On the Cartan invariants of p-solvable groups

Yasushi Ninomiya*

*Okayama University

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ON THE CARTAN INVARIANTS OF $p$-SOLVABLE GROUPS

YASUSHI NINOMIYA

Throughout the present paper, $k$ will represent an algebraically closed field of characteristic $p > 0$. Let $G$ be a finite $p$-solvable group, and $B$ a block ideal of defect $d$ of the group algebra $kG$. In [3], Fong proved that each Cartan invariant of $B$ is always bounded above by $p^d$. On the other hand, Koshitani [6] proved that the nilpotency index of the Jacobson radical of $B$ is bounded above by $p^d$, that is, the Loewy length of each projective indecomposable $B$-module is not greater than $p^d$. In this paper, we consider the possibility that the composition length of each projective indecomposable $B$-module is not greater than $p^d$. In other words, we consider the possibility that

\[(*) \quad \text{each row-sum of the Cartan matrix of } B \text{ is bounded above by } p^d.\]

In §1, we consider the case that $G$ has $p$-length 1, and prove that the Cartan matrix of every block ideal of $kG$ has property $(*)$ if and only if $G/O_{p^r}p(G)$ is abelian. Furthermore, we prove that if every irreducible $B$-module has $k$-dimension a power of $p$, then the Cartan matrix of $B$ has property $(*)$. Now, let $G$ be an arbitrary finite group, and $H$ a normal subgroup of $G$. Let $B$ and $b$ be block ideals of $kG$ and $kH$, respectively, such that $B$ covers $b$. In §2 (resp. §3), we consider the case that $[G:H] = p$ (resp. $[G:H] = q$, a prime number different from $p$), and the relationship between the Cartan invariants of $B$ and those of $b$ will be investigated. As a consequence, we show that if $[G:H]$ is a power of $p$ and the Cartan matrix of $b$ has property $(*)$, then the Cartan matrix of $B$ also has property $(*)$. However, in general, the converse need not be true; a counterexample will be given in §4.

Throughout this paper, all modules are assumed to be finitely generated right modules. We denote by $P_G(M)$ the projective cover of a $kG$-module $M$. If $H$ is a subgroup of $G$, then $M|_H$ is a $kH$-module obtained from $M$ by restricting the domain of operators to $kH$. Given a $kH$-module $L$, we denote by $L^G$ the induced module $L \otimes_{kH} kG$. The Jacobson radical of $kG$ is denoted by $J_G$. Given a block ideal $B$ of $kG$, we denote by $C_B$ and $\delta(B)$ the Cartan matrix of $B$ and a defect group of $B$, respectively.

1. Let $G$ be a $p$-solvable group, and $B$ an arbitrary block ideal of
In [9], Schwarz proved that if $G/O_{p'}(G)$ is abelian then each row-sum of $C_B$ is equal to $|\delta(B)|$. Furthermore, the converse of this fact has been proved in [8]. First, by making use of these results, we prove the following

**Theorem 1.** Let $G$ be a $p$-solvable group of order $p^a m$ ($a \geq 1$, $p \nmid m$). If $G$ has $p$-length 1, then the following are equivalent:

1. $G/O_{p'}(G)$ is abelian.
2. If $B$ is an arbitrary block ideal of $kG$, then each row-sum of $C_B$ is bounded above by $|\delta(B)|$.
3. If $B$ is an arbitrary block ideal of $kG$, then each row-sum of $C_B$ is equal to $|\delta(B)|$.
4. If $B_0$ is the principal block ideal of $kG$, then each row-sum of $C_{B_0}$ is bounded above by $p^a$.
5. If $B_0$ is the principal block ideal of $kG$, then each row-sum of $C_{B_0}$ is equal to $p^a$.

**Proof.** In view of [9, Satz 6.3] and [8, Theorem 5], it suffices to show that (4) implies (1).

Suppose that (4) holds. Since $G$ has $p$-length 1, $G/O_{p'}(G)$ has a normal Sylow $p$-subgroup. As is well known, $B_0$ is isomorphic to $kG/O_{p'}(G)$. Hence, we may assume that $O_{p'}(G) = 1$ and $G$ has a normal Sylow $p$-subgroup. Then $kG$ itself is the principal block ideal of $kG$. Now, let $(F_1, F_2, \ldots, F_s)$ be a full set of non-isomorphic irreducible $kG$-modules, where $F_i$ is a trivial $kG$-module. We put $f_i = \dim_k F_i$ and $u_i = \dim_k P_C(F_i) (1 \leq i \leq s)$. Then $p$ does not divide $f_i$ because $G$ has a normal Sylow $p$-subgroup. As is well known, $P_C(F_i)$ is isomorphic to a direct summand of $F_i \otimes_k P_C(F_i)$. Hence, by [4, Theorem 2B], we have $u_i = p^a f_i$ for all $i$. Now, we may assume that $f_s$ is a maximal one among $f_i$'s, and that $f_1 = f_{t+1} = \cdots = f_s$. Suppose that $f_s > 1$, and so $t > 1$. Let $c_{ij}$ be the $(i,j)$-entry of the Cartan matrix $C$ of $kG$ (the multiplicity of $F_j$ as a composition factor of $P_C(F_i)$). Then, by our assumption, there holds that

$$p^a f_i = u_i = \sum_{1 \leq j \leq s} c_{ij} f_i \leq (\sum_{1 \leq j \leq s} c_{ij}) f_i \leq p^a f_i \quad (t \leq l \leq s).$$

This implies that if $c_{il} \neq 0$ then $f_i = f_l$. Hence we have $c_{il} = 0$ provided $t \leq i \leq s$ and $1 \leq j \leq t - 1$. However, this is impossible, because $C$ is indecomposable. Hence $f_s = 1$. This implies that every irreducible $kG$-module has $k$-dimension 1, and hence $G/O_p(G)$ is abelian, proving (1).

Next, we prove the following
Proposition 2. Let $G$ be a $p$-solvable group, and $B$ a block ideal of $kG$. If every irreducible $B$-module has $k$-dimension a power of $p$, then each row-sum of $C_B$ is bounded above by $|\delta(B)|$.

Proof. Let $\{F_1, F_2, \cdots, F_s\}$ be a full set of non-isomorphic irreducible $B$-modules. Then, by assumption, $\dim_k P(CF_i) = p^a$ for all $i$, where $p^a$ is the order of a Sylow $p$-subgroup of $G$ ([4, Theorem 2B]). Now, we put $\dim_k F_i = p^{e_i} (1 \leq i \leq s)$. We may assume that $e_1$ is minimal among $e_i$'s. Let $c_{i,j}$ be the $(i,j)$-entry of $C_B$. Then we have

$$p^a = \dim_k P(CF_i) = \sum_{j=1}^s c_{i,j} \dim_k F_j \geq (\sum_{j=1}^s c_{i,j}) p^{e_1}.$$ 

Since $|\delta(B)| = p^{a-e_1}$, the above implies that

$$|\delta(B)| = p^a / p^{e_1} \geq \sum_{j=1}^s c_{i,j}$$

for all $i$, proving the assertion.

2. Let $H$ be a normal subgroup of a finite group $G$, and $b$ a block ideal of $kH$. We denote by $T_C(b)$ the inertial subgroup of $b$:

$$T_C(b) = \{ g \in G \mid g^{-1}fg = f \},$$

where $f$ is a central primitive idempotent of $kH$ such that $b = fhH$. Given an irreducible $b$-module $L$, we denote by $T_C(L)$ the inertial subgroup of $L$:

$$T_C(L) = \{ g \in G \mid L \otimes_{kH} g \cong L \text{ as } kH\text{-modules} \}.$$

One may remark that $T_C(L)$ is contained in $T_C(b)$. Now, let $\{g_i \mid 1 \leq i \leq t\}$ be a right transversal of $T_C(b)$ in $G$. Then $e = \sum_{i=1}^t g_i^{-1}fg_i$ is a central idempotent of $kG$. If $e = e_1 + e_2 + \cdots + e_m$ is the decomposition of $e$ into (orthogonal) central primitive idempotents of $kG$, then we say that each block ideal $e_i kG$ covers $b$.

Throughout the subsequent study in this section, we suppose that $[G:H] = p$. Our objective is to find some relationship between Cartan invariants of $b$ and those of a block ideal of $kG$ which covers $b$. We notice here that if $L$ is an irreducible $kH$-module then $T_C(L)$ is either $H$ or $G$.

At first, we prove the following

Lemma 3. Let $L$ be an irreducible $kH$-module. Then there holds the following:

(1) If $T_C(L) = H$, then $L^G$ is an irreducible $kG$-module and $P_C(L^G) \cong P_H(L)^G$.

(2) If $T_C(L) = G$, then there exists a unique (up to isomorphism) irreducible $kG$-module $W$ such that $W|_H \cong L$; and then $P_C(W) \cong P_H(L)^G$. 

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Proof. (1) It is well known that $L^g$ is an irreducible $kG$-module ([2, Chap. III. (2.11)]). Since $P_h(L)^g$ is a projective $kG$-module and 
$$P_h(L)^g/(P_h(L)I_h)^g \cong (P_h(L)/P_h(L)I_h)^g \cong L^g,$$
we see that $P_c(L)^g \cong P_h(L)^g$.

(2) It is well known that there exists a unique irreducible $kG$-module $W$ such that $W|_h \cong L$ ([2, Chap. III. (3.16)]) and that $P_h(L)^g$ is a projective indecomposable $kG$-module ([2, Chap. III. (3.13)]). Since $W$ is isomorphic to an irreducible submodule of $L^g$ and $L^g$ is isomorphic to a submodule of $P_h(L)^g$, $W$ is isomorphic to the socle of $P_h(L)^g$. This implies that $P_c(W) \cong P_h(L)^g$.

Now, let $L_1$, $L_2$, $V_1$ and $V_2$ be irreducible $kH$-modules such that $T_c(L_i) = H$ and $T_c(V_i) = G$ $(i = 1, 2)$. We put $M_i = L_i^g$. Further, we denote by $W_i$ an irreducible $kG$-module such that $W_i|_h \cong V_i$. Let $\sigma$ be an element of $G$ such that $(\sigma, \sigma^2, \cdots, \sigma^{n-1})$ is a right transversal of $H$ in $G$. Given a $k$-space $X$ and a positive integer $n$, we denote by $nX$ a direct sum of $n$ copies of $X$. Then, in virtue of Frobenius reciprocity theorem and the preceding lemma, we can easily see the next

Lemma 4. (1) $\text{Hom}_{kG}(P_c(M_1), P_c(M_2)) \cong \bigoplus_{i=0}^{n-1} \text{Hom}_{kH}(P_h(L_1), P_h(L_2) \otimes_{kH} \sigma^i).$

(2) $\text{Hom}_{kG}(P_c(M_1), P_c(W_1)) \cong \text{Hom}_{kH}(P_h(L_1), P_h(V_1)) \cong \text{Hom}_{kH}(P_h(L_1) \otimes_{kH} \sigma^i, P_h(V_1)).$

(3) $\text{Hom}_{kG}(P_c(W_1), P_c(M_1)) \cong \bigoplus_{i=0}^{n-1} \text{Hom}_{kH}(P_h(V_1), P_h(L_1) \otimes_{kH} \sigma^i) \cong \text{Hom}_{kH}(P_h(V_1), P_h(L_1)).$

(4) $\text{Hom}_{kG}(P_c(W_1), P_c(W_2)) \cong \text{Hom}_{kH}(P_h(V_1), P_h(V_2)).$

By [1, Theorem 54.16, we see that if $X$ and $Y$ are irreducible $kG$-modules, then $\dim_k \text{Hom}_{kG}(P_c(X), P_c(Y))$ is equal to the multiplicity of $X$ as a composition factor of $P_c(Y)$. Hence, the above gives a linkage between the Cartan invariants of $kG$ and those of $kH$.

The next can be proved by [2, Chap.V. (3.5)], [5, Proposition 4.2] and [7, Theorem 6.11].

Lemma 5. If $b$ is a block ideal of $kH$, then $b$ is covered by a unique block ideal $B$ of $kG$, and there holds the following:

(1) If $T_c(b) = H$ then $\delta(B) \subseteq \delta(b)$.
(2) If $T_c(b) = G$ then $\delta(B) \cap H \subseteq \delta(b)$ and $|\delta(b)| = \rho|\delta(b)|$.

Now, let $b$ be a block ideal of $kH$ such that $T_c(b) = H$. Then $T_c(L)$...
$= H$ for every irreducible $b$-module $L$. Let $B$ be a block ideal of $kG$
which covers $b$, and $M$ an irreducible $B$-module. Then $M|_{|H}$ is a completely reducible $kH$-module by Clifford’s theorem, and there exists a composition factor $L$ of $M|_{|H}$ belonging to $b$. Since $L_c$ is an irreducible $kG$-module (Lemma 3 (1)) and $\text{Hom}_{kG}(L_c, M) \cong \text{Hom}_{kH}(L, M|_{|H}) \neq 0$, we have $L_c \cong M$. Now, let $\{L_1, L_2, \cdots, L_s\}$ be a full set of non-isomorphic irreducible $b$-modules. We put $b_i = \sigma^{-i-1}b\sigma^{i-1}$, $L_{ji} = L_j \otimes_{kH{\sigma}^{i-1}} M_i = L_{ij}$ ($1 \leq i \leq p$; $1 \leq j \leq s$). Then, Lemmas 4 and 5 together with the above fact imply the following result which is a special case of [2, Chap. V. (2.5)].

**Proposition 6.** (1) $\{M_1, M_2, \cdots, M_s\}$ is a full set of non-isomorphic irreducible $B$-modules, and $\{L_{1i}, L_{2i}, \cdots, L_{si}\}$ is a full set of non-isomorphic irreducible $b_i$-modules ($1 \leq i \leq p$).

(2) $B$ and $b$ have the same Cartan matrix, and have a defect group in common.

Next, suppose that $b$ is a block ideal of $kH$ with $T_G(b) = G$. Then the inertial subgroup $T_G(L)$ of any irreducible $b$-module $L$ is either $H$ or $G$. Let $B$ be a block ideal of $kG$ which covers $b$. If $M$ is an irreducible $B$-module, then there exists a composition factor $L$ of a completely reducible $kH$-module $M|_{|H}$ belonging to $b$. If $T_G(L) = H$ then, as stated just before Proposition 6, $L_c$ is isomorphic to $M$. On the other hand, if $T_G(L) = G$ then $M|_{|H} \cong L$ by Lemma 3 (2). Now, let $\{L_{11}, \cdots, L_{1p}; \cdots; L_{r1}, \cdots, L_{rp}; V_1, V_2, \cdots, V_t\}$ be a full set of non-isomorphic irreducible $b$-modules, where $T_G(L_{1i}) = H$, $L_{ij} = L_{ij} \otimes_{kH{\sigma}^{i-1}} (1 \leq i \leq r; 1 \leq j \leq p)$ and $T_G(V_l) = G$ ($1 \leq l \leq t$). Put $M_l = L_{1l}$ ($1 \leq i \leq r$), and choose an irreducible $kG$-module $W_i$ such that $W_i|_{|H} \cong V_i$ ($1 \leq l \leq t$). Then $\{M_1, \cdots, M_r; W_1, \cdots, W_t\}$ is a full set of non-isomorphic irreducible $B$-modules. Given irreducible $kG$-modules $X$, $Y$ (resp. irreducible $kH$-modules $A$, $B$), we denote by $c_{XY}$ (resp. $\tilde{c}_{AB}$) the multiplicity of $Y$ (resp. $B$) as a composition factor of $P_G(X)$ (resp. $P_H(A)$). Then, by Lemma 4, we have the following

**Proposition 7.** (1) $c_{M_1M_1} = \sum_{i=1}^{p} c_{L_{1i}L_{1i}} = \sum_{i=1}^{p} \tilde{c}_{L_{1i}L_{1i}} = \cdots = \sum_{i=1}^{p} \tilde{c}_{L_{1p}L_{1i}} (1 \leq i, j \leq r)$.

(2) $c_{W_1M_1} = \sum_{i=1}^{p} c_{V_1L_{1i}} (1 \leq i \leq t; 1 \leq j \leq r)$.

(3) $\tilde{c}_{M_1W_1} = p\tilde{c}_{L_{11}V_1} = p\tilde{c}_{L_{12}V_1} = \cdots = p\tilde{c}_{L_{1p}V_1} (1 \leq i \leq r; 1 \leq j \leq t)$.

(4) $\tilde{c}_{W_1W_1} = p\tilde{c}_{V_1V_1} (1 \leq i, j \leq t)$.

We are now in a position to state the following
Theorem 8. Let \( N \) be a normal subgroup of \( G \) such that \( G/N \) is a \( p \)-group. Let \( b \) be a block ideal of \( kN \). If \( B \) is a block ideal of \( kG \) which covers \( b \), then there holds the following:

1. If each row-sum of \( C_b \) is bounded above by \( |\delta(b)| \), then each row-sum of \( C_B \) is bounded above by \( |\delta(B)| \).
2. The converse of (1) is true, provided one of the following conditions holds:
   (i) \( T_G(b) = H \).
   (ii) \( T_G(L) = G \) for every irreducible \( b \)-module \( L \).

Proof. (1) By Propositions 6, 7 and Lemma 5 (2).
(2) If (i) holds, then the converse is true by Proposition 6. On the other hand, if (ii) holds then \( C_B = [G:N]C_b \) by Proposition 7. Since \( |\delta(B)| = [G:N]|\delta(b)| \) (Lemma 5 (2)), the converse of (1) is also true.

The next is a combination of Theorems 1 and 8.

Corollary 9. Let \( G \) be a group such that \( G = O_{p'}pp(G) \) and \( O_{p'}pp(G)/O_{p'}pp(G) \) is abelian, and let \( B \) be a block ideal of \( kG \). Then each row-sum of \( C_B \) is bounded above by \( |\delta(B)| \).

3. Throughout this section, we assume that \( H \) is a normal subgroup of \( G \) with \( [G:H] = q \), a prime different from \( p \). We notice here that if \( L \) is an irreducible \( kh \)-module and \( T_G(L) \neq H \) then \( T_G(L) = G \). We establish first three lemmas which correspond to Lemmas 3, 4 and 5, respectively.

Lemma 10. Let \( L \) be an irreducible \( kh \)-module. Then there holds the following:

1. If \( T_G(L) = H \), then \( L^G \) is an irreducible \( kG \)-module and \( P_G(L^G) \cong P_H(L)^G \).
2. If \( T_G(L) = G \), then there exist \( q \) non-isomorphic irreducible \( kG \)-modules \( W_1, W_2, \ldots, W_q \) such that \( W_i|_H \cong L \); and then \( P_G(W_i)|_H \cong P_H(L) \).

Proof. (1) It is well known that \( L^G \) is an irreducible \( kG \)-module ([2, Chap. III, (2.11)]). It is also clear that \( P_H(L)^G \) is a projective \( kG \)-module. Noting that \( J_G = J_hkG \), we get

\[
P_H(L)^G/P_H(L)^GJ_G \cong (P_H(L)/P_H(L)J_H)^G \cong L^G.
\]

Hence, \( P_H(L)^G \cong P_G(L^G) \).

(2) The existence of such \( W_i \)'s is well known ([10, Lemma 1]). Observing \( J_G = J_hkG \), we get
Let $L_1$, $L_2$, $V_1$ and $V_2$ be irreducible $kH$-modules such that $T_G(L_i) = H$ and $T_G(V_i) = G$ $(i = 1, 2)$. We put $M_i = L_i^G$, and choose an irreducible $kG$-module $W_i$ such that $W_i|_H \cong V_i$. Let $r$ be an element of $G$ such that \{1, $\tau$, $\ldots$, $\tau^{q-1}$\} is a right transversal of $H$ in $G$.

**Lemma 11.** (1) $\varprojlim \text{Hom}_{kG}(P_G(M_1), P_G(M_2)) \cong \bigoplus_{i=0}^{\infty} \text{Hom}_{kH}(P_H(L_1), P_H(L_2) \otimes_{kH} \tau^i) \cong \bigoplus_{i=0}^{\infty} \text{Hom}_{kH}(P_H(L_1) \otimes_{kH} \tau^i, P_H(L_2))$.
(2) $\text{Hom}_{kG}(P_G(M_1), P_G(W_1)) \cong \text{Hom}_{kH}(P_H(L_1), P_H(V_1))$.
(3) $\text{Hom}_{kG}(P_G(W_1), P_G(M_1)) \cong \text{Hom}_{kH}(P_H(V_1), P_H(L_1))$.
(4) If $W_{11}$, $\ldots$, $W_{1q}$ (resp. $W_{21}$, $\ldots$, $W_{2q}$) are non-isomorphic irreducible $kG$-modules such that $W_{i1}|_H \cong V_1$ (resp. $W_{21}|_H \cong V_2$), and if $1 \leq l \leq q$, then
$$\sum_{i=1}^{q} \dim_k \text{Hom}_{kG}(P_G(W_{1i}), P_G(W_{2l})) = \dim_k \text{Hom}_{kH}(P_H(V_1), P_H(V_2)).$$

**Proof.** (1), (2) and (3) are clear by Frobenius reciprocity theorem and Lemma 10.
(4) Observing that $J_G = J_H kG$, we get
$$P_H(V_2) J_H^l / P_H(V_2) J_H^{l+1} \cong (P_G(W_{2i})|_H) J_H^l / (P_G(W_{2i})|_H) J_H^{l+1}$$
$$\cong (P_G(W_{2i}) J_G^l / P_G(W_{2i}) J_G^{l+1})|_H$$
where $r$ is an arbitrary non-negative integer and $1 \leq l \leq q$. This shows that the multiplicity of $V_1$ as a composition factor of $P_H(V_2) J_H^l / P_H(V_2) J_H^{l+1}$ coincides with that of $W_{i1}$ as a composition factor of $P_G(W_{2i}) J_G^l / P_G(W_{2i}) J_G^{l+1}$ $(1 \leq i \leq q)$. Since $\dim_k \text{Hom}_{kH}(P_H(V_1), P_H(V_2))$ (resp. $\dim_k \text{Hom}_{kG}(P_G(W_{1i}), P_G(W_{2i}))$) is equal to the multiplicity of $V_1$ (resp. $W_{1i}$) as a composition factor of $P_H(V_2)$ (resp. $P_G(W_{2i})$), the assertion (4) follows immediately.

The next is obvious by [5, Proposition 4.2].

**Lemma 12.** Let $B$ and $b$ be block ideals of $kG$ and $kH$, respectively. If $B$ covers $b$, then $B$ and $b$ have a defect group in common.

Now, we consider the case that the inertial subgroup $T_G(b)$ of a block ideal $b$ of $kH$ coincides with $H$.

**Lemma 13.** Let $b$ be a block ideal of $kH$. If $T_G(b) = H$ then $b$ is covered by a uniquely determined block ideal of $kG$.

**Proof.** Suppose that more than one block ideal of $kG$ covers $b$, and
let $B_1, B_2, \ldots, B_m$ be all such block ideals of $kG$. By an argument similar to that employed in the paragraph preceding Proposition 6, we see that if $M$ is an irreducible $B$-module then there exists an irreducible $b$-module $L$ such that $M \cong L^G$. So, we let $L_1, L_2, \ldots, L_m$ be irreducible $b$-modules such that $L_i^G$ belongs to $B_i$ $(1 \leq i \leq m)$. Now, if $L$ and $L'$ are irreducible $b$-modules such that $L^G$ and $L'^G$ belong to different block ideals of $kG$, then $\text{Hom}_{kG}(P_G(L^G), P_G(L'^G)) = 0$, and so $\text{Hom}_{kH}(P_H(L_i), P_H(L'_j)) = 0$ by Lemma 11 (1). Thus, we see that $\tilde{c}_{L_i,L_j} = 0$ for $i \neq j$, where $\tilde{c}_{L_i,L_j}$ is the multiplicity of $L_1$ as a composition factor of $P_H(L_i)$. But this is impossible, because $C_b$ is indecomposable. Hence $b$ is covered by a uniquely determined block ideal of $kG$.

Let $b$ be a block ideal of $kH$ such that $T_G(b) = H$, and $B$ a block ideal of $kG$ which covers $b$. Let $(L_1, L_2, \ldots, L_s)$ be a full set of non-isomorphic irreducible $b$-modules. Now, putting $b_i = \tau^{-(i-1)}bt_i^{-1}$, $L_{ji} = L_j \otimes_{kH} \tau_i^{-1}$ and $M_j = L_j^G (1 \leq i \leq q; 1 \leq j \leq s)$, by Lemmas 10—13 we get the following which is a special case of [2, Chap. V, (2.5)].

**Proposition 14.** (1) $(M_1, M_2, \ldots, M_s)$ is a full set of non-isomorphic irreducible $B$-modules, and $(L_1, L_2, \ldots, L_s)$ is a full set of non-isomorphic irreducible $b$-modules $(1 \leq i \leq q)$.

(2) $B$ and $b$ have the same Cartan matrix and have a defect group in common.

Next, suppose that $b$ is a block ideal of $kH$ with $T_G(b) = G$. Then, for any irreducible $b$-module $L$, $T_G(L)$ is either $H$ or $G$. Let $(B_1, B_2, \ldots, B_m)$ be a full set of block ideals of $kG$ covering $b$. We put $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$. If $M$ is an irreducible $B$-module, then there exists a composition factor $L$ of a completely reducible $kH$-module $M|_H$ belonging to $b$. If $T_G(L) = H$ then, as in the paragraph preceding Proposition 6, we see that $M \cong L^G$; and if $T_G(L) = G$ then $M|_H \cong L^G$ by Lemma 10 (2). Now, let $(L_1, \ldots, L_1; \ldots; L_r, \ldots, L_r; V_1, V_2, \ldots, V_t)$ be a full set of non-isomorphic irreducible $b$-modules, where $T_G(L_i) = H$, $L_{ij} = L_i \otimes_{kH} \tau_j^{-1}$ $(1 \leq i \leq r; 1 \leq j \leq q)$ and $T_G(V_l) = G (1 \leq l \leq t)$. We put $M_l = L_l^G (1 \leq i \leq r)$, and we let $(W_1, \ldots, W_t)$ be a full set of non-isomorphic irreducible $kG$-modules such that $W_l|_H \cong V_l (1 \leq l \leq t)$. Then $(M_1, M_2, \ldots, M_r; W_1, \ldots, W_1; \ldots; W_r, \ldots, W_t)$ is a full set of non-isomorphic irreducible $B$-modules. Further, according to Lemma 11, we can prove the following proposition which corresponds to Proposition 7.
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Proposition 15. (1) \( c_{M_iM_j} = \sum_{t=1}^{q} c_{L_1L_{t-1}} = \sum_{j=1}^{q} c_{L_{j-1}L_{j}} = \cdots = \sum_{r=1}^{q} c_{L_{r-1}L_{r}} \) (1 \( \leq i, j \leq r \)).

(2) \( c_{W_{i}M_j} = c_{W_{j}M_j} = \cdots = c_{W_{k}M_j} = c_{V_{t}M_j} = c_{V_{t}L_{t+1}} = c_{V_{t}L_{t+2}} = \cdots = c_{V_{t}L_{t+r}} \) (1 \( \leq i \leq t \); 1 \( \leq j \leq r \)).

(3) \( c_{M_iW_{j}} = c_{M_jW_{j}} = \cdots = c_{M_kW_{j}} = c_{L_1V_j} = c_{L_2V_j} = \cdots = c_{L_{r}V_j} \) (1 \( \leq i \leq r \); 1 \( \leq j \leq t \)).

(4) \( \sum_{i=1}^{r} c_{W_{i}W_{j}} = \sum_{j=1}^{q} c_{W_{j}W_{j}} = \cdots = \sum_{j=1}^{q} c_{W_{j}W_{j}} = c_{V_{i}V_{j}} \) (1 \( \leq i, j \leq t \)).

We are now in a position to state the following

Theorem 16. Let \( H \) be a normal subgroup of \( G \) with \( [G:H] = q \). Let \( b \) be a block ideal of \( kH \), and \( B \) a block ideal of \( kG \) which covers \( b \). Then there holds the following:

(1) Suppose that, for every irreducible \( b \)-module, its inertial subgroup coincides with \( G \). If each row-sum of \( C_b \) is bounded above by \( |\delta(b)| \), then that of \( C_B \) is bounded above by \( |\delta(B)| \).

(2) Suppose that, for every irreducible \( b \)-module, its inertial subgroup coincides with \( H \). Then, \( B \) is the unique block ideal covering \( b \), and the following statements are equivalent:

(i) Each row-sum of \( C_b \) is bounded above by \( |\delta(b)| \).

(ii) Each row-sum of \( C_B \) is bounded above by \( |\delta(B)| \).

Proof. (1) By Proposition 15.

(2) In the same way as in the proof of Lemma 13, we can see that \( b \) is covered uniquely by a block ideal of \( kG \), even if \( T_G(b) \) is different from \( H \). The rest of the assertion follows from Propositions 14 and 15.

4. In this section, we assume \( p = 3 \) and give a counterexample which shows that the converse of Theorem 8 (1) need not be true.

Let \( U = \langle u \rangle \times \langle v \rangle \) be an elementary abelian group of order \( 3^2 \). We look upon \( U \) as a vector space over GF(3). Then \( \text{SL}(2,3) \) acts naturally on \( U \). We denote by \( G \) a semi-direct product of \( U \) by \( \text{SL}(2,3) \) with respect to this action. We notice that \( |\text{SL}(2,3)| = 24 \) and a Sylow 2-subgroup \( Q \) of \( \text{SL}(2,3) \) is a quaternion group. We let \( Q = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle \), and denote by \( \langle s \rangle \) a Sylow 3-subgroup of \( \text{SL}(2,3) \). Then we may, and shall assume that \( G = \langle u, v, a, b, s \rangle \) and

\[
a^{-1}ua = u^2 v, \ b^{-1}ub = uv, \ a^{-1}va = uv, \ b^{-1}vb = uv^2.
\]
\[
s^{-1}us = uv, \ s^{-1}vs = v, \ s^{-1}as = b, \ s^{-1}bs = ba.
\]

In what follows, we put \( X = U \langle a^2 \rangle \), \( Y = U \langle a \rangle \) and \( H = UQ \). Now, by
making use of Propositions 7 and 15, we shall determine the Cartan matrices of $kH$ and $kG$.

To begin with, we shall determine the Cartan matrix of $kX$. Put $\varepsilon_1 = -(1 + a^2)$ and $\varepsilon_2 = -1 + a^2$. Then $1 = \varepsilon_1 + \varepsilon_2$ is a decomposition of 1 into orthogonal primitive idempotents in $kX$. By a brief computation, we can see that $\{e_i, e_iue_i, e_iuve_i, e_iuve^2_i\}$ is a $k$-basis of $e_i kX e_i$ ($i = 1, 2$) and that $\{e_1ue_2, e_1ve_2, e_1uve_2, e_1uve^2_2\}$ is a $k$-basis of $e_1 kX e_2$. Hence, we have

**Lemma 17.** *The Cartan matrix of $kX$ is given by $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$.*

Next, we shall determine the Cartan matrix of $kY$. Put $e_1 = 1 + a + a^2 + a^3$, $e_2 = 1 - a + a^2 - a^3$, $e_3 = 1 + \xi a - a^2 - \xi a^3$ and $e_4 = 1 - \xi a - a^2 + \xi a^3$, where $\xi$ is a primitive 4-th root of 1 in $k$. Then $1 = e_1 + e_2 + e_3 + e_4$ is a decomposition of 1 into orthogonal primitive idempotents in $kY$. Put $L_i = e_i kX/e_i J_X$ ($i = 1, 2$) and $M_j = e_j kY/e_j J_Y$ ($1 \leq j \leq 4$). Then it is easy to see that $T_Y(L_i) = T_Y(L_2) = Y, M_1|_X \cong M_2|_X \cong L_1$ and $M_3|_X \cong M_4|_X \cong L_2$. By Lemma 17 and Proposition 15, we get the following:

- $C_{M_1 M_1} + C_{M_1 M_2} = C_{M_2 M_1} + C_{M_2 M_2} = 5$.
- $C_{M_1 M_3} + C_{M_1 M_4} = C_{M_2 M_3} + C_{M_2 M_4} = 4$.
- $C_{M_3 M_1} + C_{M_3 M_2} = C_{M_4 M_1} + C_{M_4 M_2} = 4$.
- $C_{M_3 M_3} + C_{M_3 M_4} = C_{M_4 M_3} + C_{M_4 M_4} = 5$.

On the other hand, we can see that $\{e_i, e_iue_i, e_iuve_i\}$ is a $k$-basis of $e_i kYe_i$ ($i = 1, 3$) and $\{e_1ue_3, e_1ue_3\}$ is a $k$-basis of $e_1 kYe_3$. Thus, $C_{M_1 M_1} = C_{M_3 M_3} = 3$ and $C_{M_1 M_2} = 2$. Noting here that the Cartan matrix is symmetric, we get $C_{M_i M_i} = 3$ ($1 \leq i \leq 4$) and $C_{M_i M_j} = 2$ ($1 \leq i \neq j \leq 4$).

**Lemma 18.** *The Cartan matrix of $kY$ is given by $\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$.*

Now, we determine the Cartan matrix of $kH$. Put

- $f_1 = -(1 + a + a^2 + a^3)(1 + b)$,
- $f_2 = -(1 + a + a^2 + a^3)(1 - b)$,
- $f_3 = -(1 - a + a^2 - a^3)(1 + b)$,
- $f_4 = -(1 - a + a^2 - a^3)(1 - b)$,
- $f = -(1 - a^2)$.

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Noting that $kH/U \cong kQ$, we see that $f_1, f_2, f_3$ and $f_4$ are orthogonal primitive idempotents of $kH$ and $f$ can be decomposed into two orthogonal primitive idempotents of $kH$, say $f_5$ and $f_6$. Thus, $1 = f_1 + f_2 + f_3 + f_4 + f_5 + f_6$ is a decomposition of $1$ into orthogonal primitive idempotents in $kH$. Let $N_i = f_i kH / f_i J_H$ ($1 \leq i \leq 6$). Then, it is easy to see that $T_H(M_1) = T_H(M_2) = H$, $T_H(M_3) = T_H(M_4) = Y$, $N_1|_Y \cong N_2|_Y \cong M_1$, $N_3|_Y \cong N_4|_Y \cong M_2$ and that $N_5 \cong N_6 \cong M_5^H \cong M_6^H$. Now, Proposition 15 together with Lemma 18 yields the following:

\[
\begin{align*}
C_{N_1N_2} &= C_{N_1N_1} + C_{N_2N_2} = 3, \\
C_{N_1N_3} &= C_{N_1N_1} + C_{N_3N_3} = 2, \\
C_{N_1N_4} &= C_{N_1N_1} + C_{N_4N_4} = 2, \\
C_{N_2N_3} &= C_{N_2N_2} + C_{N_3N_3} = 2, \\
C_{N_2N_4} &= C_{N_2N_2} + C_{N_4N_4} = 2, \\
C_{N_3N_4} &= C_{N_3N_3} + C_{N_4N_4} = 2, \\
C_{N_5N_5} &= 5.
\end{align*}
\]

On the other hand, we can see that $\{f_i, f_i uf_i\}$ is a $k$-basis of $f_i kH f_i$ ($i = 1, 3$) and $\{f_i uf_i\}$ is a $k$-basis of $f_i kH f_i$. Hence, $C_{N_1N_1} = C_{N_3N_3} = 2$ and $C_{N_1N_3} = 1$.

Now, we can find all the Cartan invariants of $kH$ as in the next lemma.

\begin{center}
Lemma 19. The Cartan matrix of $kH$ is given by \[
\begin{pmatrix}
2 & 1 & 1 & 1 & 2 \\
1 & 2 & 1 & 2 & 1 \\
1 & 1 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 5
\end{pmatrix}.
\]
\end{center}

In conclusion, we determine the Cartan matrix of $kG$. It is easy to see that $T_G(N_1) = G$ and $T_G(N_2) = T_G(N_3) = T_G(N_4) = H$. Now, suppose that $T_G(N_5) = H$. Then $N_5 \otimes_{kH} S$ and $N_5 \otimes_{kH} S^2$ are non-isomorphic irreducible $kH$-modules. But this is impossible, because $N_5$ is the only one (up to isomorphism) irreducible $kH$-module with $k$-dimension $2$. Hence $T_G(N_5) = G$. Thus, we see that $f_1 kG$, $f_2 kG$ and $f_3 kG$ are non-isomorphic projective indecomposable $kG$-modules (Lemma 3). Putting $F_1 = f_1 kG / f_1 J_G$, $F_2 = f_2 kG / f_2 J_G$ and $F_3 = f_3 kG / f_3 J_G$, we see that $F_1|_H \cong N_1$, $F_3|_H \cong N_5$ and $F_2 \cong N_5^2 \cong N_5 \cong N_4^2$. Hence, by Lemma 19 and Proposition 7, we can get the Cartan matrix of $kG$.

\begin{center}
Theorem 20. The Cartan matrix of $kG$ is given by \[
\begin{pmatrix}
6 & 3 & 6 \\
3 & 4 & 6 \\
6 & 6 & 15
\end{pmatrix}.
\]
\end{center}
Obviously, each row-sum of the Cartan matrix of $kG$ is not greater than 27, the order of a Sylow 3-subgroup of $G$. However, the 5-th row-sum of the Cartan matrix of $kH$ exceeds 9, the order of a Sylow 3-subgroup of $H$.

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SHINSHU UNIVERSITY AND OKAYAMA UNIVERSITY

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