On separable polynomials of degree 2 in skew polynomial rings

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ON SEPARABLE POLYNOMIALS OF DEGREE 2
IN SKEW POLYNOMIAL RINGS

Dedicated to Professor Mikao Moriya on his 70th birthday

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In [1], K. Kitamura studied free quadratic extensions of commutative rings and its isomorphism classes. In [4], K. Kishimoto studied some quadratic extensions of non-commutative rings and its isomorphism classes which are some partial generalization of [1]. In these studies, the set of polynomials of degree 2 plays an important rôle. Indeed, any quadratic extension (considered in [1] and [4]) is determined by some polynomial ring and some polynomial of degree 2. The purpose of this note is to study (separable) polynomials of degree 2 in a skew polynomial ring and is to present some generalization and sharpening of the result of [4].

Throughout this paper, $B$ will mean a (non-commutative) ring with identity element 1, and all ring extensions of $B$ will be assumed to have the the (common) identity element 1. Now, let $\rho$ be an automorphism of $B$, and $D$ a derivation of $B$, so that $D(x+y)=D(x)+D(y)$ and $D(xy)=D(x)y+xD(y)$ ($x, y \in B$). As in [2]-[4], by $B[X; \rho]$ (resp. $B[X; D]$), we denote the ring of all polynomials $\sum b_i X^i$ ($b_i \in B$) with an indeterminate $X$ whose multiplication is defined $bX = \rho(b)$ (resp. $bX = Xb + D(b)$) for each $b \in B$. For a while, let $B[X; *]$ be one of $B[X; \rho]$ and $B[X; D]$. By $B[X; *]_{\alpha}$, we denote the subset of $B[X; *]$ of all polynomials $f = x^2 - xa - b$ with $fB[X; *] = B[X; *]f$ and $xa = aX$. A polynomial $f = x^2 - xa - b \in B[X; *]_{\alpha}$ is called to be separable (resp. Galois) if the factor ring $B[X; *]/fB[X; *]$ is a separable (resp. Galois) extension of $B$ in the sense of [5]. For $f = x^2 - xa - b \in B[X; *]_{\alpha}$, we denote $a^2 + 4b$ by $\delta(f)$, which will be called the discriminant of $f$. Moreover, for $f, g \in B[X; *]_{\alpha}$, if the factor rings $B[X; *]/fB[X; *]$ and $B[X; *]/gB[X; *]$ are $B$-ring isomorphic then we write $f \sim g$. Clearly the relation $\sim$ is an equivalence relation in $B[X; *]_{\alpha}$. By $B[X; *]_{\bar{\alpha}}$, we denote the set of equivalence classes of $B[X; *]_{\alpha}$ with respect to the relation $\sim$. As is easily seen, quadratic extensions considered in [1] and [4] can be written as factor rings $B[X; *]/fB[X; *]$ for some $B, *$, and $f \in B[X; *]_{\alpha}$. Hence $B[X; *]_{\bar{\alpha}}$ can be regarded as the set of (B-ring) isomorphism classes of free quadratic extensions with multiplication defined by *. The details will be given in §2 and §3.
In §1, we shall make some remarks on free quadratic (Galois) extensions, which contains preliminary Lemmas.

In §2, we consider $B[X; \rho]_{\infty}$ and prove that for $f \in B[X; \rho]_{\infty}$, $f$ is Galois if and only if $\delta(f)$ is invertible in $B$. Moreover, we shall study the set $B[X; \rho]_{\infty}$ and shall prove that if there exists a Galois polynomial in $B[X; \rho]_{\infty}$ then $B[X; \rho]_{\infty}$ forms an abelian semigroup (under some composition) such that for $g \in B[X; \rho]_{\infty}$, $g$ is Galois if and only if the element $C$ of $B[X; \rho]_{\infty}$ with $C \equiv g$ is invertible in the semigroup. This study contains the results of Kishimoto [4, §2].

In §3, we consider $B[X; D]_{\infty}$ and study the separable (Galois) polynomials of $B[X; D]_{\infty}$ and the structure of $B[X; D]_{\infty}$. In the study, the discriminants of polynomials in $B[X; D]_{\infty}$ play important roles. Indeed, if there exists a separable polynomial in $B[X; D]_{\infty}$ whose discriminant is invertible in $B$ then for $g \in B[X; D]_{\infty}$, $g$ is separable if and only if $\delta(g)$ is invertible in $B$. Moreover, we shall present some conditions that polynomials in $B[X; D]_{\infty}$ are separable (Galois). Further, we shall study the sets $B[X; D]_{\infty}$ and $B[X; D]_{\infty} = \{ C \in B[X; D]_{\infty} : C \equiv X^2 - xa - b (a \text{ is fixed}) \}$ which form abelian groups under some conditions. The study also contains some generalizations of the results of Kishimoto [4, Th. 3.5 and its corollary].

1. Preliminary lemmas. For a ring extension $A/B$ and a set $\mathfrak{G}$ of automorphisms in $A$, we shall use the following conventions: $J(\mathfrak{G}, A) = \{ a \in A ; \sigma(a) = a \text{ for all } \sigma \in \mathfrak{G} \}$, and $\mathfrak{G}(B, \mathfrak{G}) = \{ \sigma \in \mathfrak{G} ; \sigma(b) = b \text{ for all } b \in B \}$. Moreover, a ring extension $A/B$ is called $\mathfrak{G}$-Galois if $\mathfrak{G}$ is a group of $B$-ring automorphisms in $A$, $J(\mathfrak{G}, A) = B$, and there are elements $a_1, \ldots, a_n, a_1^*, \ldots, a_n^*$ such that $\sum a_i \sigma(a_i^*) = \delta_{i, \sigma}$ (Kronecker’s delta) for all $\sigma$ in $\mathfrak{G}$ (cf. [5]).

First, we shall prove the following

Lemma 1.1. Let $A$ be a ring extension of $B$ with $A = xB + B$, and assume that there is a $B$-ring automorphism $\sigma$ in $A$ such that $x - \sigma(x)$ is invertible in $A$. Then $\{ x, 1 \}$ is a right free $B$-basis of $A$. and $J(\sigma, A) = B$.

Proof. Let $0 = xb_1 + b_0$ where $b_1, b_0 \in J(\sigma, A)$. Then $0 = (xb_1 + b_0) - \sigma(xb_1 + b_0) = (x - \sigma(x))b_1$. Since $x - \sigma(x)$ is inversible, we have $b_1 = 0$, and so, $b_0 = 0$. Thus $\{ x, 1 \}$ is right $J(\sigma, A)$-free. Since $B \subset J(\sigma, A)$ and $A = xB + B$, it follows that $\{ x, 1 \}$ is a right free $J(\sigma, A)$-basis of $A$, and $B = J(\sigma, A)$. 

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Lemma 1.2. Let $A$ be a $\mathfrak{G}$-Galois extension of $B$ such that $A = \mathfrak{G}$-Galois extension of $B$. Then $\mathfrak{G}$ is of order 2, and for $\sigma \neq 1$ in $\mathfrak{G}$, $x - \sigma(x)$ is invertible in $A$. Moreover, $\{x, 1\}$ is a right free $B$-basis of $\mathfrak{G}$.

Proof. Since $A$ is $\mathfrak{G}$-Galois over $B$, there exist elements $u_1, \ldots, u_n, v_1, \ldots, v_n$ in $A$ such that

$$\sum_i u_i \tau(v_i) = \delta_{i, 1} \text{ for all } \tau \in \mathfrak{G}.$$ 

Hence $\sum_i \tau^{-1}(u_i)v_i = \delta_{i, 1}$ for all $\tau \in \mathfrak{G}$, that is,

$$\sum_i \tau(u_i)v_i = \delta_{i, 1} \text{ for all } \tau \in \mathfrak{G}.$$ 

Now, we set $v_i = b_i x + c_i$, where $b_i, c_i \in B$, $i = 1, \ldots, n$. Then for $\tau \in \mathfrak{G}$, we have

$$\sum_i u_i \tau(v_i) = \sum_i u_i \tau(b_i x + c_i) = (\sum_i u_i b_i) \tau(x) + \sum_i u_i c_i.$$ 

Let $\tau \neq 1$ be an arbitrary element of $\mathfrak{G}$. Then

$$1 = \sum_i u_i v_i - \sum_i u_i \tau(v_i)$$

$$= \left( (\sum_i u_i b_i)x + \sum_i u_i c_i \right) - \left( (\sum_i u_i b_i) \tau(x) + \sum_i u_i c_i \right)$$

$$= \left( \sum_i u_i b_i \right) (x - \tau(x)).$$

This implies that $x - \tau(x)$ has a left inverse. By a similar method, we see that $x - \tau(x)$ has a right inverse. Hence $x - \tau(x)$ is invertible in $A$. Next, let $\tau_1, \tau_2 \in \mathfrak{G}$ and $\tau_i \neq 1$ ($i = 1, 2$). Then $x - \tau_i(x) = (\sum_i u_i b_i)^{-1} = x - \tau_2(x)$, and whence $\tau_1(x) = x - (\sum_i u_i b_i)^{-1} = \tau_2(x)$. Since $A = \mathfrak{G}$-Galois extension of $B$, and the $\tau_i$ are $B$-ring automorphisms of $A$, it follows that $\tau_1 = \tau_2$. Hence the order of $\mathfrak{G}$ is not greater than 2. Noting $A \neq B$, we see that $\mathfrak{G}$ is of order 2. The other assertions are direct consequences of Lemma 1.1.

Lemma 1.3. Let $A$ be a ring extension of $B$ such that $A = \mathfrak{G}$-Galois extension of $B$ and $x \mathfrak{G} \cap B = \{0\}$. Let $S$ be a subset of $B$ such that $BSB = \{0\}$ and $xs = sx$ for all $s \in S$. Then $ASA = \mathfrak{G}$-Galois extension of $B$. The factor ring $A/ASA$ is a ring extension of $B/(A + ASA)/ASA \cong B/BSB$. If $A$ is separable over $B$ then $A/ASA$ is separable over $B$.

Proof. We set $x^2 = xa + b$. Then, for each $s \in S$, we have $xBSB + BsB \subset AsA = (xB + B)s(xB + B) = (xB + B)(sxB + BsB)$, where $xB + B \subset xB + B$, $BsB \subset x(x + b)BsB$, and hence $AsA = xBSB + BsB$. This implies that $ASA = xBSB + BsB$. Since $x \mathfrak{G} \cap B = \{0\}$, $ASA \cap B = BsB$. Thus the factor ring $A/ASA$ is a ring extension of $B$. Now, set
\[ \tilde{A} = A/\text{ASA}, \quad \tilde{u} = u + \text{ASA} \quad \text{for each} \quad u \in A, \quad \text{and assume that} \quad A \quad \text{is separable over} \quad B. \quad \text{Then, there exists a (left)B-(right)B-homomorphism} \quad \phi : A \otimes_B A \to A \quad \text{such that} \quad \phi(a_1 \otimes a_2) = a_1a_2 \quad (a_1, a_2 \in A) \quad \text{and} \quad \phi \quad \text{splits.} \quad \text{Then, the} \quad \phi \quad \text{induces a (left)B-(right)B-homomorphism} \quad \tilde{\phi} : \tilde{A} \otimes_B \tilde{A} \to \tilde{A} \quad \text{with} \quad \tilde{\phi}(\tilde{a}_1 \otimes \tilde{a}_2) = \tilde{a}_1\tilde{a}_2 \quad (\tilde{a}_1, \tilde{a}_2 \in \tilde{A}). \quad \text{Clearly, the} \quad \tilde{\phi} \quad \text{splits. Hence} \quad \tilde{A} \quad \text{is separable over} \quad B. \]

**Lemma 1.4.** Let \( A \) be a \( \mathfrak{G} \)-Galois extension of \( B \) such that \( A = xB + B \neq B \) and \( \mathfrak{G} = \{1, \sigma\} \). Let \( S \) be a subset of \( B \) such that \( \text{BSB} \neq B \) and \( sx = xs \) for all \( s \in S \). Then, \( ASA = xBSB + BSB = BSBx + BSB, \quad ASA \cap B = BSB, \quad \text{and the factor ring} \quad A/ASA \quad \text{is a Galois extension of} \quad (B + ASA)/ASA = B/BSB \quad \text{with Galios group} \quad \mathfrak{G} = \{1, \sigma\} \quad \text{where} \quad 1 \quad \text{and} \quad \sigma \quad \text{are automorphisms of} \quad A/ASA \quad \text{induced by} \quad 1 \quad \text{and} \quad \sigma. \)

**Proof.** By Lemma 1.2, \( \{x, 1\} \) is a right free \( B \)-basis of \( A \) and is also a left free \( B \)-basis of \( A \). Hence by Lemma 1.3, we have \( ASA = xBSB + BSB = BSBx + BSB \) and \( ASA \cap B = BSB \). Now, we set \( \tilde{A} = A/ASA, \quad \tilde{B} = (B + ASA)/ASA, \quad B_0 = J(\mathfrak{G}, \tilde{A}) \) and \( \tilde{a} = a + ASA \) for each \( a \in A \). By Lemma 1.2, \( x - \sigma(x) \) is invertible in \( A_{\tilde{B}} \) and so is \( x - \sigma(\tilde{x}) \) in \( \tilde{A} \). Noting \( \tilde{B} \subset \tilde{B}_0 \), we have \( \tilde{A} = \tilde{xB}_0 + \tilde{B}_0 = \tilde{xB} + \tilde{B} \). Hence, by Lemma 1.1, we obtain \( \tilde{B} = \tilde{B}_0 \). Since \( A \) is Galois over \( B \), there exist elements \( u_1, \ldots, u_n, v_1, \ldots, v_n \) in \( A \) such that \( \sum_i u_i \sigma(v_i) = \delta_{i, \tau} \) for all \( \tau \in \mathfrak{G} \). This implies \( \sum_i \tilde{u}_i \tilde{\sigma}(\tilde{v}_i) = \delta_{i, \tau} \) for all \( \tau \in \mathfrak{G} \). Therefore, we conclude that \( \tilde{A} \) is a Galois extension of \( \tilde{B} \).

Next, we shall prove the following

**Lemma 1.5.** Let \( A \) be a ring extension of \( B \) such that \( A = xB + B = Bx + B \) and \( \{x, 1\} \) is a right free \( B \)-basis. Let \( x^2 = xa + b = ax + b \) and \( ab = ba \). Moreover, write \( \beta x = x\beta^* + \beta^* \) for each \( \beta \in B \) and assume \( \beta^* = \beta^* \) for all \( \beta \in B \). If \( \beta^* + 4b \) is invertible in \( B \) then \( A \) is Galois over \( B \).

**Proof.** For any \( \beta \in B \), we have \( \beta(xa + b) = (x\beta^* + \beta^*)a + \beta b = x\beta^*a + (\beta^*a + \beta b) \), and \( \beta x^2 = (x\beta^* + \beta^*)x = x\beta^*x + \beta^*x = x(x\beta^* + \beta^*) + x\beta^* + \beta'' = x\beta^* + x(\beta^* \beta^*) + \beta'' = (xa + b)\beta^* + 2x\beta^* + \beta'' = x(\beta^* + 2\beta^*) + \beta'' \). This implies \( \beta^*a = a\beta^* + 2\beta^* \). Since \( \{\beta^* + 4b \in B \} = B \), it follows that \( \beta^*a = a\beta^* + 2\beta^* \) for all \( \beta \in B \). Hence we obtain that for any \( \beta \in B \),

\[
\beta(a - x) = \beta a - \beta x = \beta a - x\beta^* - \beta^* = -x\beta^* + (a\beta^* + 2\beta^*) - \beta^* = (a - x)\beta^* + \beta^*.
\]

Moreover, we have
\[(a-x)^2 = a^2 - 2ax + x^2 = a^2 - 2ax + xa + b = (a-x)a + b.\]

Hence the mapping \(\sigma : xb_1 + b_2 \rightarrow (a-x)b_1 + b_2 (b_1, b_2 \in B)\) is a \(B\)-ring automorphism of \(A\). Now, we assume that \(a^2 + 4b\) is invertible in \(B\). Then, since \((a-2x)^2 = a^2 - 4ax + 4x^2 = a^2 + 4b\), the difference \(a-2x = (a-x) - x\) is invertible in \(A\). Hence \(\sigma \neq 1\) and \(\sigma^2 = 1\). If \(xb_1 + b_2 \in \mathcal{J}(\sigma, A) (b_1, b_2 \in B)\) then \(xb_1 + b_2 = \sigma(x)b_1 + b_2 = (a-x)b_1 + b_2\), and so, \((a-2x)b_1 = 0\) which implies that \(b_1 = 0\). Now, we set \(u_1 = (a-2x)^{-1}(a-x), \ u_2 = (a-2x)^{-1}, v_1 = 1, \) and \(v_2 = -x\). Then \(\sum \tau \omega(v_\tau) = \delta_{1, \tau}\) for all \(\tau \in \{1, \sigma\}\). Hence, it follows that \(A\) is a Galois extension of \(B\) with Galois group \(\{1, \sigma\}\), completing the proof.

In the rest of this paper, \(Z\) will mean the center of \(B\). Moreover, \(U(B)\) denotes the set of all invertible elements in \(B\), and for any subset \(S\) of \(B\), \(U(S)\) denotes the intersection of \(S\) and \(U(B)\). Clearly \(U(Z)\) coincides with the set of all invertible elements in \(Z\). Further, for any \(b \in B, b_t\) (resp. \(b_r\)) denotes the left (resp. right) multiplication of \(B\) determined by \(b\).

2. On \(B[X; \rho]_{(\alpha)}\). In this section, we study \(B[X; \rho]_{(\alpha)}\), the subset of \(B[X; \rho]\). As in [4], we shall use the following conventions: \(B_1 = f(\rho, B); Z_1 = Z \cap B_1; B(\rho^\alpha) = \{u \in B; \alpha u = u\alpha(\alpha)\}\) for all \(\alpha \in B\) (where \(n\) is any integer); \(B_1(\rho^\alpha) = B(\rho^\alpha) \cap B_1\).

Now, let \(f = X^2 - Xa - b \in B[X; \rho]_{(\alpha)}\). Then, the factor ring \(A = B\{X; \rho\}/fB[X; \rho]\) is a quadratic extension of \(B\) such that for \(x = x + fB[X; \rho]\),

\[
A = B + XB, \ \alpha x = x\alpha(\alpha) \quad \text{for all} \ \alpha \in B,
\]

\[
x^2 = xa + b = ax + b, \ \text{and}
\]

\[
\{1, x\} \text{ is a right free } B\text{-basis.}
\]

Then we have \(xa = ax, x^3 = x(xa + b) = (xa + b)x,\) and \(\alpha x^2 = x^2\rho^2(\alpha)\) for all \(\alpha \in B\). From these equalities, it follows that \((2, 0)\) \(a \in B_\rho(\rho)\) and \(b \in B_\rho(\rho^2)\).

Clearly \(ab = ba, xa = ax,\) and \(xb = bx\). Conversely, if a system \(\{\rho, a, b\} (a, b \in B)\) satisfies the condition \((2, 0)\) then

\[
X^2 - Xa - b \in B[X; \rho]_{(\alpha)}.
\]

Hence, it follows that

\[
(2, 1) \ B[X; \rho]_{(\alpha)} = \{X^2 - Xa - b; a \in B_\rho(\rho) \text{ and } b \in B_\rho(\rho^2)\}.
\]

For any \(f = X^2 - Xa - b \in B[X; \rho]_{(\alpha)},\) we denote the factor ring \(B[X; \rho]_{(\alpha)}/fB[X; \rho]\) by \(B[x; \rho, a, b]\) (or, by \(B[x_\alpha]\)) where \(x = x, X = +\)
$fB[X; \rho]$. First, we shall prove the following

**Lemma 2.1.** Let $f = X^2 - Xa - b \in B[X; \rho]_{\mathbb{C}}$. Then $f$ is separable over $B$ if and only if there exist elements $b_1, b_2, b_3$ and $b_4$ in $B$ such that

$(2, \text{ii}) \quad 1 = bb_1 + b_4$

$(2, \text{iii}) \quad ab_1 + b_2 + b_3 = 0$

$(2, \text{iv}) \quad bb_1 = ab_2 + \rho(b_4)$

$(2, \text{v}) \quad \rho(b_2) = b_3$

$(2, \text{vi}) \quad b_4 \in Z$

$(2, \text{vii}) \quad b_1 \in B(\rho^{-1})$

$(2, \text{viii}) \quad b_2 \in B(\rho^{-1})$

**Proof.** We set $A = B[x; \rho, a, b]$ and assume $f$ is separable over $B$. Then, the (left)$B$-(right)$B$-homomorphism

$\phi: A \otimes_B A \rightarrow A \quad (\sum a_i \otimes b_i \rightarrow \sum a_i b_i)$

splits. Hence there exists an element $e$ in $A \otimes_B A$ such that $\phi(e) = 1$ and $(c \otimes 1)e = e(1 \otimes c)$ for all $c \in A$. Since $A \otimes_B A = (x \otimes x)B + (x \otimes 1)B + (1 \otimes x)B + (1 \otimes 1)B$, we may write

$e = (x \otimes x)b_1 + (x \otimes 1)b_2 + (1 \otimes x)b_3 + (1 \otimes 1)b_4$

where $b_i \in B$, $i = 1, \ldots, 4$. The equality $\phi(e) = 1$ implies

$x(ab_1 + b_2 + b_3) + bb_1 + b_4 = 1$.

Moreover, we have

$(x \otimes 1)e = (x \otimes x)(ab_1 + b_3) + (x \otimes 1)(ab_2 + b_4) + (1 \otimes x)bb_1 + (1 \otimes 1)bb_2$

$e(1 \otimes x) = (x \otimes x)(a \rho(b_1) + \rho(b_2)) + (x \otimes 1)b \rho(b_1) + (1 \otimes x)(a \rho(b_3) + \rho(b_4)) + (1 \otimes 1)b \rho(b_3)$, and for each $\alpha \in B$,

$(\alpha \otimes 1)e = (x \otimes x)\rho^2(\alpha)b_1 + (x \otimes 1)\rho(\alpha)b_2 + (1 \otimes x)\rho(\alpha)b_3 + (1 \otimes 1)b_4$.

Hence we obtain

(a) $ab_1 + b_2 + b_3 = 0$

(b) $bb_1 + b_4 = 1$

(c) $ab_1 + b_3 = a \rho(b_1) + \rho(b_2)$

(d) $ab_2 + b_4 = b \rho(b_1)$

(e) $bb_1 = a \rho(b_3) + \rho(b_4)$

(f) $bb_2 = b \rho(b_3)$

(g) $\rho^2(\alpha)b_1 = b_1 \alpha$

(h) $\rho(\alpha)b_2 = b_2 \alpha$

(i) $\rho(\alpha)b_3 = b_3 \alpha$

(j) $\alpha b_4 = b_4 \alpha$

where $\alpha$ runs over all the elements of $B$. Conversely, if there exist elements $b_1, b_2, b_3$ and $b_4$ in $B$ which satisfy the conditions (a)−(j) then the map $\phi$ (stated earlier) splits, that is, $A$ is separable over $B$. Hence it suffices to prove that the system of conditions (2, ii−viii) is equivalent

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to that of conditions (a)—(j). Assume (2, ii—viii). Then, (2, vi—viii) imply (g), (h), (j), and (2, v, viii) imply that for each \( \alpha \in B \), \( \rho(\alpha)b_3 = \rho(\alpha)p(b_3) = p(\rho(b_3)) = p(b_3 \rho^{-1}(\alpha)) = p(b_3)\alpha = b_3 \alpha \). Hence we have (i), that is, (2, ix)
\[
b_3 \in B(\rho^{-1}).
\]
Moreover, (2, vi—ix) and (2, i, ii) imply that for each \( i = 1, \ldots, 4 \),
\[
(2, x) \quad a b_i = b_i a = a \rho(b_i) = \rho(a b_i) = \rho(b_i a) = \rho(b_i) a \quad (2, xi) \quad b b_i = b_i b = \rho^2(b_i) = \rho^2(b_i b) = \rho^2(b_i b) = \rho^2(b_i) b, \]
\[
b b_i = 1 - b_i = 1 - \rho^2(b_i). \]
As is easily seen, (2, x) and (2, iii, v) imply \( a b_i + \rho(b_i) + \rho(b_i) = 0 \) and (2, xii)
\[
\rho(b_3) = b_3, \quad \rho(b_2) = b_2, \quad a b_2 = a \rho(b_2) = a b_3 = a \rho(b_3). \]
Further, (2, x—xii) and (2, iv) imply (c)—(f). Thus, (a)—(j) are contained in (2, ii—viii). Conversely, assume (a)—(j). Then (g)—(j) imply (2, vi—viii). Moreover, (g)—(j) and (2, i) imply (2, x). As is easily seen, (2, x), (a) and (c) imply (2, xii) which contains (2, v). Clearly, (2, xii) and (e) imply (2, iv). Thus, (2, ii—viii) are contained in (a)—(j). This completes the proof.

Now, by (2, vii, ix, xii) and (2, xi), we have
\[
(2, xiii) \quad b_2^2 = b_2 b_3 = b_3^2 = b_3 b_2, \quad (2, xiv) \quad \rho^2(b_i) = b_1. \]
For \( i = 2 \) and \( 3, \quad b_1 b_i = \rho^2(b_i) b_1 = b b_1 \) (by (2, vii, xii)). This and (2, vi, x—xiv) imply
\[
(2, xv) \quad u v = v u \text{ for each pair } u, v \subseteq \{a, b, b_1, b_2, b_3, b_4, \rho(b_i)\}, \quad r s = s r \text{ for each pair } r, s \subseteq \{a, b, \rho(b_1), b_2, b_3, b_4, \rho(b_i)\}. \]
Moreover, (2, iii, xii, xiii) imply
\[
(2, xvi) \quad a^2 b_1^2 = 4 b_2^2 = 4 b_3^2, \quad a^2 b_1 + 2 a b_2 = a b_1 + 2 a b_3 = 0 \quad \text{and, (2, ii, iv, xii, xiv) imply} \]
\[
(2, xvii) \quad 1 = b b_1 + b_1 = b_1 + a b_1 + b_2 = b(b_1 + a b_1) - a b_2. \]
These equalities will be used lately in our study.

Next, we shall prove the following

**Lemma 2.2.** Let \( f = X^2 - Xa - b \in B[X; \rho]_{(\alpha)} \) and \( f \) separable over \( B \). Let \( \{b_1, b_2, b_3, b_4\} \) be a system of elements of \( B \) which satisfies the conditions (2, ii—viii). Then \( \delta(f)B = B \delta(f) \) (\( \delta(f) = a^2 + 4b \)), and
\[
(2, xviii) \quad 2(b b_1 + \rho(b_i)) = \delta(f) b_1
\]
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\[(2, \text{xix}) \quad 4 = \delta(f)(b_1 + \rho(b_1)), \quad \text{and} \quad a = \delta(f)(b_1b_4 - b_2^2a).\]

\textbf{Proof.} For each \(\alpha \in B\), one will easily see that \(\alpha \delta(f) = \alpha(a^2 + 4b) = (a^2 + 4b)\rho^2(\alpha) = \delta(f)\rho^2(\alpha)\). Hence \(\delta(f)B = B\delta(f)\). Now, we shall prove (2, xviii) and (2, xix).

\[
\delta(f)b_1 = (a^2 + 4b)b_1 = a^2b_1 + 4bb_1 = a(ab_4) + 4(bb_1)
\]

\[
= a(-b_1 - b_4) + 4(\rho(b_4) + ab_2) \quad \text{(by (2, iii, iv))}
\]

\[
= -2ab_2 + 4\rho(b_4) + 4ab_2
\]

\[
= 2ab_2 + 4\rho(b_4) = 2(ab_2 + 2\rho(b_4))
\]

\[
= 2(bb_1 - \rho(b_4) + 2\rho(b_4)) \quad \text{(by (2, iv))}
\]

\[
= 2(bb_1 + \rho(b_4)).
\]

\[
\delta(f)(b_1 + \rho(b_4)) = \delta(f)b_1 + \delta(f)\rho(b_4) = \delta(f)b_1 + \rho(\delta(f)b_4)
\]

\[
= 2(bb_1 + \rho(b_4)) + 2\rho(bb_1 + \rho(b_4)) \quad \text{(by (2, xviii))}
\]

\[
= 2bb_1 + 2\rho(b_4) + 2\rho(bb_1) + 2b_4
\]

\[
= 2(bb_1 + b_4) + 2(\rho(bb_1) + \rho(b_4))
\]

\[
= 2 + 2 = 4 \quad \text{(by (2, i)).}
\]

\[
\delta(f)(b_1b_4 - b_2^2a) = (a^2 + 4b)(b_1b_4 - b_2^2a)
\]

\[
= a(a^2 + 4b)b_1b_4 - a(a^2 + 4b)b_2^2 \quad \text{(by (2, xv))}
\]

\[
= 2a(\rho(b_4) + bb_4)b_1 - a(a^2b_2^2 + b(4b_2^2)) \quad \text{(by (2, xviii))}
\]

\[
= 2ab_4 + bb_4b_1 - a(a^2b_2^2 + b^2b_1^2) \quad \text{(by (2, x, xvii))}
\]

\[
= 2ab_4 + b(a^2b_2) + b(b_1^2)
\]

\[
= 2ab_4 - a(ab_2^2 + b(-2ab_2))b_1
\]

\[
= (2b_4 - a(ab_2 - 2bb_1)b_2)
\]

\[
= a(2b_4 - a(bb_1 - \rho(b_4) - 2bb_1)b_1) \quad \text{(by (2, iv))}
\]

\[
= a(2b_4 - a(-b_1 - bb_1)b_2)
\]

\[
= a(2b_4 + ab_2)
\]

\[
= a(b_4 + \rho(b_4) + ab_2)
\]

\[
= a(b_4 + bb_1) = a \quad \text{(by (2, ii, iv)).}
\]

This completes the proof.

\textbf{Lemma 2.3.} Let \(f = X^2 - b \in B[X; \rho]_{\alpha}\). Then

(i) \(f\) is separable over \(B\) if and only if \(b\) is invertible and there exists an element \(z\) in \(Z\) such that \(z + \rho(z) = 1\).

(ii) \(f\) is Galois over \(B\) if and only if \(2b\) is invertible.

\textbf{Proof.} (i). Assume that \(f\) is separable over \(B\). Then, by
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(2, xvii) and (2, xv), we have $1 = b(b_i + \rho(b_i)) = (b_i + \rho(b_i))b$. Hence $b$ is invertible in $B$. Since $\rho(bb_i) = b\rho(b_i)$, $1 = bb_i + \rho(bb_i)$. Moreover, by (2, ii, vi), we have $bb_i = 1 - b_i \in Z$. Conversely, assume that $b$ is invertible in $B$ and there exists an element $z$ in $Z$ such that $z + \rho(z) = 1$. Then $1 = b(b^{-1}z) + \rho(z)$, $b(b^{-1}z) = \rho\rho(z)$, $\rho(z) \in Z$, and $b^{-1}z \in B(\rho^{-1})$. Hence, the system $\{b_1 = b^{-1}z, b_2 = b_3 = 0, b_0 = \rho(z)\}$ satisfies the conditions (2, ii—viii). Thus $f$ is separable over $B$. (ii) We set $A = B[x; \rho, 0, b]$. First, we assume that $2b$ is invertible in $B$. Then $\delta(f) = 4b$ is invertible in $B$. Hence by Lemma 1.5, $A$ is Galois over $B$. Conversely, assume that $A$ is $\emptyset$-Galois over $B$. By Lemma 1.2, $\emptyset$ is of order 2. Hence we may write $\emptyset = \{1, \sigma\}$ ($1 \neq \sigma$). Then, by [5, Th. 1.5], $A$ is separable over $B$, and whence, by (i), $b$ is invertible in $B$. Now, we suppose that $2$ is not invertible in $B$. Then $2A$ is a proper ideal of $A$. We set $\overline{A} = A/2A$ (the factor ring of $A$ modulo $2A$), $\overline{B} = (B + 2A)/2A$, $\overline{u} = u + 2A$ for each $u \in A$, and $\emptyset = \{\overline{1}, \overline{\sigma}\}$ will mean the group of automorphisms of $\overline{A}$ induced by $\{1, \sigma\}$. Then, by Lemma 1.4, $\overline{A}$ is a Galois extension of $\overline{B}$ with Galois group $\emptyset \neq \{\overline{1}\}$, and $\overline{A} = \overline{xB} + \overline{B} = \overline{Bx} + \overline{B} \neq \overline{B}(\overline{2} = \overline{0})$. Hence by Lemma 1.2, $\{\overline{x}, \overline{1}\}$ is a right free $\overline{B}$-basis of $\overline{A}$, and the difference $\overline{x} - \overline{\sigma}(\overline{x}) = \overline{x} + \overline{\sigma}(\overline{x})$ is invertible in $\overline{A}$. We set here $\overline{d} = \overline{x} - \overline{\sigma}(\overline{x})$. Then $\overline{\sigma}(\overline{d}) = \overline{\sigma}(\overline{x} + \overline{\sigma}(\overline{x})) = \overline{\sigma}(\overline{x}) + \overline{x} = \overline{d}$. Hence $\overline{d} \in \overline{B}$, and so, we may consider $d$ as an element of $B$. Since $\overline{x}^2 = \overline{b}$, we have $\overline{b} = \overline{\sigma}(\overline{x})^2 = (\overline{x} + \overline{d})^2 = \overline{x}^2 + \overline{x}(\overline{d} + \overline{\rho}(\overline{d})) + \overline{d}^2 = \overline{x}(\overline{d} + \overline{\rho}(\overline{d})) + \overline{b} + \overline{d}^2$. Since $\{\overline{x}, \overline{1}\}$ is right $\overline{B}$-free, it follows that $\overline{d}^2 = \overline{0}$, and whence $\overline{d}$ is not invertible in $\overline{A}$. This is a contradiction. We conclude therefore that $2$ is invertible in $A$.

**Remark 2.4.** We shall present an example of $B[X; \rho]_{(2)}$ such that it contains a separable polynomial which is not Galois. Let $F = GF(2^2)$. Since the extension $F/GF(2)$ is Galois, there exists an automorphism $\rho$ of $F$ such that $\rho^2 = 1$ and $\rho \neq 1$. Hence there is an element $z \in F$ with $z + \rho(z) = 1$. Consider here $F[X; \rho]$. Then $F[X; \rho]_{(2)} = \{X^2, X^2 + 1\}$ By Lemma 2.3, $X^2$ is not separable over $B$, and $X^2 + 1$ is separable over $B$ but it is not Galois over $B$.

Now, we shall prove the following theorem which is one of the main results of this section.

**Theorem 2.5.** For $f \in B[X; \rho]_{(2)}$, the following conditions are equivalent.

(a) $f$ is Galois over $B$.
(b) $\delta(f)$ is invertible in $B$.
(b') $f'B[X; \rho] + fB[X; \rho] = B[X; \rho]$ where $f'$ is the derivative.
of $f$.

Proof. Let $f = X^2 - Xa - b \in B[X; \rho]_{\mathbb{C}}$, and set $A = B[x; \rho, a, b]$. Then $f' = 2X - a$ and $(2x-a)^2 = 4x^2 - 4xa + a^2 = a^2 - 4xa + 4(xa + b) = a^2 + 4b = \delta(f)$. This shows that (b) and (b') are equivalent. Moreover, by Lemma 1.5, we see that (b) implies (a). To see the converse, we assume (a) and suppose that $\delta(f)$ is not invertible in $B$. Then, by Lemma 1.2, $A$ is a Galois extension of $B$ with Galois group of order 2, and by [5, Th. 1.5], $f$ is separable over $B$. Hence by Lemma 2.2, the elements 4 and $a$ are contained in $\delta(f)B = B\delta(f) \neq B$. Since $\delta(f) \in B$, we set here $\bar{A} = A/\delta(f)A$ (the factor ring of $A$ modulo $\delta(f)A$), $\bar{B} = (B + \delta(f)A)/\delta(f)A$, and $\bar{u} = u + \delta(f)A$ for each $u \in A$. Then from Lemma 1.4, it follows that $\bar{A}$ is Galois over $\bar{B}$ and $\bar{A} = \bar{x}\bar{B} + \bar{B} = \bar{B}\bar{x} + \bar{B} \neq B$. Hence by Lemma 1.2, $\bar{B}$ is a right free $B$-basis of $A$. Since $\rho(\delta(f)) = \delta(f)$, $\rho$ induces an automorphism of $\bar{B}$, which will be denoted by $\bar{\rho}$. Then $\bar{\alpha}x = \bar{x}\bar{\rho}(\bar{\alpha})$ for each $\bar{\alpha} \in \bar{B}$, and $\bar{x}^2 = \bar{b}$. Hence by (2. i), $X^2 - \bar{b}$ is a polynomial of $\bar{B}[X; \rho]_{\mathbb{C}}$, which is Galois over $B$. Therefore, it follows from Lemma 2.3 (ii) that $2$ is invertible in $B$. However, since $4 \in \delta(f)B$, we have $0 = \bar{4} = 2^2$. This is a contradiction. Thus we conclude that $\delta(f)$ is invertible in $B$.

Corollary 2.6. Let $f = X^2 - Xa \in B[X; \rho]_{\mathbb{C}}$. Then the following conditions are equivalent.

(a) $f$ is Galois over $B$.
(b) $a$ is invertible in $B$.
(c) $f$ is separable over $B$.

Proof. By Th. 2.5, (a) and (b) are equivalent. By [5, Th. 1.5], (a) implies (c). Moreover, by (2, xvii), we see that (c) implies (b). Thus we obtain the assertion.

Next, we consider the following conditions.

(C1) $2$ is invertible in $B$.
(C2) $\rho | Z = 1$.
(C3) $\rho$ is an inner automorphism of $B$.
(C4) $B[\rho]$ contains an invertible element of $B$.
(C4') $B[X; \rho]_{\mathbb{C}}$ contains an element $f = X^2 - Xa$ which is separable over $B$.

In case (C1), it follows from the result of K. Kishimoto [4, Th. 1.2] that for $f \in B[X; \rho]_{\mathbb{C}}$, $f$ is separable over $B$ if and only if $f$ is Galois over $B$, and this is equivalent to that $\delta(f)$ is invertible in $B$. 

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Clearly (C_3) implies (C_4), and (C_3) is equivalent to (C_3'). Moreover, by Cor. 2.6, (C_3') and (C_3'') are equivalent.

Now, we shall prove the following

**Theorem 2.7.** Assume that there holds one of the conditions \((C_1)\) -- \((C_3'')\). Then, for \(f \in B[X; \rho]_{\{\}^\circ}\), the following conditions are equivalent.

(a) \(f\) is Galois over \(B\).

(b) \(\partial(f)\) is invertible in \(B\).

(c) \(f\) is separable over \(B\).

**Proof.** In virtue of Th. 2.5 and [5, Th. 1.5], it suffices to prove that (c) implies (b). Assume (c). Let \(f = X^2 - x - a - b \in B[X; \rho]_{(\{\})}\), and set \(A = B[x; \rho, a, b]\).

Case (C_1). The assertion follows from the result of [4, Th. 1.2]. However, this is also obtained as an easy application of Lemma 2.3 which is as follows. Set \(y = x - 1/2\). Then \(y^2 = (a^2 + 4b)b^4\), and \(A = yB + B\). Clearly \(g = Y^2 - (a^2 + 4b)/4 \in B[Y; \rho]_{\{\}^\circ}\), and this is separable over \(B\). Hence, it follows from Lemma 2.3 that \((a^2 + 4b)/4\) is invertible in \(B\), and hence, \(a^2 + 4b (\not= \partial(f))\) is invertible in \(B\).

Case (C_2). Let \(b_1, b_2, b_3, b_4\) be a system of elements of \(B\) which satisfies the conditions (2, ii -- viii). Then, by (2, xvi, xvii), we have

\[1 = 2b_4 + ab_2\] and \[2 = \partial(f)b_1\]. Since \(a = \partial(f)(b_1b_4 - b_2^2)a\) (2, xix), we have

\[1 = 2b_4 + ab_2 = \partial(f)(b_1b_4 + (b_1b_4 - b_2^2)ab_2).\] This shows that \(\partial(f)\) is invertible in \(B\), completing the proof.

In the rest of this section, we shall deal with the set of \(B\)-ring isomorphism classes of the extensions \(B[x_\gamma]/B\) \((f \in B[X; \rho]_{\{\}})\). For elements \(g, g_1 \in B[X; \rho]_{\{\}}\), if \(B[x_\gamma] \cong B[x_\gamma_1]\) \((B\)-ring isomorphic\) then we write \(g \sim g_1\). Clearly, the relation \(\sim\) is an equivalence relation in \(B[X; \rho]_{\{\}}\). By \(B[X; \rho]_{\{\}}\), we denote the set of equivalence classes of \(B[X; \rho]_{\{\}}\) with respect to the relation \(\sim\), and we write \(C = \langle g \rangle\) if \(C \subseteq B[X; \rho]_{\{\}}\) and \(g \in C\).

Now, we consider the following conditions.

(D_1) There is a Galois polynomial in \(B[X; \rho]_{\{\}}\).

(D_2) \(\rho^2\) is an inner automorphism of \(B\) determined by some element in \(U(B_1)\).

If \(f\) is a Galois polynomial in \(B[X; \rho]_{\{\}}\), then, by Th. 2.5, \(\partial(f) \in U(B) \cap B_1(\rho^2)\), which implies \(\rho^2 = \partial(f)^{-1}\partial(f)\). This shows that (D_1) implies (D_2). In case (C_1), if \(\rho^2 = \theta_1^{-1}\theta_1\) for some \(\theta\) in \(U(B_1)\) then \(\theta \in B(\rho^2) \cap U(B_1)\), and by Th. 2.5, the polynomial \(X^2 - \theta\) is a Galois polynomial.
polynomial in $B[X; \rho]_{\mathbb{Z}}$; whence, the conditions (D_1) and (D_2) are equivalent. In [4], K. Kishimoto proved that if the conditions (C_i) and (D_2) are satisfied then the set $B[X; \rho]_{\mathbb{Z}}$ forms an abelian semi-group which is isomorphic to the (multiplicative) factor semi-group of $Z_1$ modulo the subgroup $\{z\rho(z); z \in U(Z)\}$. In our consideration, it will be showed that if the condition (D_1) is satisfied then $B[X; \rho]_{\mathbb{Z}}$ forms an abelian semi-group under some composition which is somewhat interest.

First, as a preliminary lemma, we shall prove the following

**Lemma 2.8.** Assume $\rho^2 = \theta \cdot \theta$, for some $\theta \in U(B_1)$. Let $n$ be an arbitrary integer. Then $B(\rho^n) = Z$, $B(\rho^n)B(\rho^n) \subseteq B(\rho^n) = \theta^n Z$, $B_1(\rho^n) = \theta^n Z_1$, $\rho(B(\rho^n)) = B(\rho^n)$, and $J(\rho^2, B) \supseteq B(\rho^n)$.

**Proof.** It is obvious that $B(\rho^n) = Z$. Now, let $c$ be an arbitrary element of $B$. If $\beta$ and $\beta' \in B(\rho)$ then $c\beta\beta' = \beta\rho(c)\beta' = \beta\beta'\rho^2(c)$. This implies $B(\rho)B(\rho) \subseteq B(\rho^n)$, and so, $B(\rho^n)B(\rho^n) \subseteq B(\rho^n)$. If $v \in B(\rho^2)$ then $cv = v\rho^2(c) = v\theta^{-1}c\theta$ which shows $\theta^{-1}c \in Z$, that is, $v \in \theta Z$. Conversely, if $z \in Z$ then $c(\theta z) = (c\theta)z = (\theta\rho^2(c))z = (\theta z)\rho^2(c)$, and hence $\theta z \in B(\rho^n)$. Thus we obtain $B(\rho^n) = \theta Z$. Since $\rho^{-1} = \theta^{-1}\theta$, it follows that $B(\rho^n) = \theta^n Z$, and $B_1(\rho^n) = B_1(\rho^n) = B_1(\theta^n Z_1)$ Next, let $\beta$ be in $B(\rho^n)$. Then $\beta^{-1}(\rho^n) = \beta\rho^n(\beta) = \beta\rho^n(\beta)$. This implies $\rho(\beta) \in B(\rho^n)$. Moreover, since $\rho(\beta)\beta = \beta\rho^n(\beta)$, we have $\rho^{-1}(\beta) = \rho^{-1}(\beta)\rho^2(\beta)$ which shows $\rho^{-1}(\beta) \in B(\rho^n)$. Hence it follows that $\rho(B(\rho^n)) = B(\rho^n)$. Finally, since $\theta\beta = \theta\rho^2(\theta) = \theta\theta$, $\beta = \theta^{-1}\theta = \theta\beta$, which shows that $B(\rho^n) \supseteq J(\rho^2, B)$.

**Corollary 2.9.** Assume $\rho^2 = \theta \cdot \theta$, for some $\theta \in U(B_1)$. Let $u, u' \in B(\rho) \cup B(\rho^n) \cup B(\rho^n) \cup Z$, $u \in B(\rho)$, $\beta \in B(\rho)$. Then $uu' = u'u$, $u\beta = \beta u = \beta\rho(u) = \rho(u)\beta$, $u\beta = \beta u = \beta(\rho(u)) = \rho(\beta)u$, and $\beta^2 = \beta\rho(\beta) = \rho(\beta)\beta$.

**Proof.** Let $v$ and $v'$ be in $B(\rho^n) \cup B(\rho^n) \cup Z$. Then, by Lemma 2.8, we have $vv' = v'v$. Since $\rho^2(\beta) = \beta$, $\theta\beta = \beta\theta$, and whence $v\beta = v\beta$. Moreover $u\beta = \beta(\rho(u)) = \beta u = \beta(\rho(u)) = \rho(\beta)\rho(u)$, $\beta u = \beta\rho(u) = \rho(\beta)u$. Hence $uu' = u\beta = \beta\rho(u) = \rho(\beta)u$, and $uu' = u\beta = \beta\rho(u) = \rho(\beta)\beta$. Further, by Lemma 2.8, we have $\beta^2 = \beta\rho(\beta) = \rho(\beta)\rho(\beta) = \rho(\beta)\beta$.

Next, we shall prove the following

**Lemma 2.10.** Assume $\rho^2 = \theta \cdot \theta$, for some $\theta \in U(B_1)$. Let $g = X^2 - X + 1$, $g_1 = X^2 - X + 1 \in B[X; \rho]_{\mathbb{Z}}$. Then, $g \sim g_1$ if and only if there exist elements $\alpha, \beta \in B$ such that $\alpha \in U(Z)$, $\beta \in B(\rho)$, $u = u_1\alpha + \beta + \rho(\beta)$, and $v = v_1\alpha\rho(\alpha) - u_1\alpha\beta - \beta^2$.
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Proof. We consider $B[x]$ and $B[x_i]$. For the convenience, we set $x=x_i$ and $y=x_j$. First, we assume that there are elements $\alpha, \beta \in B$ such that $\alpha \in U(Z)$, $\beta \in B(\rho)$, $u = u_i \alpha + \beta + \rho(\beta)$, and $v = v_i \alpha \rho(\alpha) - u_i \alpha \beta - \beta^2$. Then, for any $c \in B$, $c(y \alpha + \beta) = y \rho(c) \alpha + c \beta = y \alpha \rho(c) + \beta \rho(c) = (y \alpha + \beta) \rho(c)$, and $(y \alpha + \beta)^2 = y(u_i \alpha + \beta + \rho(\beta)) \alpha + v_i \alpha \rho(\alpha) + \beta^2 = y u_i \alpha + v + u_i \alpha \beta + \beta^2 + \beta^2 = y u_i \alpha + v + (u_i \alpha + \beta + \rho(\beta)) \beta = y u_i \alpha + v + u_i \beta = (y \alpha + \beta) u + v$ (by Cor. 2.9). Hence, noting $\alpha \in U(Z)$, the mapping $xc_1 + c_2 \rightarrow (y \alpha + \beta)c_1 + c_2 (c_1, c_2 \in B)$ is a $B$-ring isomorphism of $B[x]$ to $B[y]$. Thus we obtain $g \sim g_i$. Conversely, we assume that there is a $B$-ring isomorphism $\phi : B[x] \rightarrow B[y]$. Then $\phi(x) = y \alpha + \beta$ for some $\alpha, \beta \in B$. Since $y = (y \alpha + \beta)c_1 + c_2$ for some $c_1$ and $c_2 \in B$, the element $\alpha$ is invertible in $B$. Now, for any $c \in B$, $\phi(cx) = \phi(x) \rho(c) = (y \alpha + \beta) \rho(c) = y \alpha \rho(c) + \beta \rho(c)$, and $\phi(cx) = c \phi(x) = c(y \alpha + \beta) = y \rho(c) \alpha + c \beta$. Hence $\alpha \rho(c) = (\rho(c) \alpha$ and $\beta \rho(c) = c \beta \in B(\rho)$. This implies $\alpha \in U(Z)$ and $\beta \in B(\rho)$. Next, we note $\phi(x^2) = \phi(xu + v) = (y \alpha + \beta)(u + v) = y u_i \alpha + v + u_i \beta + v$ and $\phi(x^2) = \phi(x) \phi(x) = (y \alpha + \beta)^2 = u_i \alpha \rho(\alpha) + \beta^2$ (by Cor. 2.9). Then $aa = (u_i \alpha + \beta + \rho(\beta)) \alpha$ and $\beta \rho(\alpha) = v_i \alpha \rho(\alpha) + \beta^2$. Hence it follows that $u = u_i \alpha + \beta + \rho(\beta)$, and $v = v_i \alpha \rho(\alpha) + \beta^2 - \beta u = v_i \alpha \rho(\alpha) + \beta^2 - (u_i \alpha + \beta + \rho(\beta)) = v_i \alpha \rho(\alpha) - u_i \alpha \beta - \beta^2$ (by Cor. 2.9). This completes the proof.

Throughout the rest of this section, $\overline{\rho}$ will mean the restriction of $\rho$ to $Z$. Then, one will easily see that for $\rho \in Z$, $\rho \in Z(\overline{\rho})$ if and only if $zp = p \rho^n(z)$ for all $z \in Z$, where $n$ is any integer. From this and Lemma 2.10, we obtain the following

Corollary 2.11. Assume $\rho = \theta_i \theta_i^\rho$, for some $\theta \in U(B_i)$. Then $\overline{\rho} = 1$, $Z(\overline{\rho}) = \{ p \in Z : z p = \rho(z) p \}$ for all $z \in Z$, and $Z(X; \overline{\rho}) = 1$, $\overline{\rho} = (X^2 - X u_i - v_i) p \in Z(X; \overline{\rho})$. Moreover, for $h = X^2 - X r_i - s_i$ and $h_1 = X^2 - X r_i - s_i \in Z(X; \overline{\rho}), h \sim h_1$ if and only if there exist elements $\alpha, \beta \in Z$ such that $\alpha \in U(Z)$, $\beta \in Z(\overline{\rho})$, $r = \alpha + \beta + \rho(\beta)$ and $s = \alpha \rho(\alpha) - r \alpha \beta - \beta^2$.

Remark 2.12. Assume $\rho = \theta_i \theta_i^\rho$, for some $\theta \in U(B_i)$. Let $S$ denote the subring of $B$ generated by $B_i(\rho)$, and set $T = X^2 - X u_i - v_i \in B_X \cup B(\rho)$. Then by Cor. 2.9, $S$ is a commutative ring. Let $f_i = X^2 - X u_i - v_i \in T (i = 1, 2)$ and $s_i \in S$. We write $f_1(f_2) = f_2(f_1) = (f_1)(f_2) = (f_1 \times f_2)(f_2 \times f_1) = s_1 \times (f_1 \times f_2)$. Then, this composition is commutative and associative, i.e., $f_1 \times f_2 = f_2 \times f_1$, $s_1 \times f_1 = f_1 \times s_1$, and for $f, \rho \in T$, $(f \times f)(f \times f) = (f \times f)(f \times f) = (f \times f)(f \times f) = (f \times f)(s \times f) = f_1 \times (f_2 \times s_1) = (f_1 \times f_2)(s_1 \times f_1)$.
Hence this composition makes $T$ an abelian S-semigroup. Further, as is easily seen, we have $\delta(f_1 \times f_2) = \delta(f_1) \delta(f_2)$ and $\delta(f_1 \times s_1) = \delta(f_1)s_1^2$. We identify the ring $Z[X; \rho]$ with the subring of $B[X; \rho]$ consisting of polynomials with coefficients in $Z$. Then $B[X; \rho]_{(e)} \cup Z[X; \rho]_{(e)} \subset T$.

Now, we shall prove the following

**Lemma 2.13.** Assume $\rho^2 = \theta^{-1}\theta$, for some $\theta \in U(B_1)$. Let $g, g_1 \in B[X; \rho]_{(e)}$, $h, h_1 \in Z[X; \rho]_{(e)}$, and $\xi \in U(Z_1)$. Then $g \sim g \times \xi \in B[X; \rho]_{(e)}$, and in particular, $h \sim h \times \xi \in Z[X; \rho]_{(e)}$. Moreover, $g \times g_1 \times \theta^{-1} \in \xi Z[X; \rho]_{(e)}$, $g \times h \in B[X; \rho]_{(e)}$, and $h \times h_1 \in Z[X; \rho]_{(e)}$.

**Proof.** Let $g = X^2 - Xu - v$ and $g_1 = X^2 - Xu_1 - v_1$. Then $g \times \xi = X^2 - Xu\xi - v\xi$, $u \in B_1(\rho)$, and $v \in B_1(\rho^2)$. Hence $g \times \xi \in B[X; \rho]_{(e)}$. If we set $x = \xi$ and $\beta = 0$ then $u = u + \beta$ and $\frac{\delta^2}{\delta\beta^2} = u\beta\beta + u\alpha\beta + \beta^2$. Hence by Lemma 2.10, we have $g \sim g$. Next, we consider $g \times g_1 \times \theta^{-1} = X^2 - Xu_1\theta - (u^2v_1 + v^2u_1 + 4u_1v_1)\theta^{-1}$. Then, by Lemma 2.8, we have $u_1 \in \theta Z_1$ and $u^2v_1 + v^2u_1 + 4u_1v_1 \in \theta^2 Z_1$. Hence $u_1 \theta^{-1} \in Z_1$ and $(u^2v_1 + v^2u_1 + 4u_1v_1)\theta^{-1} \in Z_1$. Moreover, by Cor. 2.9, we have that $(z - \rho(z))u_1 \theta^{-1} = 0$ for all $z \in Z$. Hence by Cor. 2.11, we obtain $g \times g_1 \times \theta^{-1} \in Z[X; \rho]_{(e)}$. The other assertions will be easily seen from Lemma 2.8 and the fact that the sets $B_1(\rho), B_1(\rho^2), Z_1(\rho)$ and $Z_1(\rho^2)$ are $Z_1$-submodules of $B$. This completes the proof.

If there is a Galois polynomial $f$ in $B[X; \rho]_{(e)}$ then $\delta(f) \in U(B_1)$, $\rho^2 = \delta(f)^{-1}\delta(f)$ (Th. 2.5); and whence, by Lemma 2.13, we have the mappings

\[
\begin{align*}
\nu_f : & \quad B[X; \rho]_{(e)} \to Z[X; \rho]_{(e)}; \quad \nu_f(g) = f \times \delta(f)^{-1} \times g, \text{ and } \\
\nu_f : & \quad Z[X; \rho]_{(e)} \to B[X; \rho]_{(e)}; \quad \nu_f(h) = f \times h.
\end{align*}
\]

Now, we shall prove the following lemma which plays an important rôle in our study.

**Lemma 2.14.** Assume that there is a Galois polynomial $f$ in $B[X; \rho]_{(e)}$. Then

(i) If $g \sim g_1$ in $B[X; \rho]_{(e)}$ then $\nu_f(g) \sim \nu_f(g_1)$.

(ii) If $h \sim h_1$ in $Z[X; \rho]_{(e)}$ then $\nu_f(h) \sim \nu_f(h_1)$.

(iii) For each $g \in B[X; \rho]_{(e)}$, $g \sim \nu_f\nu_f(g)$, and moreover, $g$ is Galois over $B$ if and only if $\nu_f(g)$ is Galois over $Z$.

(iv) For each $h \in Z[X; \rho]_{(e)}$, $h \sim \nu_f\nu_f(h)$, and moreover, $h$ is Galois over $Z$ if and only if $\nu_f(h)$ is Galois over $B$.

**Proof.** Let $f = X^2 - Xa - b$, and set $\theta = \delta(f) (= a^2 + 4b)$. (i).
Let \( g = X^2 - Xu - v, \ g_1 = X^2 - Xu_1 - v_1 \in B[X; \rho], \) and \( g \sim g_1. \) Then, by Lemma 2.10, there exist elements \( \alpha, \beta \in B \) such that \( \alpha \in U(Z), \) \( \beta \in B(\rho), \) \( u = u_1 \alpha + \beta + \rho(\beta), \) and \( v = v_1 \alpha \rho(\alpha) - u_1 \alpha \beta - \beta^2. \) By Lemma 2.8 and Cor. 2.9, we have \( au^{-1} = (a_0 \alpha + a_\beta + a \rho(\beta)) \theta^{-1} = au \theta^{-1} \alpha + 2a \theta^{-1} \alpha \) and \( a \theta^{-1} \in Z(\rho). \) Next, we note \( a^2 v + bu^2 + 4bv = a^2 (v_1 \alpha \rho(\alpha) - u_1 \alpha \beta - \beta^2) + b (u_1 \alpha + \beta + \rho(\beta))^2 + 4b (v_1 \alpha \rho(\alpha) - u_1 \alpha \beta - \beta^2) = a^2 v_1 \alpha \rho(\alpha) - a^2 u_1 \alpha \beta - a^2 \beta^2 + b (u_1 \alpha \alpha \rho(\alpha) + u_1 \alpha \beta + u_1 \alpha \beta + \beta^2 + \rho(\beta) + \rho(\beta) + \rho(\beta) + \rho(\beta) + \rho(\beta)) + b (4v_1 \alpha \rho(\alpha) - 4u_1 \alpha \beta - 4 \beta^2). \) Then, by Cor. 2.9, we obtain \( (a^2 v + bu^2 + 4bv) \theta^{-1} = (a^2 v_1 + bu_1^2 + 4bv_1) \theta^{-1} = a \beta^2 - (a \beta \theta^{-1})^2. \) This implies \( \psi(g) \sim \phi(g_1) \) (Cor. 2.11). (ii.) \( h = X^2 - Xr - s, \ h_1 = X^2 - Xr_1 - s_1, \) \( \in Z[X; \rho] \) and \( h \sim h_1. \) Then, by Cor. 2.11, there exist elements \( \alpha, \beta \in Z \) such that \( \alpha \in U(Z), \beta \in Z(\rho), \) \( r = r_1 \alpha + \beta + \rho(\beta), \) and \( s = s_1 \alpha \rho(\alpha) - r_1 \alpha \beta - r_1 \beta^2. \) By Cor. 2.9, we have \( ar = ar_1 \alpha + 2a \beta, \) \( a \beta \in B(\rho), \) \( a^2 s + b r^2 + 4bs = a^2 (s_1 \alpha \rho(\alpha) - r_1 \alpha \beta - r_1 \beta^2) + b (r_1 \alpha + \beta + \rho(\beta))^2 + 4b (s_1 \alpha \rho(\alpha) - r_1 \alpha \beta - r_1 \beta^2) = (a^2 s_1 + b r^2 + 4bs_1) \alpha \rho(\alpha) - (ar_1 \alpha \beta - r_1 \beta^2). \) This implies \( \phi(h) \sim \phi(h_1) \) (Lemma 2.10). (iii.) \( g = X^2 - Xu - v \in B[X; \rho]. \) Then, by Remark 2.12, we have \( \psi(g) = f \times (f \times \theta^{-1}) \times g = ((f \times f) \times g) \times \theta^{-1} = X^2 - Xa^2 v u^{-1} - (a^2 v + (2a^2 b + 4b^2)u + 4(2a^2 b + 4b^2)u) \theta^{-1} = X^2 - Xa^2 u^{-1} - ((a^2 + 4b)^2 u + 2(a^2 + 4b)bu - 4b^2 u^2') \theta^{-1} = X^2 - Xa^2 u^{-1} - (u - (2b u^2) - (2b u^2)^2) \theta^{-1} \) (note \( a^2 + 4b = 0). \) Moreover, since \( a^2 \theta^{-1} + 4b \theta^{-1} = 1, \) we have \( a^2 u^{-1} = u + 2(-2bu^2) \) and \(-2bu^2 \in B(\rho). \) This implies \( g \sim \phi(g) \) (Lemma 2.10). By Remark 2.12, we have \( \phi(g) = (\phi(f \times \theta^{-1}) \times g = \phi(g) \theta^{-1} \phi(g) = \theta^{-1} \phi(g), \) which shows that \( \phi(g) \) is inversible in \( B \) if and only if \( \phi(g) \) is inversible in \( Z. \) Hence by Th. 2.5, \( g \) is Galois over \( B \) if and only if \( \phi(g) \) is Galois over \( Z. \) (iv.) \( h = X^2 - Xr - s \in Z[X; \rho]. \) Then \( \psi(h) = (f \times \theta^{-1}) \times (f \times h) = \theta^{-1} = X^2 - Xa^2 \theta^{-1} - (a^2 s + (2a^2 b + 4b^2)^2 r^2 + 4(2a^2 b + 4b^2) s \theta^{-1} = X^2 - Xa^2 \theta^{-1} - (s - (r - (2b \theta^{-1}) - (2b \theta^{-1}))^2, \) \( a^2 r \theta^{-1} = r + 2(-2b \theta^{-1}), \) and \(-2b \theta^{-1} \in Z(\rho). \) (Lemma 2.8). This shows that \( h \sim \psi(h) \) (Cor. 2.11). Since \( \phi(h) = \phi(f \times h) = \phi(h), \) \( h \) is Galois over \( Z \) if and only if \( \psi(h) \) is Galois over \( B. \) (Remark 2.12 and Th. 2.5.) This completes the proof.

Next, we consider the sets \( B[X; \rho], \) \( Z[X; \rho], \) \( Z[X; \rho] \) (the sets of equivalence classes in \( B[X; \rho], \) \( Z[X; \rho], \) \( Z[X; \rho] \) (respectively) with respect to the relations \( \sim). \) For the convenience, if \( C \in B[X; \rho], \) \( g \in C (C \subseteq Z[X; \rho], \) \( h \in C'), \) then we write \( C = \langle g \rangle (C' = \langle h \rangle). \)

Now, if there is a Galois polynomial \( f \) in \( B[X; \rho], \) then, by Lemma 2.14, we have the (well defined) mappings

\[
\begin{align*}
\phi_f & : B[X; \rho] \rightarrow Z[X; \rho], \quad \phi_f(\langle g \rangle) = \langle \phi_f(g) \rangle, \quad \text{and} \\
\psi_f & : Z[X; \rho] \rightarrow B[X; \rho], \quad \psi_f(\langle h \rangle) = \langle \psi_f(h) \rangle
\end{align*}
\]
such that $\psi_f^{-1} = 1$ and $\psi_f \psi_f = 1$. Hence we obtain the following

**Corollary 2.15.** Assume that there is a Galois polynomial $f$ in $B[X; \rho]_{(\overline{\rho})}$. Then $\psi_f$ is one-to-one, and $\psi_f^{-1} = \psi_f$. Moreover, for $\langle g \rangle \in B[X; \rho]_{(\overline{\rho})}$ and $h \in \phi_f(\langle g \rangle)$, $g$ is Galois over $B$ if and only if $h$ is Galois over $Z$.

Next, we shall prove the following

**Theorem 2.16.** Assume that there is a Galois polynomial $f$ in $B[X; \rho]_{(\overline{\rho})}$. Then $Z[X; \overline{\rho}]_{(\overline{\rho})}$ forms an abelian semigroup under the composition $\langle h \rangle \cdot \langle k \rangle = \langle h \times k \rangle$ which has the identity element $\psi_f(\langle f \rangle) = \langle f \times f \times \delta(\overline{\rho})^{-1} \rangle$, and the subset $\langle h \rangle \in Z[X; \overline{\rho}]_{(\overline{\rho})}$; $h$ is Galois over $Z$ coincides with the set of all invertible elements in the semigroup $Z[X; \overline{\rho}]_{(\overline{\rho})}$, which is a group of exponent 2.

**Proof.** Let $k = X^2 - Xp - q$, $h = X^2 - Xr - s$, $h_1 = X^2 - Xr_1 - s_1 \in Z[X; \overline{\rho}]_{(\overline{\rho})}$, and $h \sim h_1$. Then, by Lemma 2.13, $k \times h$ and $k \times h_1$ are contained in $Z[X; \overline{\rho}]_{(\overline{\rho})}$. Moreover, by Cor. 2.11, there exists elements $\alpha$, $\beta \in Z$ such that $\alpha \in U(Z)$, $\beta \in Z(\overline{\rho})$, $r = r_1\alpha + \beta + \rho(\beta)$, and $s = s_1\alpha + \beta + \rho(\beta)$. Hence we have $k \times h = X^2 - Xpr - (p^2s + q + q^2 + 4qs) = X^2 - Xp(r_1\alpha + \beta + \rho(\beta)) - (p^2s_1 + q + q^2 + 4qs_1)\alpha + \beta + \rho(\beta) - (p^2s_1 + q + q^2 + 4qs_1)(\alpha + \beta + \rho(\beta))$. This implies $k \times h \sim k \times h_1$ (Cor. 2.11). Further, if $k \sim h \in Z[X; \overline{\rho}]_{(\overline{\rho})}$ then $k \times h \sim h \times k = h_1 \times h_1 \times h_1$. Thus, the composition $\langle k \rangle \cdot \langle h \rangle = \langle k \times h \rangle$ in $Z[X; \overline{\rho}]_{(\overline{\rho})}$ is well defined. Moreover, as is easily seen, this composition is associative and commutative. Hence this makes $Z[X; \overline{\rho}]_{(\overline{\rho})}$ into an abelian semigroup. Now, for any $h \in Z[X; \overline{\rho}]_{(\overline{\rho})}$, we have $h \sim \mu_f(h) = (f \times \delta(\overline{\rho})^{-1}) \times (f \times h) = (f \times f \times \delta(\overline{\rho})^{-1}) \times h = \nu_f(f) \times h$, that is, $\langle h \rangle = \langle \nu_f(f) \times h \rangle = \langle \nu_f(f) \rangle \langle h \rangle$. Hence $\langle \nu_f(f) \rangle$ is the identity element of $Z[X; \overline{\rho}]_{(\overline{\rho})}$. Similarly, for any Galois polynomial $g \in B[X; \rho]_{(\overline{\rho})}$, $\langle \nu_g(g) \rangle$ is the identity element of $Z[X; \overline{\rho}]_{(\overline{\rho})}$, and so, $\langle \nu_f(f) \rangle = \langle \nu_g(g) \rangle = \langle g \times g \times \delta(\overline{\rho})^{-1} \rangle$. Next, let $h$ be an arbitrary Galois polynomial in $Z[X; \overline{\rho}]_{(\overline{\rho})}$. Then, by Lemma 2.14, $\nu_f(h)$ is a Galois polynomial in $B[X; \rho]_{(\overline{\rho})}$. Hence $\langle \nu_f(f) \rangle = (\nu_f(h) \times \nu_f(h) \times \delta(\overline{\rho})^{-1}) = (\nu_f(h) \times \delta(\overline{\rho})^{-1}) = (f \times f \times \delta(\overline{\rho})^{-1}) \times (h \times h \times \delta(\overline{\rho})^{-1}) = (f \times h \times h \times \delta(\overline{\rho})^{-1}) = (f \times h \times \delta(\overline{\rho})^{-1}) = (f \times h \times \delta(\overline{\rho})^{-1}) = (f \times h \times \delta(\overline{\rho})^{-1}) = \langle f \rangle \times \langle h \rangle \times \langle h \rangle = \langle h \rangle^2$. This shows that $\langle h \rangle$ is an invertible element of $Z[X; \overline{\rho}]_{(\overline{\rho})}$ of order 2. Conversely, assume that $\langle h \rangle$ is invertible in $Z[X; \overline{\rho}]_{(\overline{\rho})}$. Then $\langle \mu_f(f) \rangle = \langle h \rangle \langle h \rangle = \langle h \rangle \times \langle h \rangle$ for some $k \in Z[X; \overline{\rho}]_{(\overline{\rho})}$. By Cor. 2.15, $\mu_f(f)$ is Galois over $Z$, and so is $h \times k$. Hence $\delta(h \times k) = \delta(h) \delta(k)$ is invertible.
in $Z$, and so is $\delta(h)$. Thus $h$ is Galois over $Z$. This completes the proof.

Now, we shall prove the following theorem which is one of our main results.

**Theorem 2.17.** Assume that there is a Galois polynomial $f$ in $B[X; \rho]_{\Omega}$. Then $B[X; \rho]_{\Omega}$ forms an abelian semigroup under the composition $\langle g \rangle \langle g_1 \rangle = \langle f \times \delta(f)^{-1} \times (g \times g_1) \rangle$ with the identity element $\langle f \rangle$, and the subset $\langle g \rangle \in B[X; \rho]_{\Omega}$; $g$ is Galois over $B$ coincides with the set of all inversible elements in the semigroup $B[X; \rho]_{\Omega}$, which is a group of exponent 2. Moreover, this semigroup is isomorphic to the semigroup $Z[X; \rho]_{\Omega}$ by the map $\Phi_{\rho}$.

**Proof.** By Th. 2.16, $\Phi(\langle f \rangle) = \langle \mu(f) \rangle$ is the identity element of the semigroup $Z[X; \rho]_{\Omega}$. Now, let $C = \langle g \rangle$ and $C_1 = \langle g_1 \rangle$ be any elements of $B[X; \rho]_{\Omega}$. Then, by Lemma 2.13, we see that $g \times g_1 \times \delta(f)^{-1} \in Z[X; \rho]_{\Omega}$, and $\nu(g \times g_1 \times \delta(f)^{-1}) = f \times g \times g_1 \times \delta(f)^{-1} = f \times \delta(f)^{-1} \times (g \times g_1) \in B[X; \rho]_{\Omega}$. Moreover, we have $\Phi(\nu(C) \nu(C_1)) = \Phi(\nu(g) \nu(g_1)) = \Phi(\nu(g) \nu(g_1)) = \Phi(\nu(g) \nu(g_1)) = \Phi(\nu(g) \nu(g_1))$. This shows that $B[X; \rho]_{\Omega}$ forms an abelian semigroup under the composition $\langle g \rangle \langle g_1 \rangle = \langle f \times \delta(f)^{-1} \times (g \times g_1) \rangle$ with the identity element $\langle f \rangle$ which is isomorphic to the semigroup $Z[X; \rho]_{\Omega}$ by the map $\Phi_{\rho}$. The other assertion follows from the results of Cor. 2.15 and Th. 2.16. This completes the proof.

By virtue of Ths. 2.7 and 2.17, we obtain the following

**Corollary 2.18.** Assume that $\rho = 1$ and there is a separable polynomial $f \in B[X; \rho]_{\Omega}$. Then $B[X; \rho]_{\Omega}$ forms an abelian semigroup such that $B[X; \rho]_{\Omega} \cong Z[X; 1]_{\Omega}$ and for $g \in B[X; \rho]_{\Omega}$, $g$ is separable over $B$ if and only if $\langle g \rangle$ is inversible in the semigroup $B[X; \rho]_{\Omega}$.

Lastly, for the case (C), we shall prove the following

**Theorem 2.19.** Assume $\rho = u^{-1} u$, for some $u \in U(B)$. Then, there exists a one-to-one correspondence between $B[X; \rho]_{\Omega}$ and $Z[X; 1]_{\Omega}$ in the sense of the following: $g \mapsto \mu(g) = g \times u^{-1}$. In this case, there holds that for $g$ and $g_1 \in B[X; \rho]_{\Omega}$,

(i) $g \sim g_1$, if and only if $\mu(g) \sim \mu(g_1)$.

(ii) $g$ is separable over $B$ if and only if $\mu(g)$ is separable over $Z$.

(iii) $B[x_\Omega] \cong B \otimes Z[x_\mu g]$ (B-ring isomorphic).
Proof. As is easily seen, we have that $f = X^2 - Xu \in B[X; \rho]_{(\mathcal{O})}$, and it is Galois over $B$. Moreover, for any $g \in B[X; \rho]_{(\mathcal{O})}$, $\mu(g) = g \times u^{-1} = f \times \delta((f)^{-1}) \times g = \mu(f)g$, and $\nu\mu(g) = g$. Further, for any $h \in Z[X; 1]_{(\mathcal{O})}$, $\mu\nu(h) = h$. Hence $\mu = \mu_{(\mathcal{O})}$ is one-to-one, and by Lemma 2.14, the map $\mu$ satisfies (i) and (ii). Now, for any $g = X^2 - Xc - d \in B[X; \rho]_{(\mathcal{O})}$, we consider $B[x_c]$ and set $y = x_c u^{-1}$. Then, for each $\alpha \in B$, we have $
abla \alpha y = \alpha(x_c u^{-1}) = x_c \rho(\alpha) u^{-1} = (x_c u^{-1} + cu u^{-1})(x_c u^{-1}) = y \alpha$. Moreover, $y^3 = (x_c u^{-1})^3 = x_c^3 u^{-2} = x_c^2 cu^{-1} + du^{-1}$. Hence, it follows that $yZ + Z \cong Z[x_c u]$ (Z-ring isomorphic), and $B[x_c] = yB + B = By + B = B(yZ + Z) \cong B \otimes_B Z[x_c u]$ (B-ring isomorphic). This shows (iii).

3. On $B[X; D]_{(\mathcal{O})}$. In this section, we study $B[X; D]_{(\mathcal{O})}$, the subset of $B[X; D]$ (where $D$ is a derivation of $B$ with $D(xy) = D(x)y + xD(y)$ for all $x$ and $y$ in $B$). As in [4], we shall use the following conventions: $B_0 = \{b \in B; D(b) = 0\}$; $Z_0 = Z \cap B_0$; $I_0 = b - b_0$ (an inner derivation determined by $b \in B$); $B(D^2 - a, D) = \{b; I_0 = D^2 - a, D\}$; $B(2D) = \{b; I_0 = 2D\}$; $B_0(D^2 - a, D) = B(D^2 - a, D) \cap B_0$; $B_0(2D) = B(2D) \cap B_0$.

Now, let $f = X^2 - Xa - b \in B[X; D]_{(\mathcal{O})}$. Then, the factor ring $A = B[X; D]/fB[X; D]$ is a quadratic extension of $B$ such that for $x = f + fB[X; D]$, $A = xB + B, \alpha x = \alpha x + D(\alpha)$ for all $\alpha \in B$, $x^2 = xB + B$, and $\{1, x\}$ is a right free $B$-basis. Clearly $x^2 = x(xa + b) = (x + b)x$ and $(\alpha x)x = \alpha(xa + b)$ for all $\alpha \in B$. From these equalities, it follows that

\[(3, 0) \quad a \in B_0(2D) \text{ and } b \in B_0(D^2 - a, D).\]

Particularly $ab = ba$, $ax = xa$, and $bx = xb$. Conversely, if a system \{(D, a, b) \mid (a, b \in B)\} satisfies the condition (3, 0) then $X^2 - Xa - b \in B[X; D]_{(\mathcal{O})}$.

Hence, it follows that

\[(3, i) \quad B[X; D]_{(\mathcal{O})} = \{X^2 - Xa - b; a \in B_0(2D) \text{ and } b \in B_0(D^2 - a, D)\}.

For any $f = X^2 - Xa - b \in B[X; D]_{(\mathcal{O})}$, we denote the factor ring $B[X; D]/fB[X; D]$ by $B[x; D, a, b]$ (or, by $B[x]$) where $x = x_r = X + fB[X; D]$. First, we shall prove the following

\[\text{Lemma 3.1. Let } f = X^2 - Xa - b \in B[X; D]_{(\mathcal{O})}. \text{ Then } f \text{ is separable over } B \text{ if and only if there exist elements } b_1, b_2, b_3, \text{ and } b_4 \in B \text{ such that}\]

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(3, ii) \( bb_1 + b_4 = 1 \) \hspace{1cm} (3, iii) \( ab_1 + b_2 + b_3 = 0 \)
(3, iv) \( ab_2 + b_1 = bb_1 + D(b_2) \) \hspace{1cm} (3, v) \( b_1 \in Z \)
(3, vi) \( b_3 - b_2 = D(b_1) \) \((\in Z)\) \hspace{1cm} (3, vii) \( I_{b_2} = -(b_1)_vD \)
(3, viii) \( I_{b_4} = I_{b_1}D - a_1I_{b_2} \).

**Proof.** We set \( A = B[x; D, a, b] \) and assume that \( A \) is separable over \( B \). Then, the \((\text{left})B-(\text{right})B\)-homomorphism

\[ \phi : A \otimes_B A \rightarrow A \] \[ \phi : (\sum a_i \otimes b_i) \rightarrow \sum_i a_i b_i \]

splits. Hence there exists an element \( e \) in \( A \otimes_B A \) such that \( \phi(e) = 1 \) and \( (c \otimes 1)e = e(\otimes x)c \) for all \( c \in A \). Since \( A \otimes_B A = (x \otimes x)B + (x \otimes 1)B + (1 \otimes x)B + (1 \otimes 1)B \), we may write

\[ e = (x \otimes x)b_1 + (x \otimes 1)b_2 + (1 \otimes x)b_3 + (1 \otimes 1)b_4 \]

where the \( b_i \)'s are in \( B \). The equality \( \phi(e) = 1 \) implies

\[ x(ab_1 + b_2 + b_3) + bb_1 + b_4 = 1. \]

Moreover, we have

\( (x \otimes 1)e = (x \otimes x)(ab_1 + b_2) + (x \otimes 1)(ab_2 + b_4) + (1 \otimes x)bb_1 + (1 \otimes 1)bb_2, \)

\[ e(1 \otimes x) = (x \otimes x)(ab_1 + D(b_1) + b_2) + (x \otimes 1)(bb_1 + D(b_2)) + \]

\( (1 \otimes x)(ab_2 + D(b_3) + b_4) + (1 \otimes 1)(bb_3 + D(b_4)), \) and for each \( \alpha \in B, \)

\( (x \otimes 1)e = (x \otimes x)\alpha b_1 + (x \otimes 1)(D(\alpha)b_1 + ab_2) + (1 \otimes x)(D(\alpha)b_1 + ab_2) + \)

\( (1 \otimes 1)(D(\alpha)b_1 + D(\alpha)b_2 + D(\alpha)b_3 + ab_4), \)

\[ e(1 \otimes \alpha) = (x \otimes x)b_1 + (x \otimes 1)b_2 + (1 \otimes x)b_3 + (1 \otimes 1)b_4. \]

Hence we obtain

(a) \( ab_1 + b_2 + b_3 = 0 \) \hspace{1cm} (b) \( bb_1 + b_4 = 1 \)
(c) \( ab_1 + b_3 = ab_1 + D(b_1) + b_2 \) \hspace{1cm} (d) \( ab_2 + b_4 = bb_1 + D(b_2) \)
(e) \( bb_1 = ab_3 + D(b_3) + b_4 \) \hspace{1cm} (f) \( bb_2 = bb_3 + D(b_4) \)
(g) \( ab_1 = \alpha b_1 \) \hspace{1cm} (h) \( D(\alpha)b_1 + ab_2 = b_2 + \alpha \)
(i) \( D(\alpha)b_1 + \alpha b_3 = b_3 + \alpha \) \hspace{1cm} (j) \( D(\alpha)b_1 + D(\alpha)b_2 + D(\alpha)b_3 + \alpha b_4 = b_4 + \alpha \)

where \( \alpha \) runs over all the elements of \( B \). Conversely, if there exist elements \( b_1, b_2, b_3 \) and \( b_4 \) in \( B \) which satisfy the conditions (a)\( \rightarrow \)(j) then the map \( \phi \) (stated earlier) splits, that is, \( A \) is separable over \( B \). Hence it suffices to prove that the system of conditions (3, ii\( \rightarrow \)viii) is equivalent to that of the conditions (a)\( \rightarrow \)(j). Assume (a)\( \rightarrow \)(j). Then (g), (c) and (i) imply (3, v, vi)\( \rightarrow \)(j). Moreover, (j), (a) and (3, v, vii) imply \( I_{b_2} = -(b_1)_vD^2 - (b_2 + b_3), D = I_{b_2}D + (ab_1), D = I_{b_2}D - a_1I_{b_2}. \) Hence (3, ii\( \rightarrow \)viii) are contained
in (a) -- (j). Conversely, we assume (3, ii--viii). Then, as is easily seen, we have (a) -- (d), (g) -- (j) and \( bb_2 = bb_2 + D(1) = bb_2 + D(bb_1 + b_4) = bb_2 + bD(b_1) + D(b_2) = bb_2 + b(b_3 - b_2) + D(b_1) = bb_2 + D(b_1) \). We set here

\[
(a) = D(b_1) \in \mathbb{Z}.
\]

Then, since \( I_a = 2D \) and \( I_s = D^2 - aD \), we have \( D(z) = D^2(b_1) = (I_a + aD)(b_1) = aD(b_1) = az \) and \( 2D(b_3) = I_s(b_3) = (I_s + aD)(b_3) = bD(b_3) + az + b_4 = ab_3 + D(b_3) + b_4 \). Thus (a) -- (j) are contained in (3, ii--viii). This completes the proof.

Now, since \( b \in \mathbb{Z} \) (3, vii), we see that \( z \) is rational and \( z = \frac{b_2}{b_3 - b_2} = \frac{b_2}{b_3 - b_2} = \frac{bb_2}{b_3 - b_2} = bD(b_1) + D(b_2) = -I_s(b_2) = 0 \). Thus we obtain

\[
(3, xi) \quad 0 = 2z = z^2 = b_1z = b_1D(b_2) = D(b_1)D(b_2).
\]

Moreover, by (3, viii), we have \( 0 = I_s(a) = I_s(a) = I_s(b) = I_s(b) \). Thus we obtain

\[
(3, xii) \quad uv = vu \quad \text{for each pair} \quad u, v \in \{a, b, b_1, b_2, b_3, b_4\}.
\]

Next, we shall prove the following.

**Lemma 3.2.** Let \( f = X^2 - Xa - b \in B[X ; D] \), and \( f \) separable over \( B \). Let \( b_1, b_2, b_3, b_4 \) be a system of elements of \( B \) which satisfies the conditions (3, ii--viii). Then

\[
(3, xiii) \quad 2 = \delta(f)(b_1 + b_2b_3 - b_2^2), \quad 4 = \delta(f)(2b_3)
\]

where \( \delta(f) = a^2 - 4b \), moreover

\[
(3, xiv) \quad a^2 = \delta(f)(b_4 - bb_1)
\]

\[
(3, xv) \quad \delta(f)B = B\delta(f), \quad D\delta(f)B \subset \delta(f)B
\]

\[
(3, xvi) \quad I_a = 0 \quad (\text{mod } \delta(f)B)
\]

\[
(3, xvii) \quad b_1 \equiv 0, \quad b_2 \equiv b_3, \quad b_4 \equiv 1 \quad (\text{mod } \delta(f)B)
\]

\[
(3, xviii) \quad ab_2 + D(b_2) \equiv 1, \quad D(b_2) \equiv 1 \quad (\text{mod } \delta(f)B)
\]

\[
(3, xix) \quad I_{b_2} = 0 \quad (\text{mod } \delta(f)B).
\]

**Proof.** We set \( z = b_3 - b_2 \). Then

\[
\delta(f)(b_4^2 - bb_1^2) = \delta(f)(b_4 + (a^2 - 4b)b_2), \quad \text{by (3, xi)}
\]
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\[ = (a^2 + 4b)b_1 + az + az(1 + ab_2) \quad \text{by (2, xi, xii)} \]
\[ = a^2b_1 + 4bb_1 + az + az(bb_1 + b_4 + ab_2) \quad \text{by (3, ii)} \]
\[ = a(ab_1 + z) + 4bb_1 + az(2bb_1 + D(b_2)) \quad \text{by (3, iv)} \]
\[ = a(-2b_2 + 2bb_1 + 2(1 - b_4) + azD(b_2)) \quad \text{by (3, ii, iii, xi)} \]
\[ = 2 - 2ab_2 + 2bb_1 - 2b_4 + aD(b_1)D(b_2) \quad \text{by (3, ix)} \]
\[ = 2 - 2D(b_2) + aD(b_1)D(b_2) = 2 \quad \text{by (3, iv, x, xi).} \]
\[ \delta(f)(2b_1) = \delta(f)(2b_1 + 2b_2z) = 2\delta(f)(b_1 + b_2z) = 4 \quad \text{by (3, xi).} \]
\[ \delta(f)(b_1 - bb_1) = \delta(f)(1 - 2bb_1) = \delta(f) - \delta(f)(2b_1) \quad \text{by (3, ii)} \]
\[ = a^2 + 4b - 4b = a^2 \quad \text{by (3, xiii)).} \]
\[ \delta(f)B = (a^2 + 4b)B \subset a^2B + 2B \subset a(Ba - 2D(B)) + 2B \quad \text{by (3, i)} \]
\[ \subset (aB)a - 2B \subset Ba^2 + B \subset B\delta(f) \quad \text{by (3, xii, xiii, xiv)).} \]

Similarly, we have \( B\delta(f) \subset B\delta(f)B \), and hence \( \delta(f)B = B\delta(f) \). Moreover, since \( D(\delta(f)) = 0 \), it follows that \( D(\delta(f)B) = \delta(f)D(B) \subset \delta(f)B \).

In the rest of this proof, \( h \equiv k \) denotes the congruence \( h \equiv k \pmod{\delta(f)B} \) in \( B \). Then \( I_a(B) = 2D(B) = 0 \) (by (3, i)), that is, \( I_a = 0 \). Further,

\[ b_1 \equiv b_1(bb_1 + b_4) \equiv b_1(ab_2 + b_4 - D(b_2) + b_4) \quad \text{by (3, ii, iv)} \]
\[ \equiv b_1ab_2 + 2b_2b_4 - b_1D(b_2) \equiv b_1ab_2 \quad \text{by (3, xii)} \]
\[ \equiv (b_1ab_2)b_2 \equiv a^2b_2b_2^2 = 0 \quad \text{by (3, xiv)).} \]

Hence by (3, vi, ii, iv, xiii, xiv, vii), we have \( b_2 \equiv b_2 = D(b_2) \equiv 0 \), \( b_4 \equiv 1 \), \( ab_2 \equiv 1 + D(b_2) \), \( D(b_2)^2 \equiv (ab_2 - 1)^2 \equiv a^2b_2^2 - 1 \equiv 1 \), and \( I_{a_2}(B) = D(B)b_1 = 0 \).

This completes the proof.

**Lemma 3.3.** Let \( f = X^2 - Xa - b \) and \( g = X^2 - Xu - v \) be in \( B[X; D]_{(2)} \), and let \( f \) be separable over \( B \). Then

(i) \( a = u \pmod{\delta(f)B} \) and \( \delta(f)B \supset \delta(g)B \).

(ii) If \( g \) is separable over \( B \) then \( \delta(f)B = \delta(g)B \).

(iii) If \( aB = B \pmod{\delta(f)B} \) then \( \delta(f) \subset U(B) \), and conversely.

(iv) If \( D \) is inner then \( \delta(f) \subset U(B) \).

(v) If \( 2 \in U(B) \) then \( \delta(f) \subset U(B) \).

(vi) If \( 2 = 0 \) and \( D \mid Z = 0 \) then \( a \), \( \delta(f) \subset U(B) \).

**Proof.** Let \( \{ b_1, b_2, b_3, b_4 \} \) be a system of elements of \( B \) which satisfies the conditions (3, ii - viii). In the proof, \( x \equiv y \) denotes the congruence \( x \equiv y \pmod{\delta(f)B} \) in \( B \). Since \( I_a = D^2 - aD \) and \( I_v = D^2 - uD \), we have

\[ a = D(b_2)D(b_2)a = D(b_2)(I_a(b_2) + D(b_2)a) \quad \text{by (3, xvii, xix))} \]
\[\equiv D(b_2)(I_s + a, D)(b_2) \equiv D(b_2)D^2(b_2) \equiv D(b_2)(I_s + u, D)(b_2)\]
\[\equiv D(b_2)(I_s(b_2) + D(b_2)u) \equiv D(b_2)D(b_2)u \equiv u \] (by (3, xviii, xix)).
Moreover, \( \delta(g) \equiv u^2 + 4v \equiv u^2 \equiv a^2 \equiv 0 \) (by (3, xiii, xiv)). Thus we obtain (i).

If \( g \) is separable over \( B \) then, by (i), we have also \( \delta(g)B \supseteq \delta(f)B \), and (ii), so, \( \delta(g)B = \delta(f)B \). To see (iii), let \( aB = B \). Then, since \( \delta(g) \equiv 0 \) \( I_s \equiv 0 \), we have \( B = (aB)^2 \equiv a^2B \equiv 0 \), and this shows \( B = \delta(f)B \).

Conversely, if \( B = \delta(f)B \) then \( B = (a^2 + 4b)B = a^2B = a(aB) \) (by (3, xiii)), and this implies \( B = aB \). Next, if \( D \) is an inner derivation \( I_s \), then, by (3, xviii, xix), we have \( 1 \equiv D(b_2)^2 = (I_s(b_2))^2 \equiv (-I_s(d))^2 \equiv 0 \), and whence \( \delta(f)B = B \). If \( 2 \in U(B) \), then \( D = 2^{-1}I_s = I_{2^{-1}}I_s \) and hence \( \delta(f) \in U(B) \) by (iv). Finally, let \( 2 = 0 \) and \( D|Z = 0 \). Then

\[b_1 = (bb_1 + b_4)b_1 = (ab_2 + b_4 - D(b_2) + b_4)b_1 \] (by (3, ii, iv))
\[= (ab_2 + 2b_4 - D(b_2))b_1 = ab_2b_1 = (ab_1)b_2 \] (by (3, xi))
\[= -(b_2 + b_4)b_2 = D(b_2)b_2 = 0 \] (by (3, iii, v, vii)).

Hence by (3, viii), we have \( I_s = -(b_4)D = 0 \). This implies \( b_2 \in Z \), and so, \( D(b_2) = 0 \). Thus we obtain \( ab_2 = bb_1 + D(b_2) - b_4 = bb_1 + b_4 = 1 \) (by (3, ii, iv)). Therefore, it follows that \( aB = B \), and so, \( \delta(f) \in U(B) \) (by (iii)). This completes the proof.

Next, we consider the following conditions.

(C\(_1\)) \( B[X; D]_{\alpha} \) contains an element \( f = X^2 - X a - b \) such that either \( \delta(f) \) or \( a \) is invertible in \( B \).

(C\(_2\)) \( D \) is an inner derivation.

(C\(_3\)) \( 2 \) is invertible in \( B \).

(C\(_4\)) \( 2 = 0 \) and \( D|Z = 0 \).

Now, we shall prove the following theorem which is one of the main results of this section.

**Theorem 3.4.** Assume that there holds one of the conditions (C\(_1\)) – (C\(_4\)). Then, for \( g \in B[X; D]_{\alpha} \), the following conditions are equivalent.

(a) \( g \) is Galois over \( B \).

(b) \( \delta(g) \) is invertible in \( B \).

(c) \( g \) is separable over \( B \).

**Proof.** By Lemma 1.5 and [5, Th. 1.5], we see that (b) implies (a), and (a) implies (c). Hence it suffices to prove that (c) implies (b). We assume (c), and set \( g = X^2 - Xu - v \). If \( \delta(f) \) is invertible in \( B \) then,
by Lemma 1.5 and [5, Th. 1.5], \( f \) is separable over \( B \), and hence, by Lemma 3.3 (ii) and (3, xv), \( B = \delta(g)B = B\delta(g) \), which implies \( \delta(g) \in U(B) \). Next, let \( a \) be invertible in \( B \). Then \( aB = B \). By Lemma 3.3(i), we have \( a \equiv \alpha \) (mod \( \delta(g) \)). This implies \( uB \equiv aB \equiv B \) (mod \( \delta(g) \)). Hence by Lemma 3.3(ii), we obtain \( \delta(g) \in U(B) \). The other assertions follow from the results of Lemma 3.3(iv, v, vi). This completes the proof.

Now, for the cases (C_4) and (C_5), we shall prove the following

**Theorem 3.5.** Let \( D = I_0 \), an inner derivation. Then, there exists a one-to-one correspondence between \( B[X; D]_{(\alpha)} \) and \( Z[X; 0]_{(\alpha)} \) (= \( Z[X; D = 0]_{(\alpha)} \)) in the sense of the following

\[ f = X^2 - Xa - b \rightarrow \phi(f) = X^2 - X(a - 2c) - (b + c(a - 2c) + c^2) \]

such that for \( f, f_1, \) and \( f_2 \in B[X; D]_{(\alpha)} \),

(i) \( f \) is separable over \( B \) if and only if \( \phi(f) \) is separable over \( Z \),

(ii) \( B[x_r] \cong B \otimes Z[x_{\alpha}(r)] \) (\( B \)-ring isomorphic),

(iii) \( B[x_r] \cong B[x_q] \) (\( B \)-ring isomorphic) if and only if \( Z[x_{\alpha}(r)] \cong Z[x_{\alpha}(q)] \) (\( Z \)-ring isomorphic).

**Proof.** Let \( f = X^2 - Xa - b \in B[X; D]_{(\alpha)} \). Since \( I_{a + 2c} = I_{a - 2c} = 0 \), we have \( a - 2c \in Z \). We set here \( z = a - 2c \). Then

\[
0 = I_0 - I_0^2 + a.I_e = I_0 + (a - I_0)I_e = I_0 + ((2c + c)(c - c))I_e
\]

\[
= I_0 + (c + c)(c - c)I_e + z.I_e = I_0 + (c + c)(c - c)I_e
\]

\[
= I_0 + I_0^2 + I_e = I_0 + I_0^2 + ce.
\]

This implies \( b + c^2 + cz \in Z \). Moreover, it is obvious that if \( f_1 \neq f_2 \in B[X; D]_{(\alpha)} \), then \( \phi(f_1) \neq \phi(f_2) \). Hence the \( \phi \) is an injective map of \( B[X; D]_{(\alpha)} \) to \( Z[X; 0]_{(\alpha)} \). Conversely, let \( g = X^2 - Xa' - b' \in Z[X; 0]_{(\alpha)} \). We set here \( a = a' + 2c \) and \( b = b' + c^2 - ca' \). Then \( I_a = I_{a' + 2c} = I_{a'} + I_0^2 = 2I_e \). Moreover, we have

\[
I_b = I_{b' + 2c} = I_{b'} - I_0^2 - I_0 = - I_0^2 - I_0
\]

\[
= - ((c + c)(c - c) - a'I_e = - (c + c)(c - c)I_e - a'I_e
\]

\[
= ((c + c) - 2c - a'I_e = ((c - c) - a'I_e = I_0^2 - a'I_e.
\]

Hence we have \( f = X^2 - Xa - b \in B[X; I_a]_{(\alpha)} \) and \( \phi(f) = g \). Thus, the \( \phi \) is surjective, and so, this is one-to-one. Now, let \( f = X^2 - Xa - b, f_1, f_2 \in B[X; D]_{(\alpha)} \). Then \( \delta(f) = a^2 + 2b = (a - 2c)^2 + 4(c(a - 2c) + b + c^2) = \delta(\phi(f)) \). Hence the assertion (i) follows from Th. 3.4. Next, we consider \( B[x_r] \) and set \( y = x_r - c \). Then, for each \( \alpha \in B \),

\[
ay = ax_r - ac = x_r\alpha + I_0(\alpha) - ac = x_r\alpha - c\alpha = (x_r - c)\alpha = y\alpha.
\]
Moreover, we have
\[ y^2 = (x - c)^2 = x^2 - 2xc + c^2 = x(a + b - 2x)c + c^2 = y(a - 2c) + (c(a - 2c) + b + c^2). \]
Hence \( y^2 = x^2 + Z \equiv Z[\alpha, \beta]. \) Thus, it follows that \( B[X; D] \cong B_Y = B_Y = B(yZ + Z) \equiv B \otimes Z \mathbb{Z}[X; D]. \) This shows (ii). If \( B[X; D] \cong B[x] \) (B-ring isomorphic) then \( Z \cong (\text{the centralizer of } B \text{ in } B[x]) \cong (\text{the centralizer of } B \text{ in } B[x]). \) The converse is obvious. Thus we obtain (iii). This completes the proof.

Now, for elements \( g \) and \( g_1 \in B[X; D] \) if \( B[x] \cong B[x_1] \) (B-ring isomorphic) then we write \( g \sim g_1. \) Moreover, by \( B[X; D] \) we denote the set of equivalence classes of \( B[X; D] \) with respect to the relation \( \sim, \) and we write \( C = \langle g \rangle \) if \( C \subseteq B[X; D] \) and \( g \in C. \)

In virtue of Th. 3.5 and Th. 2.16, we obtain the following corollary which contains the result of [4, Cor. 1 (2)].

**Corollary 3.6.** Let \( D \) be an inner derivation. Then \( B[X; D] \) forms an abelian semigroup under the composition \( \langle g_1 \rangle \langle g_2 \rangle = \langle \phi^{-1}(\phi(g_1) \times \phi(g_2)) \rangle \) where \( \phi \) is as in Th. 3.5 and \( \phi(g_2) = (g_2) \times \phi(g_2) \) is as in Remark 2.12, so that this group is isomorphic to the group \( Z[X; D] = 0 \) in Remark 2.12,\( Z[X; D] = 0 \). For the case \( C \), we have the result of Cor. 3.18 which will be verified later.

Next, we shall prove the following

**Theorem 3.7.** Assume \( 2 = 0, \) and let \( f = X^2 - Xa - b \in B[X; D], \) Then

(i) \( f \) is separable over \( B \) if and only if there exist elements \( b_1, b_2 \) in \( B \) such that \( b_1 \in Z, \ D(b_1) + ab_1 = 0, \ D(b_2) + ab_2 = 1, \) and \( I_2 = (b_1). \)

(ii) \( f \) is Galois over \( B \) if and only if there exists an element \( s \) in \( U(Z) \) such that \( D(s) + as = 1. \)

**Proof.** (i). Assume that \( f \) is separable over \( B. \) Then, there exists a system \( \{b_1, b_2, b_3, b_4\} \) of elements in \( B \) which satisfies the conditions (3, iii - viii). Then, we have \( b_1 \in Z \) (by (3, vi)), \( D(b_1) + ab_1 = b_2 - b_1 = 0 \) (by (3, iii, vi)), and \( D(b_2) + b_2 = ab_2 + b_1 - bb_1 + ab_2 = b_4 + bb_1 = 1 \) (by (3, ii, iv)). Moreover, we have \( I_2 = (b_1) = (b_2) = (b_1) = D \) (by (3, viii)). Conversely, we assume that there exist elements \( b_1, b_2 \) in \( B \) such that \( b_1 \in Z, \ D(b_1) + ab_1 = 0, \ D(b_2) + ab_2 = 1, \) and \( I_2 = (b_1). \) We set here
b_2 = ab_1 + b_2 and b_4 = bb_1 + 1. Then \( ab_2 + b_4 = ab_2 + bb_1 + 1 = ab_2 + bb_1 + D(b_2) + ab_2 = bb_1 + D(b_2) \). Moreover, since \( I_a = 2D = 0 \) and \( I_b = D^2 - aD \), we have \( I_{ab} = I_{b^2} = I_{b_2} = (b_1)(D^2 - aD) = (ab_2 + b_1) + D(b_2) \). Hence, the system \( \{ b_1, b_2, b_3, b_4 \} \) satisfies the conditions (3, ii - viii). Hence \( f \) is separable over \( B \). (ii). Set \( A = B[x; D, a, b] \) where \( x = x_0 \), and assume that \( f \) is Galois over \( B \). Then, by Lemma 1.2, \( A \) is a Galois extension of \( B \) with Galois group \( \{ 1, D \} \) and \( x - \sigma(x) = x + \sigma(x) \in U(B) \). We set here \( s = (x - \sigma(x))^{-1}, y = xs, \) and \( c = y^2 + y \). Then \( y + \sigma(y) = 1 \), and \( \sigma(c) = c \). Hence \( c \in B \). Let \( \alpha \) be an arbitrary element of \( B \). Then \( \sigma(\alpha y - y \alpha) = \alpha y - y \alpha \), and so, this is contained in \( B \). Hence the mapping \( \alpha \rightarrow \alpha y - y \alpha = E(\alpha) (\alpha \in B) \) is a derivation of \( B \). Moreover \( \alpha y = \alpha (xs) = (\alpha x) s = (\alpha x + D(\alpha)s)s = xs + D(\alpha)s \) and \( \alpha y = y + E(\alpha) = (xs) + E(\alpha) \). Hence \( \sigma(\alpha y - y \alpha) = \alpha y - y \alpha \). This implies \( c \in B \). Now, we have \( B \supset c = y \). Set \( s = (x^2 + xs = (xs)(xs) + x(3x - D(s)) + xs = x^2 s + xD(s)s + xs = x(a + b)s^2 + xD(s)s + s = x(a + b)s^2 + xD(s)s + s = x(a + b)s^2 + xD(s)s + s = 0 \). Since \( s \) is invertible in \( B \), we obtain \( D(s) + as = 1 \). Conversely, we assume that \( D(s) + as = 1 \) for some invertible element \( s \) in \( Z \). Then, we see that \( (xs)^2 + x = bs^2 \). Moreover, one will easily see that the mapping \( \alpha \rightarrow D(\alpha)s (\alpha \in B) \) is a derivation of \( B \), \( \alpha s = (xs) + D(\alpha)s (\alpha \in B) \) and \( \{ xs, 1 \} \) is a right free \( B \)-basis of \( A \). By Th. 3.5, \( g = Y^2 - Y - bs^2 \) is a Galois polynomial of \( B[Y; sD] \). Hence \( A \) is a Galois extension of \( B \). Thus \( f \) is Galois over \( B \), completing the proof.

**Corollary 3.8.** Assume \( 2 = 0 \), and let \( f = X^2 - Xa - b \in B[X; D] \), so that \( a \) is nilpotent. Then, \( f \) is separable over \( B \) if and only if there is an element \( s \) in \( Z \) such that \( D(s) + as = 1 \); and in this case, \( D(s) \) is invertible in \( B \).

**Proof.** Assume that \( f \) is separable over \( B \). Then, by Th. 3.7, there exist elements \( b_1, b_2 \) in \( B \) such that \( b_1 \in Z \), \( D(b_1) + ab_1 = 0 \), \( D(b_2) + ab_2 = 1 \), and \( I_{ab} = (b_1) \). Since \( a^2 = 0 \) for some integer \( n > 0 \), it follows that for an integer \( 2^m \geq n, 1 = (D(b_2) + ab_2)^m = D(b_2)^m + (ab_2)^m = D(b_2)^m \), and whence \( D(b_2) \) is invertible in \( B \). Hence, noting that \( 0 = I_{ab} = (b_1) \) \( D(b_2) = D(b_2)b_1 \), we obtain \( b_1 = 0 \). This implies \( I_{ab} = 0 \), that is, \( b_2 \in Z \). Conversely, we assume that there is an element \( s \) in \( Z \) such that \( D(s) + as = 1 \). Then, for \( b_1 = 0 \) and \( b_2 = s \), we have \( b_1 \in Z \), \( b_2 = 0 \), \( D(b_2) + ab_2 = 1 \), and \( I_{ab} = (b_1) \). Hence by Th. 3.7, \( f \) is separable over \( B \). This completes the proof.

Next, we shall prove the following
Corollary 3.9. Let $f = X^2 - Xa - b \in B[X; D]$. Then, $f$ is separable (resp. Galois) over $B$ if and only if $2 = 0$, $a^2 = 0$, and there exists an element $s$ in $Z$ (resp. in $U(Z)$) with $D(s) + as = 1$.

Proof. If $f$ is separable over $B$ and $\delta(f) = 0$ then, by Lemma 3.2, we have $2 = 0$ and $a^2 = 0$. Hence the corollary follows from the results of Th. 3.7 and Cor. 3.8.

The following corollaries 3.10 and 3.11 are direct consequences of Cor. 3.9.

Corollary 3.10. Let $f = X^2 - b \in B[X; D]$. Then, $f$ is separable (resp. Galois) over $B$ if and only if $2 = 0$ and there exists an element $s$ in $Z$ (resp. in $U(Z)$) with $D(s) = 1$.

Corollary 3.11. Let either $2 \neq 0$ or $D|Z = 0$. Then, for any separable polynomial $f$ in $B[X; D]$, there holds $\delta(f) \neq 0$.

Next, we shall prove the following

Theorem 3.12. Assume $2 = 0$, and let $f = X^2 - Xa - b \in B[X; D]$. If $f$ is separable (resp. Galois) over $B$ then, for all $v \in B(aD - aD)$, $X^2 - Xa - v$ is separable (resp. Galois) over $B$. Moreover, if $f$ is separable over $B$ and $a$ is nilpotent then, $\{X^2 - Xa - v ; v \in B(aD - aD)\} = B[X; D]$.

Proof. The first assertion is a direct consequence of Th. 3.7. Now, let $f$ be separable over $B$ and $a$ be nilpotent. Then, by Cor. 3.8, there exists an element $s$ in $Z$ such that $D(s) + as = 1$, and then, $D(s)$ is invertible in $B$. Since $D^2(s) = aD(s)$, it follows that $a = D^2(s)(D(s))^{-1}$.

Next, let $X^2 - Xp - q$ be an element of $B[X; D]$. Then $I_p = 0$, and $D_a = D^2 - pD$, whence $D^2(s) = D(s)p$. This shows that $p = D^2(s)(D(s))^{-1} = a$. Hence $X^2 - Xp - q \in \{X^2 - Xa - v ; v \in B(aD - aD)\}$ (not $B[X; D]$). This completes the proof.

In virtue of Cor. 3.9 and Th. 3.12, we obtain the following

Corollary 3.13. If $B[X; D]$ contains a separable polynomial $f = X^2 - Xa - b$ with $\delta(f) = 0$ then $B[X; D] = \{X^2 - Xa - v ; v \in B(aD - aD)\}$. In particular, if $B[X; D]$ contains a separable polynomial $X^2 - b$ with $\delta(f) = 0$ then $B[X; D] = \{X^2 - v ; v \in B(aD)\}$.

Now, let $e \in B$, and set $J = B e B$. Moreover, we assume $J \neq B$. Since $D(e) = 0$, we have $D(J) \subset J$. We denote the factor ring $B/J$ by $B_e$, and for any element $c$ in $B$, we denote $c + J$ by $c_e$. Further, by

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D_r, we denote the derivation of B_r induced by D. Then, for any 
\( g = X^2 - Xu - v \in B[X; D]_{(a)} \), we have 
\( X^2 - Xu - v \in B_r[X; D_r]_{(a)} \). This will be denoted by \( g_r \). Under this situation, we shall prove the following

Theorem 3.14. If there exists a separable (resp. Galois) polynomial 
\( f \) in \( B[X; D]_{(a)} \) such that \( \epsilon = \partial(f) \) is not invertible in \( B \) then, for 
any \( g \in B[X; D]_{(a)} \), the polynomial \( g_r (\in B_r[X; D_r]_{(a)}) \) is separable 
(resp. Galois) over \( B_r \), and the factor ring \( B[x \epsilon] / \epsilon B[x] \) is 
\( B_r \)-ring isomorphic to \( B_r[x \epsilon] \).

Proof. Let \( g \) be an element of \( B[X; D]_{(a)} \), and set \( A_r = B[x \epsilon; D, a, b] \). First, we assume that \( f \) is separable over \( B \). By (3.xv), we have 
\( \epsilon B = B \epsilon \neq B \). Noting \( \epsilon x \epsilon = x \epsilon \epsilon \), we see that 
\( \epsilon A_r = A_r \epsilon \) and 
\( \epsilon A_r \cap A_r = \epsilon B \). Then, as is easily seen, the factor ring \( A_r / \epsilon A_r \) is 
\( B_r \)-ring isomorphic to \( B_r[x \epsilon; D_r, a_r, b_r] \). Now, since \( A_r \) is separable 
over \( B \), \( A_r / \epsilon A_r \) is separable over \( B_r \) by Lemma 1.3. Hence \( f_r \) is a 
separable polynomial in \( B_r[X; D_r] \), and the discriminant of \( f_r \) is zero. 
Hence by Cor. 3.13, \( g_r \) is separable over \( B_r \). By a similar way, we 
see that if \( f \) is Galois over \( B \) then \( g_r \) is Galois over \( B_r \) (by Lemma 1.3 
and Cor. 1.13). This completes the proof.

Next, we shall present an example of \( B[X; D]_{(a)} \) containing separable 
polynomials whose discriminants are zero.

Remark 3.15. Let \( F \) be a field of characteristic 2 (for example, 
\( F = GF(2) \)), and \( R \) the ring of polynomials of \( x \) with coefficients in \( F \), 
where \( x \) is an indeterminate. Moreover, let \( S \) be the subring of the 
guotient field of \( R \) which is generated by \( x^{-1} \) over \( R \), and \( D \) the ordinary 
derivative of \( S \) such that 
\( D(\sum_i a_i x^i) = \sum_i i a_i x^{i-1} \). Then \( D^2 = 0 \), and 
whence \( X^2 \in S[X; D]_{(a)} \) and \( X^2 \in R[X; D]_{(a)} \). Since \( D(x) = 1 \) and 
\( x \) is invertible in \( S \), it follows from Cor. 3.10 that \( X^2 \) is a Galois 
polynomial in \( S[X; D]_{(a)} \). As is easily seen, there is not an invertible 
element \( s \) in \( R \) with \( D(s) = 1 \). Hence \( X^2 \) is not a Galois polynomial 
in \( R[X; D]_{(a)} \). However, since \( D(x) = 1 \) and \( x \in R \), it follows from 
Cor. 3.10 that \( X^2 \) is a separable polynomial in \( R[X; D]_{(a)} \). Moreover 
\( \{ b \in S ; D(b) = 0 \} = F = \{ b \in R ; D(b) = 0 \} \), and whence by Cor. 3.13, we 
have \( S[X; D]_{(a)} = \{ X^2 - b ; b \in F \} \) and \( R[X; D]_{(a)} = \{ X^2 - b ; b \in F \} \).

In the rest of this section, we assume \( B[X; D]_{(a)} \neq \emptyset \), and moreover, for a (fixed) element \( X^2 - Xa - b \in B[X; D]_{(a)} \), we shall use the 
following conventions:
\[ B[X; D]_{(3),a} = \{ X^2 - Xa - v ; v \in B(B)(D^2 - a, D) \} ( \subset B[X; D]_{(3)}) . \]

\[ B[X; D]_{(3),a} \] is the set of equivalence classes of \( B[X; D]_{(3),a} \) with respect to \( \sim \); and write \( \langle g \rangle = C \) if \( C \in B[X; D]_{(3),a} \) and \( g \in C. \) Moreover, as in \([4, \S \,3]\), we set

\[ \mathfrak{B}_a(B) = \{ \beta \in B \ such \ that \ \beta^2 + D(\beta) + \beta a \in Z_o \ and \ \beta \in B \}. \]

Now, the following lemma will be easily seen.

**Lemma 3.15.** (Cf. \([4, \text{Lemma } 3.2]\)). Let \( X^2 - Xa - b \in B[X; D]_{(3)} \). Then \( B[X; D]_{(3),a} = \{ X^2 - Xa - (b + z) ; z \in Z_o \}. \)

Next, for case \( 2 = 0 \), we have the following

**Lemma 3.16.** (Cf. \([4, \text{Lemma } 3.1]\)). Assume \( 2 = 0 \), and let \( X^2 - Xa - b \in B[X; D]_{(3)} \). Then \( Z \subset \mathfrak{B}_a(B) \), \( Z \subset \mathfrak{B}_a(B) \), \( z \in Z_o \), and \( Z \) is an additive subgroup of \( (Z_o, +) \). If \( \beta = n \beta \) for all \( \beta \), \( n \in \mathfrak{B}_a(B) \) then \( \mathfrak{B}_a(B) \) is also an additive subgroup of \( (Z_o, +) \).

**Proof.** Let \( z \) be any element of \( Z \), and set \( c = z^2 + D(z) + za \). Then \( D(c) = D(z^2) + D^2(z) + D(z)a = 2zD(z) + (D^2 - a, D)(z) = I_0(z) = 0 \). Hence \( c \in Z_o \). If we set \( \alpha = 1 \) then \( I_0 = D + \alpha, D \), \( \alpha \in U(Z) \), \( \alpha^2 = 1 \), and \( a(1 + \alpha) = D(\alpha) \). This shows \( z \in \mathfrak{B}_a(B) \), and whence \( Z \subset \mathfrak{B}_a(B) \). Thus we obtain \( Z \subset \mathfrak{B}_a(B) \). Clearly \( Z \) is an additive subgroup of \( (Z_o, +) \). The other assertion is proved by making use of the same method as in the proof of \([4, \text{Lemma } 3.1]\). This completes the proof.

Moreover, by making use of the same methods as in the proofs of \([4, \text{Lemma } 3.3 \ and \ Th. \ 2.4]\), we obtain the following lemma and theorem.

**Lemma 3.17.** (Cf. \([4, \text{Lemma } 3.3]\)). Assume \( 2 = 0 \), and let \( X^2 - Xa - b \in B[X; D]_{(3)} \). Then, for elements \( g \) and \( h \) in \( B[X; D]_{(3),a} \), \( g \sim h \) if and only if \( g - h \in \mathfrak{B}_a(B) \).

**Theorem 3.18.** (Cf. \([4, \text{Th. } 3.4]\)). Assume \( 2 = 0 \), and let \( f = X^2 - Xa - b \in B[X; D]_{(3)} \). If \( \beta \gamma = \gamma \beta \) for all \( \beta, \gamma \in \mathfrak{B}_a(B) \) then \( B[X; D]_{(3),a} \) forms an abelian group of exponent 2 under the composition \( \langle g \rangle \langle h \rangle = \langle g + h + f \rangle \) with the identity element \( \langle f \rangle \), and this group is isomorphic to the additive group \( (Z_o, +) / \mathfrak{B}_a(B) \).

Now, we shall prove the following
Theorem 3.19. Assume $2 = 0$, and let $f = X^2 - Xa - b \in B[X; D]_{\omega}$. \(\text{If } Z = \mathfrak{B}_a(B) \text{ then the group } B[X; D]_{\omega} \text{ is isomorphic to the additive group } (Z_0, +)/Z_\omega.\) Moreover, if $f$ is Galois over $B$ and $Z = \mathfrak{B}_a(B)$ then there exists an element $s$ in $U(Z)$ such that $B[X; D]_{\omega}$ is isomorphic to the additive group $(Z, +)/(K_1 + K_2)$ where $K_1 = \{z \in Z ; sD(z) = z\}$, and $K_2 = \{z^2 + z ; z \in Z\}$.

Proof. The first assertion is a direct consequence of Th. 3.18. Now, we assume that $f$ is Galois over $B$ and $Z = \mathfrak{B}_a(B)$. Then, by Th. 3.7, there exists an element $s$ in $U(Z)$ with $D(s) + as = 1$. Clearly $s^2 \in Z_0$, and the mapping $z_0 \mapsto z_0s^2 \ (z_0 \in Z_0)$ is an automorphism of $(Z_0, +)$. This shows that $(Z_0, +)/Z_\omega \cong (Z_0, +)/Z_\omega s^2$. Since $0 = D(1) = D(ss^{-1}) = D(s)s^{-1} + sD(s^{-1}) = (as + 1)s^{-1} + sD(s^{-1})$. We have $D(s^{-1}) = (as + 1)s^{-2}$. Hence $Z_\omega s^2 = \{(z^2 + az + D(z))s^2 ; z \in Z\}$ $= \{(zs^{-1})^2 + a(zs^{-1}) + D(zs^{-1})s^2 ; z \in Z\}$ $= \{z^2 + azs + (D(z)s^{-1} + zD(s^{-1}))s^2 \ ; z \in Z\}$ $= \{z^2 + azs + (D(z)s^{-1} + z(as + 1)s^{-2})s^2 = \{z^2 + z + sD(z) ; z \in Z\}$. Therefore, it follows that $(Z_0, +)/Z_\omega \cong (Z_0, +)/\{z^2 + z + sD(z) ; z \in Z\}$. Moreover, for any $z \in Z$, we have $D(z + sD(z)) = D(z^2) + D(z + sD(z)) = D(z^2 + z + sD(z)) = D(z^2 + z + sD(z)) \in D(Z_\omega s^2) = \{0\}$, which implies $z + sD(z) \in Z_0$. Next, we consider the mapping $\phi : Z \to Z_0 \ (z \mapsto z + sD(z))$. Clearly the map $\phi$ is additive, and $Z_0 \supset \phi(Z) \supset \phi(Z_0) = Z_\omega$, that is, $\phi$ is surjective. Further, we have $\ker \phi = K_1$, and for any $z \in Z$, $\phi(z + sD(z)) = z^2 + z + sD(z^2 + z) = z^2 + z + sD(z)$, which implies $\phi(K_2) = \{z^2 + z + sD(z) ; z \in Z\}$. Hence, it follows that $(Z, +)/(K_1 + K_2) \cong (Z_0, +)/\{z^2 + z + sD(z) ; z \in Z\}$. Thus we obtain $(Z, +)/(K_1 + K_2) \cong (Z_0, +)/Z_\omega \cong B[X; D]_{\omega}(\text{Th. 3.18})$, completing the proof.

For case $a \in U(Z)$, we have the following

Corollary 3.20. (Cf. [4, Cor. 1]). Assume that $2 = 0$, $D|Z = 0$, and there exists a separable polynomial $f = X^2 - Xa - b$ in $B[X; D]_{\omega}$. Then $a \in U(Z)$, and any polynomial in $B[X; D]_{\omega}$ is Galois over $B$; moreover, the group $B[X; D]_{\omega}$ is isomorphic to the additive group
$(Z, +)/\{z^2 + z; z \in Z\}$.

**Proof.** By Lemma 3.3(vi), $\delta(f) = a^2$ is invertible in $B$. Clearly, for any $g \in B[X; D]_{(\alpha, \cdot)}$, we have $\delta(g) = a^2$, and hence by Th. 3.4, $g$ is Galois over $B$. Next, let $\beta$ be any element of $\mathfrak{B}_a(B)$. Then, there exists an element $\alpha \in U(Z)$ such that $I_\alpha = D + \alpha D$ and $a(1 + \alpha) = D(\alpha)$. Since $D(\alpha) = 0$ and $\alpha \in U(Z)$, we have $1 = \alpha$, which implies $I_\beta = 0$, that is, $\beta \in Z$. Noting $Z \subset \mathfrak{B}_a(B)$ (Lemma 3.16), we obtain $Z = \mathfrak{B}_a(B)$. Moreover, for any $s \in Z$, if $z \in Z$ and $sD(z) = z$ then $z = 0$. Hence, it follows from Th. 3.19 that $B[X; D]_{(\alpha, \cdot)}(Z, +)/\{z^2 + z; z \in Z\}$, which is our desired one.

Moreover, for case $a \in Z - U(Z)$, we have the following

**Theorem 3.21.** Assume that $2 = 0$ and there exists a separable polynomial $f = X^2 - Xa - b$ in $B[X; D]_{(\alpha)}$ so that $a$ is nilpotent. Then $B[X; D]_{(\alpha)} = B[X; D]_{(\alpha, \cdot)} = (Z, +)/Z$. If, in particular, $f$ is Galois over $B$ then there exists an element $s$ in $U(Z)$ such that $B[X; D]_{(\alpha, \cdot)} = (Z, +)/(K_1 + K_2)$ where $K_1 = \{z \in Z; sD(z) = z\}$, and $K_2 = \{z^2 + z; z \in Z\}$.

**Proof.** By Cor. 3.8 and Th. 3.12, there exists an element $s$ in $Z$ with $D(s) \in U(Z)$, and $B[X; D]_{(\alpha, \cdot)} = B[X; D]_{(\alpha)}$. Now, let $\beta$ be an element of $\mathfrak{B}_a(B)$. Then $I_\beta = D + \alpha D$ for some $\alpha \in U(Z)$. Applying this to the element $s$, we have $0 = I_\beta(s) = D(s) + D(s)\alpha = D(s)(1 + \alpha)$, and hence $\alpha = 1$. Since $2 = 0$, it follows $I_\beta = 0$, that is, $\beta \in Z$. Noting $Z \subset \mathfrak{B}_a(B)$ (Lemma 3.16), we obtain $Z = \mathfrak{B}_a(B)$. Hence, by Th. 3.19, we obtain the assertion.

Lastly, as a direct consequence of Cor. 3.9 and Th. 3.21, we obtain the following

**Corollary 3.22.** Assume that there exists a separable polynomial $f = X^2 - Xa - b$ in $B[X; D]_{(\alpha)}$ with $\delta(f) = 0$. Then $B[X; D]_{(\alpha)} = B[X; D]_{(\alpha, \cdot)} = (Z, +)/Z$. If, in particular, $f$ is Galois over $B$ then there exists an element $s$ in $U(Z)$ such that $B[X; D]_{(\alpha, \cdot)} = (Z, +)/(K_1 + K_2)$ where $K_1 = \{z \in Z; sD(z) = z\}$ and $K_2 = \{z^2 + z; z \in Z\}$.

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