On generalized uniserial blocks

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Throughout $R$ will represent a (unital) Artinian algebra over a field $K$ of characteristic $p > 0$, $J(R)$ the radical of $R$, and $G$ a finite group whose order is divisible by $p$. In [7, Theorem 6], M. Osima stated that the group algebra $KG$ is uniserial if and only if $G$ is $p$-nilpotent and a Sylow $p$-subgroup of $G$ is cyclic. In §1, by making use of K. Morita [3] we formulate the same for $RG$ (Theorem 1). In §2, we consider $KG$ for a splitting field $K$. If a block $B$ of $KG$ has a cyclic defect group $D$ then Dade's theorem [1, Theorem 78. 1] and [8, Lemma 4.2] enable us to see that the nilpotency index $t(B)$ of $J(B)$ is not greater than $|D|$ (cf. [4, Remark 1]). In Theorem 2, we shall prove that $t(B) = |D|$ if and only if $B$ is a generalized uniserial ring.

1. At first we consider the case $R$ is a simple algebra over $K$. As was stated in [5, Theorem 8], by making use of [7, Theorem 1] and [3, Theorem 8] (instead of [7, Theorem 6]) we have the following

**Lemma 1.** Let $R$ be a simple algebra over $K$.

1. $RG$ is primary decomposable if and only if $G$ is $p$-nilpotent.
2. $RG$ is uniserial if and only if $G$ is a $p$-nilpotent group with a cyclic Sylow $p$-subgroup.

Now, we can prove our first theorem.

**Theorem 1.** $RG$ is uniserial if and only if $R$ is semisimple and $G$ is a $p$-nilpotent group with a cyclic Sylow $p$-subgroup.

**Proof.** We assume that $RG$ is uniserial. Since $R$ is a homomorphic image of $RG$, $R$ is uniserial. Let $R = R_1 \oplus \cdots \oplus R_t$ be a decomposition of $R$ into primary rings, and $R_i = R_i/J(R_i)$. Then, $RG$ being uniserial, $G$ is a $p$-nilpotent group with a cyclic Sylow $p$-subgroup (Lemma 1 (2)). Since $R_i$ is primary, $R_i$ is isomorphic to the matrix ring $(S_i)_{n_i}$ with some completely primary ring $S_i$. Hence, we have $R_iG \cong (S_iG)_{n_i} \cong S_iG \otimes_K (K)_{n_i}$. Then by [5, Lemma 6], $S_iG$ is uniserial. Let $P$ be a Sylow $p$-subgroup of $G$. Since $S_iP$ is a homomorphic image of $S_iG$ and $S_iP/J(S_iP) \cong S_i/J(S_i)$, $S_iP$ is a completely primary uniserial ring. If $J(S_j) \neq 0$ for some $j$, then it is obvious that $J(S_j)$ is not contained in the augmentation ideal $J$ of $S_jP$. Further, since $g - 1 \in J \setminus J(S_j)P$ for
any \( g \neq 1 \) in \( P \), we see that \( f(S)P \) and \( J \) are incomparable. This yields a contradiction that \( S \) is not uniserial. Thus, \( R \) is semisimple. The converse part is also easy by Lemma 1 (2).

2. Let \( L \) be an extension field of the \( p \)-adic completion of the rationals, and \( R \) the complete local ring whose quotient field is \( L \). Let \( K \) be the residue class field of \( R \). Throughout the present section, we assume that \( L \) is a splitting field for \( G \).

Lemma 2. If \( B \) is a block of \( KG \) with a defect group \( D \), then the following conditions are equivalent:

1. \( D \) is cyclic and the decomposition matrix of \( B \) takes the form

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & & \ddots & \\
0 & 0 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & & & \\
1 & 1 & \ldots & 1
\end{pmatrix}
\]

(1)

2. \( D \) is cyclic and the Cartan matrix of \( B \) is of the form

\[
\begin{pmatrix}
s+1 & s & \ldots & s \\
s & s+1 & \ldots & s \\
\vdots & & \ddots & \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1
\end{pmatrix}
\]

(II)

3. \( B \) is a generalized uniserial ring.

Proof. The implication (1) \( \Rightarrow \) (2) is obvious, and (2) \( \Rightarrow \) (3) is a consequence of [2, Folgerung 4]. (3) \( \Rightarrow \) (2): Since \( B \) is a generalized uniserial ring, by [6, Theorem 17] \( B \) has only a finite number of indecomposable modules. Hence, \( D \) is cyclic. The rest of the proof is evident by [3, Remark, p. 158]. (2) \( \Rightarrow \) (1): By Dade's theorem [1, Theorem 68.1], the Cartan matrix \( (c_{ij}) \) of \( B \) is of the form
where the elements in the *-parts are 0 or 1. By (2), we have $s = 1$ or $a = 0$. First we consider the case $s = 1$. Let $\{U_i | 1 \leq i \leq a + b\}$ be a complete set of representatives of isomorphic classes of principal indecomposable $B$-modules, $\overline{U}_i$, the principal indecomposable $RG$-module such that $K \otimes \overline{U}_i \cong U_i$, and $\phi_i$ the character afforded by $\overline{U}_i$. Let $\{\chi_j | 1 \leq j \leq a + b + 1\}$ be a complete set of irreducible complex characters of $B$. Since $s = 1$, each $\phi_i$ is the sum of distinct two $\chi_j$’s. When $a + b \leq 2$, it is trivial that the decomposition matrix of $B$ takes the form (I). Hence, we suppose $a + b > 2$ and $\phi_1 = \chi_1 + \chi_2$. Since $(\phi_1, \phi_2) = c_{1k} = 1$ for $k \neq 1$, $\phi_k$ contains $\chi_1$ or $\chi_2$. We may assume here that $\phi_2$ contains $\chi_1$. If one of $\phi_i$’s ($i \geq 3$), say $\phi_3$, does not contain $\chi_1$, then $\phi_3$ contains $\chi_2$. Since $(\phi_2, \phi_3) = c_{23} = 1$, $\phi_2$ and $\phi_3$ contain a character different from $\chi_1$ or $\chi_2$ in common. This yields a contradiction that $\{\chi_j\}$ is not a tree. We have therefore seen that each $\phi_i (1 \leq i \leq a + b)$ contains $\chi_1$. Thus, the decomposition matrix of $B$ takes the form (I). Next, we consider the case $s \neq 1$. Then $a = 0$ and the decomposition matrix of $B$ takes the form (I) by [1, Theorem 68.1].

By Dade’s theorem [1, Theorem 68.1] and [8, Lemma 4.2], it is easy to see that if a defect group $D$ of $B$ is cyclic then $t(B) \leq |D|$. Hence, if a Sylow $p$-subgroup $P$ of $G$ is cyclic then the nilpotency index $t(G)$ of $f(KG)$ is not greater than $|P|$. Now, our attention will be directed towards the case $t(B) = |D|$ and the case $t(G) = |P|$.

**Theorem 2.** If a defect group $D$ of a block $B$ of $KG$ is cyclic, then the following conditions are equivalent:

1. $t(B) = |D|$.
2. $B$ is a generalized uniserial ring.
Proof. (1) \implies (2): By [Theorem 68.1], the Cartan matrix \((c_{kl})\) of \(B\) is of the form (II). Therefore we have \(|D| = t(B) \leq \max_i \{\sum_k c_{ik}\} \leq a + bs + 1 \leq (a + b) s + 1 = |D|\), whence it follows that \(a + bs + 1 = (a + b)s + 1 = |D|\). Hence, we have \(s = 1\) or \(a = 0\). First we consider the case \(s = 1\). Then \(a + b = |D| - 1\). Since \(a + b\) divides \(p - 1\), we have \(|D| = p\) and \(t(B) = \sum_i c_{ii} = p\) for some \(k\). Let \(U_i, \tilde{U}_i, \phi_i (1 \leq i \leq a + b), \chi_j (1 \leq j \leq a + b + 1)\) be as in the proof of Lemma 2. Since \(s = 1\), each \(\phi_i\) is the sum of distinct two \(\chi_j's.\) We suppose \(\phi_k = \chi_1 + \chi_2.\) Since \((\phi_k, \phi_l) = c_{kl} = 1\) for \(l \neq k\), \(\phi_k\) contains \(\chi_1\) or \(\chi_2.\) We suppose that \(m \phi_i's\) contain \(\chi_1\) and \(n \phi_i's\) do \(\chi_2.\) Now, let \(M, N\) be RG-submodules of \(\tilde{U}_k\) corresponding to \(\chi_1, \chi_2\) respectively. Since \(K \otimes \tilde{U}_k\) is uniserial by \(t(B) = p\), we may assume \(K \otimes M\) contains \(K \otimes N.\) Then all composition factors of \(K \otimes N\) appear among those of \(K \otimes M.\) Thus, we have \(n = 1.\) Rearranging \(\phi_i's\) and \(\chi_j's,\) the decomposition matrix of \(B\) takes the form (I). Next, we consider the case \(s = 0.\) Then \(a = 0\) and the Cartan matrix of \(B\) is of the form (II). Thus, by Lemma 2, \(B\) is a generalized uniserial ring. (2) \implies (1): Since \(B\) is a generalized uniserial ring, the Cartan matrix of \(B\) is of the form (II) (Lemma 2). Now, let \(f\) be an arbitrary primitive idempotent of \(B.\) Since \(\sum_k c_{ii} = se + 1 = |D|\) for \(1 \leq k \leq e = \) the number of non isomorphic principal indecomposable modules of \(B,\) the length of the unique composition series of \(Bf\) is \(|D|\). Therefore \(J(B)^{\leq 1}f \neq 0\) and \(J(B)^{\leq 0}f = 0.\) Hence \(t(B) = |D|\).

Corollary. If \(G\) has a cyclic Sylow \(p\)-subgroup of order \(p^a,\) then the following conditions are equivalent:

1. \(t(G) = p^a.\)
2. There exists a generalized uniserial block of defect \(a.\)

REFERENCES


1) Recently, S. Koshitani gave a different proof by making use of the result in [H. Kupisch : Projektive Moduln endlicher Gruppen mit zyklischer \(p\)-Sylow Gruppe, J. of Algebra 10 (1968), 1—7].
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(Received April 30, 1977)