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CONDITIONS FOR ELEMENTS TO BE CENTRAL IN CERTAIN RINGS

MOHAMED N. DAIF and MOHAMED H. FAHMY

Recently, many papers, [3], [4], [7], [9], [10] and others, have been devoted to studying the conditions which force an element of a ring to be central. In this paper, we continue the investigation on such conditions, and partially generalize the results of Awtar [2], Mogami and Hongan [10] and Felzenszwalb [3].

Throughout this paper R is an associative ring with 1, and Z the center of R . Given a subset M of R , $r(M)$ and $l(M)$ denote the right annihilator and the left annihilator of M in R , respectively.

Now, let a and x be elements of R , and consider the following conditions :

(i) $[ax, xa] \in Z$.

(ii) There exists an integer $n(x) > 1$ such that

$$(xa)^k - x^k a^k \in Z, \quad k = n(x), n(x)+1, n(x)+2.$$

The main theorems of this paper are stated as follows :

Theorem 1. *Let R be a semiprime ring, and U an ideal of R with $l(U) = 0$. If (i) is satisfied for all $x \in U$, then a is central.*

Theorem 2. *Let R be a prime ring with no non-zero nil left ideals and $2R \neq 0$. If (ii) is satisfied for all $x \in R$, then a is central.*

In preparation for proving our theorems, we establish the following lemmas.

Lemma 1. *Let R be a division ring. If (i) or (ii) is satisfied for all $x \in R$, then a is central.*

Proof. Suppose, to the contrary, $a \notin Z$. If (i) is satisfied then $[x, a^{-1}xa] \in Z$ for all $x \in R$. Next, assume that (ii) is satisfied. Since

$$x[x^{k-1}a^{k-1}, ax]a = [x^k a^k, xa] = [x^k a^k - (xa)^k, xa] = 0,$$

we get $[x^{k-1}a^{k-1}, ax] = 0$, and hence

$$[(xa)^{k-1}, ax] = [(xa)^{k-1} - x^{k-1}a^{k-1}, ax] = 0, \quad k = n(x)+1, n(x)+2.$$

From the last, we obtain $(xa)^n [xa, ax] = [(xa)^{n+1}, ax] - [(xa)^n, ax]xa = 0$.

This proves that $[xa, ax] = 0$, and so $[a^{-1}xa, x] = 0$. Thus, in any case, $[x, a^{-1}xa] \in Z$. Since the inner automorphism effected by a is non-trivial, R is commutative by [11, Remark 2]. This contradiction shows that $a \in Z$.

Lemma 2. *Let R be a primitive ring, and U a non-zero ideal of R . If (i) or (ii) is satisfied for all $x \in U$, then a is central.*

Proof. Since R is primitive, it has a faithful irreducible module V which can also be regarded as a faithful irreducible U -module. By Density Theorem, U acts densely on V as a vector space over a division ring Δ . If $\dim_{\Delta} V = 1$ then R is a division ring, and $a \in Z$ by Lemma 1. So, we assume henceforth that $\dim_{\Delta} V > 1$. Let v be a non-zero vector in V . We claim that v and va are linearly dependent. First, consider the case of (i). Suppose that v, va, va^2 are linearly independent. By Density Theorem, there exist $u_1, u_2 \in U$ such that $vu_1 = v, (va)u_1 = v, (va^2)u_1 = 0; vu_2 = 0, (va)u_2 = -v, (va^2)u_2 = va$. Thus, $0 = v[[u_1a, au_1], u_2] = v$, which contradicts the choice of v . So we may assume that $va^2 = \beta v + \gamma va$, where $\beta, \gamma \in \Delta$. If v and va are linearly independent, then there are $u'_1, u'_2 \in U$ such that $vu'_1 = v, (va)u'_1 = v; vu'_2 = 0, (va)u'_2 = -v$, which yields again $v = 0$. This proves that v and va are linearly dependent. Next, assume that (ii) is satisfied for all $x \in U$. If v and va are linearly independent then there are $w_1, w_2 \in U$ such that $vw_1 = 0, (va)w_1 = v; vw_2 = va, (va)w_2 = v$. Thus,

$$0 = v[(w_1a)^k - w_1^k a^k, w_2] = -va \quad \text{for } k = n(w_1).$$

This contradiction shows that v and va are linearly dependent. Thus, in any case, for every $v \in V$ we have $va = \lambda(v)v$, where $\lambda(v) \in \Delta$. Noting that $\dim_{\Delta} V > 1$, we can easily see that $\lambda(v)$ does not depend on v , i.e., $va = \lambda v$ for all $v \in V$. Now, if $u \in U$, we have $(vu)a = \lambda vu = (va)u$. Thus $V[u, a] = 0$, that is $[u, a] = 0$ for all $u \in U$. Now, by [8, Lemma 1.1.6], we get $a \in Z$.

Lemma 3. *Let R be a semiprimitive ring, and U a two-sided ideal of R with $l(U) = 0$. If (i) or (ii) is satisfied for all $x \in U$, then a is central.*

Proof. Divide the set of all primitive ideals of R into two parts: let \mathcal{P}_1 be the set of those which contain U , and \mathcal{P}_2 the set of those which do not. Let $U_1 \equiv \bigcap_{P \in \mathcal{P}_1} P$, and $U_2 \equiv \bigcap_{P \in \mathcal{P}_2} P$. Then $U_1 \cap U_2 = 0$. Since $U_2 \subseteq l(U_1) \subseteq l(U) = 0$, R is a subdirect sum of R/P ($P \in \mathcal{P}_2$). Obviously, $(U+P)/P$ is non-zero for every $P \in \mathcal{P}_2$. Hence, by Lemma 2, $[a, x] \in \bigcap_{P \in \mathcal{P}_2} P = 0$ for all $x \in R$.

Corollary 1. *Let R be a semiprimitive ring. If (i) or (ii) is satisfied for all $x \in R$ then a is central.*

Lemma 4. *Let R be a ring without non-zero nil left ideals, and $a \in R$. Suppose that for each $x \in R$ there exists $n = n(x) > 1$ such that $(xa)^n - x^n a^n \in Z$.*

(1) *Let $b \in l(a)$. If $x, y \in R$ and $xy = 0$ then $xa^k by = 0$ for all $k > 1$.*

(2) $l(a) = \{x \in R \mid xa^m = 0 \text{ for some } m \geq 1\}$.

(3) $r(a) = l(a)$.

Proof. (1) For any $r \in R$, there exists $n > 1$ such that $(yrxa)^n = (yrxa)^n - (yrx)^n a^n \in Z$, and therefore $(yrxa)^{n+1} = [(yrxa)^n, yrx]a = 0$. Thus, $Rxay$ is a nil left ideal, and so we get $xay = 0$. Since $(ab)^2 = 0$, $t = 1 + ab$ is invertible and $tat^{-1} = a - a^2b$. By the above, $xay = 0 = t^{-1}xt \cdot a \cdot t^{-1}yt$, and so we get

$$xa^2by = -x(a - a^2b)y = -t(t^{-1}xt \cdot a \cdot t^{-1}yt)t^{-1} = 0.$$

Now, it is easy to see that $xa^k by = 0$ for all $k > 1$.

(2) It suffices to show that $xa^m = 0$ ($m > 1$) yields $xa^{m-1} = 0$. Putting $y = xa^{m-2}$, we have $ya^2 = 0$. For any $r \in R$, there exists $n > 1$ such that $(rya)^n = (rya)^n - (ry)^n a^n \in Z$, and therefore $(rya)^{n+1} = ry[a, (rya)^n] = 0$. Since R has no non-zero nil left ideals, we get $xa^{m-1} = ya = 0$.

(3) Let $x \in r(a)$, and $n = n(x)$. Since $x^n a^n = x^n a^n - (xa)^n \in Z$, there holds $x^{n+1} a^n = [x, x^n a^n] = 0$, whence it follows $x^{n+1} a = 0$ by (2). We consider the right ideal V consisting of all $v \in R$ such that $r^{n(r)} v = 0$ for all $r \in r(a)$. If $r \in r(a)$ and $r^2 = 0$ then, for any $v \in V$, $(rv)^m v = 0$ and $(r + rv)^m v = 0$, where $m = \max\{n(rv), n(r + rv)\}$. Expanding the last equation, we get $(rv)^m = 0$. Then, since R contains no nil right ideals either, it follows that $rV = 0$. Let y be an arbitrary element of V . Since $(x^n yx)^2 = 0$ and $x^n yx \in r(a)$, by what was just proved above, there holds $x^n yxy = 0$. Repeating the above argument, we obtain $x^{n-1} yxyxyxy = 0$, and eventually $(xy)^{1+2+\dots+2^{n-1}+1} = 0$. Thus, xV is a nil right ideal, and hence zero; in particular, $xa = 0$. This proves $r(a) \subseteq l(a)$.

To prove the converse inclusion, let $b \in l(a)$. Let x be an arbitrary element of R , and $n = n(x)$. Then

$$0 = [x^n a^n - (xa)^n, xa] = [x^n a^n, xa] = x(x^{n-1} a^n xa - ax^n a^n).$$

By (1),

$$0 = xa^{n+2} b(x^{n-1} a^n xa - ax^n a^n) = xa^{n+2} bx^{n-1} a^n xa,$$

and therefore again by (1)

$$(bxa^{n+2})^{n+1} = bxa^{n+2}b(xa^{n+2}b)^{n-2}xa^n \cdot a^2b \cdot xa \cdot a^{n+1} = 0.$$

Recalling that R contains no non-zero nil left ideals, we obtain $l(a)xa^{n+2} = 0$, and therefore $l(a)xa = 0$ by (2). Then, since $(xab)^2 = 0$, we see that Rab is a nil left ideal, and hence $ab = 0$, which proves $l(a) \subseteq r(a)$.

We are now ready to complete the proofs of our theorems.

Proof of Theorem 1. As in the proof of Lemma 3, we see that there exist prime ideals P_a of R such that $\bigcap_a P_a = 0$ and $U \not\subseteq P_a$. Thus, without loss of generality, we may assume that R is a prime ring and U is a non-zero ideal of R . Let $S = RC$ be the central closure of R , where C is the extended centroid of R (see [8, pp. 20–31]). Let $V = UC$. Noting that for any $u_1, u_2 \in U$,

$$[ax_1, x_2a] + [ax_2, x_1a] = [a(x_1 + x_2), (x_1 + x_2)a] - [ax_1, x_1a] - [ax_2, x_2a] \in Z,$$

we can easily see that the condition (i) carries over to S with respect to the ideal V . Suppose now that a is not in Z , and choose $u \in U$ such that $ua \neq 0$. Then, S satisfies the generalized polynomial identity $[[x_1ua, ax_1u], x_2] = 0$, and a theorem of Martindale [8, Theorem 1.3.2] shows that S is primitive, and therefore $a \in Z(S)$ by Lemma 2. Hence, $a \in Z$, a contradiction.

Corollary 2. (1) *Let R be a ring without non-zero nil ideals. If for each $x \in R$ there exists an integer $m = m(x) > 1$ such that $[x^m y, yx^m] \in Z$ for all $y \in R$, then R is commutative.*

(2) *Let R be a semiprime ring. If for each $x \in R$ there exists a polynomial $p(t) = p_x(t)$ with integer coefficients such that*

$$[(x - x^2 p(x))y, y(x - x^2 p(x))] \in Z \quad \text{for all } y \in R,$$

then R is commutative.

Proof. (1) Since R is a subdirect sum of prime rings without non-zero nil ideals, it is enough to prove the assertion for prime rings without non-zero nil ideals. Then, by Theorem 1, $x^m \in Z$. Hence, by [5, Theorem 5], R is commutative.

(2) By Theorem 1 and [6, Theorem 19].

Proof of Theorem 2. We may assume that a is non-zero. Furthermore, by Corollary 1, we may assume that the Jacobson radical J of R is non-zero. Let x be an arbitrary element of J . Then both $1+x$ and $1-x$ are units in R . Since R is prime and $r(a) = l(a)$ by Lemma 4 (3), a is regular. Thus,

we can use the same argument as in the proof of Lemma 1 to show that

$$[a,ax] + [xa,a] + [xa,ax] = [(1+x)a, a(1+x)] = 0, \text{ and} \\ -[a,ax] - [xa,a] + [xa,ax] = 0.$$

Since $2R \neq 0$, the last two equations give $[xa,ax] = 0$. Thus from Theorem 1 it follows that $a \in Z$.

As an easy consequence of Corollary 1 and Theorem 2, we have

Corollary 3. *Let R be either a semiprimitive ring or a 2-torsion free prime ring with no non-zero nil left ideals.*

(1) *If for every $x, y \in R$ there exist integers $m = m(x) > 1$ and $n = n(x,y) > 1$ such that*

$$(yx^m)^k - y^k x^{mk} \in Z, \quad k = n, n+1, n+2,$$

then R is commutative.

(2) *If for every $x, y \in R$ there exists a polynomial $p(t) = p_x(t)$ with integer coefficients and an integer $n = n(x,y) > 1$ such that*

$$(y(x - x^2 p(x)))^k - y^k (x - x^2 p(x))^k \in Z, \quad k = n, n+1, n+2,$$

then R is commutative.

We conclude this paper with the following remark.

Remark. Careful scrutiny of the proof of Lemma 4 shows that, in case $n(x)$ are bounded, the lemma turns out to be true if we just assume that R has no non-zero nil left ideals of bounded index. By a result of Levitzki, every semiprime ring has no non-zero nil left ideals of bounded index. Hence, in case (ii) is satisfied for all x and $n(x)$ are bounded, Theorem 2 is still true without the hypothesis that R has no non-zero nil left ideals.

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