On Certain periodic rings

Isao Mogami*

*Okayama University

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ON CERTAIN PERIODIC RINGS

Dedicated to Professor Katsumi Numakura on his 60th birthday

ISAO MOGAMI

Let $R$ be a ring (not necessarily with 1), $N$ the set of nilpotent elements in $R$, and $N^*$ the subset of $N$ consisting of all $a$ with $a^2 = 0$. Let $D$ be the set of right and left zero-divisors in $R$. Given a positive integer $n > 1$, we set $E_n = \{ x \in R \mid x^n = x \}$; in particular, $E = E_2$. As is well known, if $R$ is periodic then every element $x$ in $R$ can be written in the form $x = e + a$, where $a \in N$ and $e \in E_n$ for some $n$.

In this paper, we prove the following theorem, which includes all the results in [1].

**Theorem.** Let $R$ be a periodic ring with $N^*$ commutative.

1. Then $N$ coincides with the Jacobson radical of $R$ and $\overline{R} = R/N$ is a subdirect sum of fields.
2. Let $n > 1$ be a fixed positive integer. If every element $d$ in $D$ can be written in the form $d = e + a$, where $a \in N$ and $e \in E_n$, then $\overline{R}$ is either a field or $\overline{E_n}$.

**Proof.** (1) Let $J$ be the Jacobson radical of the periodic ring $R$, which is obviously a nil ideal. First, we claim that every idempotent $\overline{e}$ of $\overline{R} = R/J$ is central. Since $J$ is a nil ideal, we may assume from the beginning that $e$ is an idempotent of $R$. By hypothesis, $eR(1-e) \cdot (1-e)Re = (1-e)Re \cdot eR(1-e)$, and so $eR(1-e)Re = 0$ (where 1 is used formally). Hence, by the semiprimeness of $\overline{R}$, we get $\overline{eR}(1-\overline{e}) = 0$, and therefore $\overline{ex} = \overline{exe}$ for all $x \in R$. Furthermore, $\overline{eR}(1-\overline{e})\overline{R} = 0$ yields $(1-\overline{e})\overline{Re} = 0$, and so $\overline{ex} = \overline{exe}$ for all $x \in R$. Thus we have seen that $\overline{e}$ is central. Now, it is easy to see that every nilpotent element of $\overline{R}$ generates a nil right ideal. Hence $N$ coincides with $J$. As is easily seen, for any element $x \in R$ there exists a non-negative integer $k$ such that $x - x^{k+2} \in N$. Hence $\overline{R}$ is a subdirect sum of fields, by Jacobson’s commutativity theorem.

(2) Let $x$ be an arbitrary element of $R$. If $x \in D$ then $x = x^{k+1}$ with some non-negative integer $k$. Then $e = x^{k+1}$ is a right or left unity of $R$, and therefore $\overline{e}$ is the unity of the commutative ring $\overline{R}$ and $\overline{x}$ is a unit (see (1)). On the other hand, if $x \in D$ then $\overline{x^n} = \overline{x}$, by hypothesis.
In case $R = D$, it is clear that $\bar{R} = \bar{E}_n$. In what follows, we consider the case that $R \neq D$. Then, by the above claim, $\bar{R}$ has the unity $\bar{e}$. It suffices therefore to show that if $\bar{R}$ contains a non-unit $\bar{x} \neq 0$ then $\bar{R} = \bar{E}_n$. Actually, by the above claim, $\bar{f} = \bar{x}^{n-1}$ is an idempotent with $\bar{x}\bar{f} = 0$ and $\bar{R} = \bar{fR} \oplus (\bar{e} - \bar{f})\bar{R}$, so that $\bar{R} = \bar{E}_n$.

Combining the proof of Theorem (2) with [3, Proposition 2], we can easily see the following.

**Corollary.** Let $R$ be a periodic ring with $N^*$ commutative. If $D$ is included in the subring $\langle E \cup N \rangle$ generated by $E \cup N$, then $\bar{R}$ is either a field or a subdirect sum of finite prime fields.

**Remark.** Following [2], a ring $R$ is called an $I$-ring (resp. $I'$-ring) if every element of $R$ can be written as a product of elements in $E$ (resp. $E \cup N$). Now, let $R$ be a (not necessarily periodic) $I'$-ring with $N^*$ commutative. Then, the argument employed in the proof of Theorem (1) enables us to see that every idempotent in the factor ring of $R$ modulo its prime radical $P$ is central. If $E \neq 0$, then $R/P$ is an $I$-ring by [2, Lemma 1]. Hence $R/P$ is a Boolean ring, and $N$ coincides with $P$. Also, if $N$ is multiplicatively closed (especially, if $N$ is commutative) then $N$ forms an ideal of $R$.

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**References**


**Okayama University and Tsuyama College of Technology**

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