A note on conjugates II

Hisao Tominaga*
A NOTE ON CONJUGATES II

HISAO TOMINAGA

In this note, we use the following conventions: By a ring we mean a ring with an identity, and by a subring we mean one which contains this identity. By a simple ring we shall mean a two-sided simple ring with minimum condition for left ideals, and by a primary ring [a completely primary ring] a ring such that the (Jacobson) radical is nilpotent and the residue class ring modulo the radical is a simple ring [a division ring]. For any non-empty subset $B$ of a ring $R$, $V_k(B)$ will denote the centralizer of $B$ in $R$. If, for each element of a subring $S$ of $R$ with an inverse in $R$, the inverse is always contained in $S$, then $S$ will be called a $\pi$-subring of $R$. For example, $V_k(B)$ and each subring with minimum condition for left ideals are $\pi$-subrings of $R$. $R^*$ means the multiplicative group consisting of all regular elements of a ring $R$. And for any set $S$, $\#(S)$ will signify the cardinal number of $S$.

Recently, W.R.Scott proved the following powerful lemma [3, p.305]: Let $D$ be an infinite division ring, $S$ a proper division subring of $D$. Then $(D^* : S^*) = \#(D)$, where $(D^* : S^*)$ is the group index of $S^*$ in $D^*$. And more recently, in [1], C.C.Faith has pointed out that the following lemma by F.Kasch in [2] is a direct consequence of Scott's lemma: Let $S$ be an infinite division subring of a division ring $D$ not contained in the centre of $D$. Then every element $d \in D$ which is outside of $V_d(S)$ possesses infinitely many conjugates $xdx^{-1}$ with $x \in S^*$. On the other hand, in the previous note [4], the present author has obtained the following which contains Kasch's: Let $R$ be an infinite simple subring of a ring $U$, and $T$ the set of conjugates of an element $t \in U$ by all regular elements in $R$. Then $\#(T) = \#(R)$ or 1.

The purpose of this note is to prove a generalization of Scott's lemma, and to present, as its direct consequence, an extension of [4, Theorem].

Our fundamental lemma is the following:

**Lemma 1.** Let $R$ be a primary ring with the radical $N$ such that $\bar{R} = R/N$ is infinite, and $S$ a $\pi$-subring of $R$. If $(R^* : S^*) < \#(\bar{R})$, where $(R^* : S^*)$ is the group index of $S^*$ in $R^*$, then $S = R$.

**Proof.** Let $R = \sum e_{ij}$, where $e_{ij}$'s are matric units and $C = V_k(\{e_{ij}\})$ is a completely primary ring. If $n = 1$, then $R$ (whence $S$) is completely primary, and let $\{\bar{r}_a\}$ be a linearly independent left basis of $\bar{R}$ over $\bar{S} = 1$.
(S + N)/N, where \( \overline{r}_a \) is the residue class of \( r_a \in R \). Then, it is clear that \( r_{a \beta} \notin S^* \) if \( a \neq \beta \). Hence, by our assumption \( (R^* : S^*) < \#(\overline{R}) \), we obtain \( \#(S) = \#(\overline{R}) \). And so, for each \( r \in R \), we can choose suitable \( s_1, s_2 \) and \( s \in S^* \) such that \( s_1 \neq s_2 \) (mod \( N \)), \( r \neq s_1 \) (mod \( N \)) for \( i = 1, 2 \), and \( r - s_1 = s(r - s_2) \). We obtain therefore \( r = (1-s)^{-1}(s_1 - ss_2) \in S \), whence it follows \( S = R \). Secondly, we shall prove the case \( n > 1 \). For each \( e_{ij} \) (\( i \neq j \)), \( \{ c + e_{ij} \mid c \text{ runs over a fixed complete representative system of } \overline{C} \} \) forms a subset of \( R^* \) whose cardinal number is \( \#(\overline{C}) = \#(\overline{R}) \). And so, there exist some \( c_1, c_2 \in C^* \) and \( s \in S^* \) such that \( c_1 \neq c_2 \) (mod \( N \)), and \( c_1 + e_{ij} = s(c + e_{ij}) \) is, that is,

\[
(*) \quad s = (c_1 + e_{ij})(c_2 + e_{ij})^{-1} = c_1c_2^{-1} + (1 - c_1c_2^{-1})e_{ij}.
\]

Since, for each \( c, c' \in C^* \), \( c'e^{-1} \notin (C \cap S)^* \) yields \( c'e^{-1} \notin S^* \), we obtain \( (C^*: (C \cap S)^*) < \#(\overline{R}) = \#(\overline{C}) \). And then, \( C \cap S \) being evidently a \( \pi \)-subring of \( C \), the proof for the case \( n = 1 \) shows \( C \cap S = C \), that is, \( S \supseteq C \). Hence, noting that \( c_1 \neq c_2 \) (mod \( N \)), from (*) one will readily see that \( e_{ij} \) is contained in \( S \), accordingly so are all \( e_{ij} \)'s. And then, \( S \) being \( \sum \) \( (C \cap S) e_{ij} \) necessarily, we obtain our assertion \( S = R \).

As an easy consequence of our theorem, we obtain the following extension of [4, Theorem].

**Theorem 1.** Let \( R \) be a primary subring of a ring \( U \) such that the residue class ring \( \overline{R} \) modulo its radical is infinite, and \( T \) the set of conjugates of an element \( t \subseteq U \) by all regular elements of \( R \). Then either \( \#(T) \geq \#(R) \) or \( \#(T) = 1 \).

**Proof.** Since \( \#(T) = (R^*: V_{\pi}(t)^*) \), we can apply Lemma 1 to \( R \) and its \( \pi \)-subring \( V_{\pi}(t) \). Hence our assertion is almost clear.

Of course, our theorem may be restated in the following way.

**Theorem 1'.** Let \( R \) be a primary subring of \( U \) such that the residue class ring \( R \) modulo its radical is infinite, and \( T \) a subset of \( U \) which is transformed into itself by all regular elements of \( R \). If \( \#(T) < \#(R) \), then \( T \subseteq V_{\pi}(R) \).

Finally, as a special case of Theorem 1, we obtain

**Corollary 1.** Let \( R \) be a primary subring of \( U \) which is of characteristic zero, and \( T \) the set of conjugates of an element \( t \subseteq U \) by all regular elements of \( R \). Then \( \#(T) \) is either infinite or 1.

**Remark.** Let \( a \) be an element of a ring \( A \). If \( xax = x \) has no non-zero solutions in \( A \), then \( a \) is called a root element. All the results of
A NOTE ON CONJUGATES II

this note except Corollary 1 are still valid for such $R$ that the set of all root elements of $R$ coincides with the radical $N$ and $R/N$ is an infinite simple ring.

REFERENCES


DEPARTMENT of MATHEMATICS,
OKAYAMA UNIVERSITY

(Received November 5, 1958)