)|, 1 \leq i \leq 8u+7)$ is possibly a portion of a symmetric $2-(16u+11, 8u+5, 4u+2)$ design.

Now we prove the following proposition.

**Proposition 1.** A necessary and sufficient condition for $\mathfrak{Q}$ to be completed to a symmetric $2-(16u+11, 8u+5, 4u+2)$ design can be stated as follows.

There exist $8u+4$ subsets $\mu(j)$ of size $4u+2, 1 \leq j \leq 8u+4$, of $\sigma(1)$,
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called blocks again. such that $D(\ell) = (\sigma(1), \sigma(i), \sigma(i)^*), 1 \leq i \leq 4u+3, \\
\mu(j), 1 \leq j \leq 8u+4)$ forms a $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design, where the five parameters correspond to the usual notation $v, k, \lambda, b$ and $r$ respectively, with the following three conditions:

(1) Put $\sigma(i) = \sigma(i)$ or $\sigma(i)^*, 1 \leq i \leq 4u+3$. Then with any fixed
$\sigma(i)$ one half of the $\mu(k)$ intersects in $2u+1$ points and the other half of the
$\mu(k)$ intersects in $2u$ points.

(2) With each of any fixed $\sigma(i)$ and $\sigma(j)$ for $i \neq j$ one quarter of the
$\mu(k)$ intersects in $2u+1$ points and another quarter of the $\mu(k)$ intersects in
$2u$ points.

(3) Let $a$ be a point such that $1 \leq a \leq 8u+4$. If $a$ belongs to $\sigma(i)$,
then exactly $2u$ of the $\mu(k)$ which intersects with $\sigma(i)$ in $2u$ points contain $a$.
If $a$ does not belong to $\sigma(i)$, then exactly $2u+1$ of the $\mu(k)$ which intersects
with $\sigma(i)$ in $2u$ points contain $a$.

Proof. Necessity. Suppose that $\mathfrak{P}$ is completed to a symmetric $2-
(16u+11, 8u+5, 4u+2)$ design $D$. New blocks will be denoted by $\sigma(i)$,
for $8u+8 \leq i \leq 16u+11$. Put $\mu(i-8u-7) = \sigma(1) \cap \sigma(i)$ for $8u+8 \leq
i \leq 16u+11$. Then $D(\ell) = (\sigma(1), \sigma(i), \sigma(i)^*), 1 \leq i \leq 4u+1, \mu(j), 1 \\
\leq j \leq 8u+4)$ is a $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design.
In fact, let $a$ and $b$ be any two distinct points of $\sigma(1)$. Then $a$ belongs to
$8u+5$ blocks of $D$ including $\sigma(1)$ and $|a, b|$ is contained in $4u+2$ blocks of
$D$ including $\sigma(1)$. Hence $a$ belongs to $8u+4$ blocks of $D(\ell)$ and $|a, b|$ is
contained in $4u+1$ blocks of $D(\ell)$. So we have only to check three conditions
(1), (2) and (3) on $D(\ell)$.

If $\sigma(8u+7+k), 1 \leq k \leq 8u+4$, contains both $8u+2i+2$ and $8u+
2i+3$, where $2 \leq i \leq 4u+4$, or if it contains neither $8u+2i+2$ nor $8u+
2i+3$, then $\sigma(8u+7+k) \cap \tau(i1) = \sigma(8u+7+k) \cap \tau(i2)$. Put $|\sigma(8u+
7+k) \cap \tau(i1)| = x$. Then $4u+2 = |\sigma(2i-2) \cap \sigma(8u+7+k)| = |\sigma(i) \\
\cap \mu(k)| + x = |\sigma(2i-1) \cap \sigma(8u+7+k)| = |\sigma(i)^* \cap \mu(k)| + x$. Every
$\mu(k)$ contains the point $8u+5$. So $|\sigma(i) \cap \mu(k)| + |\sigma(i)^* \cap \mu(k)| = 
4u+1$. Hence we have a contradiction that $4u+3 = 2x$. Thus we have
that $|\sigma(8u+7+k) \cap \tau(i1)| - |\sigma(8u+7+k) \cap \tau(i2)| = 1$, and that
$|\sigma(i) \cap \mu(k)| = 2u$ or $2u+1$. Let $E$ and $F$ be the numbers of the $\mu(k)$
such that $|\sigma(i) \cap \mu(k)| = 2u$ and $2u+1$ respectively. Since every point
of $\sigma(i)$ belongs to $4u+1$ of the $\mu(k)$, we have that $(4u+2)(4u+1) = 
2u E + (2u+1)F$. Then $E$ is a multiple of $2u+1$ and this fact implies that
$E = F = 4u+2$ proving (1).
We notice that \(|a(8u+7+k) \cap \sigma(i)| = 2u+1\) if and only if \(8u+2i+2 \in a(8u+7+k)\). Let \(2 \leq i \neq j \leq 4u+4\). Then, since \(D(r)\) is a symmetric \(2-(4u+3, 2u+2, u+1)\) design, there exist \(2(u+1)-1 = 2u+1\) of the \(\tau(\ell 1)\) and \(\tau(\ell 2)\) containing the points \(8u+2i+2\) and \(8u+2j+2\). So \(4u+2-(2u+1) = 2u+1\) of the \(a(8u+7+k)\) contain the points \(8u+2i+2\) and \(8u+2j+2\), proving (2).

Let \(a \in \sigma(i)\), for \(1 \leq a \leq 8u+4\). Now there exist exactly \(2(2u+2)-1 = 4u+3\) of the \(\tau(jk)\) containing the point \(8u+2i+2\). So there exist exactly \((2u+1)+1 = 2u+2\) of the \(a(\ell)\) with \(\ell \leq 8u+7\) containing both \(a\) and \(8u+2i+2\). Hence there exist exactly \(4u+2-(2u+2) = 2u\) of the \(a(\ell)\) with \(\ell \geq 8u+8\) containing both \(a\) and \(8u+2i+2\). These are the blocks \(a(\ell)\) with \(\ell \geq 8u+8\) intersecting with \(\sigma(i)\) in \(2u\) points. The rest is similar. This proves (3).

 Sufficiency. Suppose that we have a \(2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)\) design \(D(\ell)\) satisfying (1), (2) and (3).

Clearly \(\mu(k)\) contains the point \(8u+k\), for \(1 \leq k \leq 8u+4\). Since \(\sigma(i) \cup \sigma(i)^* = a(1) \cup 8u+5\), we have that \(|\sigma(i) \cap \mu(k)| = 2u+1\) or \(2u\) according as \(|\sigma(i)^* \cap \mu(k)| = 2u\) or \(2u+1\) respectively, for \(2 \leq i \leq 4u+4\) and \(1 \leq k \leq 8u+4\).

We form a configuration consisting of \(8u+4\) blocks \(\nu(1), ..., \nu(8u+4)\) of size \(4u+3\) based on the set of points \(8u+6, 8u+7, ..., 16u+11\). \(\nu(k)\) contains the point \(8u+2+2i\) or \(8u+3+2i\) according as \(|\sigma(i) \cap \mu(k)| = 2u\) or \(|\sigma(i)^* \cap \mu(k)| = 2u\) respectively, for \(2 \leq i \leq 4u+4\) and \(1 \leq k \leq 8u+4\). Since \(\nu(k)\) contains exactly one point of \(8u+2+2i, 8u+3+2i\) for each \(i\), such that \(2 \leq i \leq 4u+4\), the size of \(\nu(k)\) equals \(4u+3\).

We put \(a(8u+7+j) = \mu(j) \cup \nu(j)\), for \(1 \leq j \leq 8u+4\), and let \(B = \{a(1), a(2), ..., a(16u+11)\}\). Then we show that \(D = (P, B)\) is a symmetric \(2-(16u+11, 8u+5, 4u+2)\) design.

First we show that \(D\) is a 1-design. Let \(a\) be a point. If \(1 \leq a \leq 8u+5\), then, since \(D(\ell)\) has replication number \(8u+4\) and since \(a\) belongs to \(a(1)\), \(a\) belongs to \((8u+4)+1 = 8u+5\) blocks of \(B\). So let \(8u+6 \leq a \leq 16u+11\). Now every point of \(D(r)\) belongs to \(2u+2\) blocks. One of these blocks say \(\tau(i)\), contains \(a\) or \(a-1\) as \(\tau(i)T\). So there exists \(2(2u+1)+1 = 4u+3\) blocks \(a(i)\) with \(i \leq 8u+7\) containing \(a\). Now by assumption (1) on \(D(\ell)\) there exist exactly \(4u+2\) of the \(\mu(k)\) such that \(|\sigma(i) \cap \mu(k)| = 2u\) or \(|\sigma(i)^* \cap \mu(k)| = 2u\), according as \(a\) is even or odd respectively. So there exist \(4u+2\) blocks \(a(i)\) with \(i \leq 8u+8\) containing \(a\).

Next we show that \(D\) is a 2-design. Let \(a\) and \(b\) be two distinct points.
If $1 \leq a, b \leq 8u+5$, then since $D(\ell)$ is a $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design and since both $a$ and $b$ belong to $\sigma(1)$, $a$ and $b$ belong to $(4u+1)+1 = 4u+2$ blocks of $B$. Let $8u+6 \leq a, b \leq 16u+11$. If $|a, b| = |8u+6, 8u+7|, |8u+8, 8u+9|,..., or |16u+10, 16u+11|$, then we may assume that $a$ is even. Only blocks $\sigma(i)$ with $2 \leq i \leq 8u+7$ may contain $|a, b|$. Since the replication number of $D(r)$ is $2u+2$, and since $a$ appears in exactly one of the $\tau(i)$ as $\tau(i)T$, $|a, b|$ is contained in $2(2u+2-1) = 4u+2$ blocks of $B$. If $|a, b| \neq |8u+6+2i, 8u+7+2i|, 0 \leq i \leq 4u+2$, then it suffices to consider the case where $a$ and $b$ are even. Then $|a, b|$ is contained in exactly $u+1$ blocks of $D(r)$. By the skew property of $T$ exactly one of these blocks of $D(r)$, say $\tau(j)$, contains $a$ or $b$ as $\tau(j)T$. So exactly $1+2(u+1-1) = 2u+1$ blocks $\sigma(i)$ with $i \leq 8u+7$ contain $|a, b|$. By assumption (2) on $D(\ell)$ and by the definition of $\nu(k)$, exactly $2u+1$ of the $\nu(k)$ contain $|a, b|$. So exactly $2u+1$ blocks $\sigma(i)$ with $i \geq 8u+8$ contain $|a, b|$. Finally let $1 \leq a \leq 8u+5$ and $8u+5 and 8u+6 \leq b \leq 16u+11$. If $a = 8u+5$, then $a$ belongs to all of the $\nu(k)\cdot 1 \leq k \leq 8u+4$. By assumption (3) on $D(\ell)$, $b$ belongs to exactly $4u+2$ of the $\nu(k)\cdot$ So $|a, b|$ is contained in exactly $4u+2$ blocks of $B$. Thus we may assume that $1 \leq a \leq 8u+4$. Again we may assume that $b$ is even. Now $b$ belongs to exactly $2u+2$ blocks of $D(r)$ and only one of these blocks, say $\tau(k)$, contain $b$ as $\tau(k)T$. Therefore $2u+1$ pairs of blocks $\tau(ij)$ contain $b$, and $\tau(k1)$, not $\tau(k2)$, contains $b$. So if $a$ belongs to $\sigma(k)$, then exactly $2u+1$ blocks $\sigma(i)$ with $i \leq 8u+7$ contain $|a, b|$. But if $a$ belongs to $\sigma(k)\star$, then exactly $2u$ blocks $\sigma(i)$ with $i \leq 8u+7$ contain $|a, b|$. Then by assumption (3) on $D(\ell)$ exactly $2u$ or $2u+1$ blocks $\sigma(i)$ with $i \geq 8u+8$ contain $|a, b|$ according as $a$ belongs to $\sigma(1)$ or $\sigma(1)\star$. This completes the proof.

**Definition.** We call a symmetric $2-(16u+11, 8u+5, 4u+2)$ design $D$ thus constructed a Hadamard design of Bush type. Furthermore we call a Hadamard matrix of order $16u+12$ associated with $D$ a Hadamard matrix of Bush type.

**Remark 1.** The main point of proposition 1 is the fact that the construction of a Hadamard matrix of Bush type of order $16u+12$ is reduced to the construction of a $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design satisfying (1), (2) and (3) for which $8u+6$ blocks are predetermined.

**Remark 2.** There exists some freedom to construct Hadamard matri-
ces of Bush type of order $16u+12$: (i) The choice of a Hadamard matrix $H$ of order $8u+4$; (ii) The choice of $4u+4$ rows from $H$; (iii) The choice of a skew Hadamard matrix of order $4u+4$; (iv) The choice of the mapping $T$; (v) The choice of $2-(8u+5, 4u+2, 4u+1, 16u+10, 8u+4)$ design and (vi) The choice of the matching between $\mathcal{L}$ and $\mathcal{M}$.

Remark 3. For $u = 0$ it is very easy to write down a design of Bush type: $a(1) = |1, 2, 3, 4, 5|$, $a(2) = |1, 2, 6, 10, 11|$, $a(3) = |3, 4, 7, 10, 11|$, $a(4) = |1, 3, 6, 7, 8|$, $a(5) = |2, 4, 6, 7, 9|$, $a(6) = |1, 4, 8, 9, 10|$, $a(7) = |2, 3, 8, 9, 11|$, $a(8) = |1, 5, 7, 9, 11|$, $a(9) = |2, 5, 7, 8, 10|$, $a(10) = |3, 5, 6, 9, 10|$, and $a(11) = |4, 5, 6, 8, 11|$. For $u = 1$ there are more than ten inequivalent Hadamard matrices of Bush type.

2. The purpose of this section is to prove the following proposition.

Proposition 2. The transpose of a Hadamard matrix of Bush type is of Bush type. More precisely, the dual of a Hadamard design of Bush type is of Bush type.

Proof. We use the notation in the proof of Proposition 1, and consider the dual $D^d$ of the Hadamard design of Bush type in § 1, $D = (P, B)$. It will suffice to recognize in $D^d$ a configuration similar to $\mathcal{B} = (P, |a(i)|, 1 \leq i \leq 8u+7)$.

Let $\beta(i)$ be the set of blocks of $B$ containing the point $i$ of $P$, $1 \leq i \leq 16u+11$. Let $P^d$ and $B^d$ denote the sets of points and blocks of $D^d$ respectively. Then $P^d = |a(i)|$, $1 \leq i \leq 16u+11|$ and $B^d = |\beta(i)|$, $1 \leq i \leq 16u+11|$.

Now the point $a(1)$, the set of points $a(i)$ with $8u+8 \leq i \leq 16u+11$ and the block $\beta(8u+5) = |a(1)|$, $a(i)$ with $8u+8 \leq i \leq 16u+11|$ play the roles of the point $8u+5$, $P(\emptyset)$ and the block $a(1)$ in $D$, respectively.

Furthermore, $\beta(8u+5) \cap \beta(8u+2i)$ and $\beta(8u+5) \cap \beta(8u+2i+1)$, where $3 \leq i \leq 4u+5$, correspond to $\sigma(i) = \sigma(1) \cap \alpha(2i-2)$ and $\sigma(i) = a(1) \cap \alpha(2i-1)$, where $2 \leq i \leq 4u+4$, respectively. Lastly $(P(r))^d = |a(2j)|$, $1 \leq j \leq 4u+3|$, $(P(r))^d \cap \beta(8u+2i)$ and $T^d$ defined by $((P(r))^d \cap \beta(8u+2i))T^d = \sigma(2i-4)$, where $3 \leq i \leq 4u+5$, correspond to $P(r)$: $\tau(i)$ and $T$ respectively, where $2 \leq i \leq 4u+4$.

The rest may be checked without difficulty.
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