On rings satisfying the identity
\[(x+x^2+\ldots+x^n)(n) = 0\]

Yasuyuki Hirano\(\ast\) Hisao Tominaga\(\dagger\)
Adil Yaqub\(\ddagger\)

\(\ast\)Okayama University
\(\dagger\)Okayama University
\(\ddagger\)University Of California
ON RINGS SATISFYING THE IDENTITY
\[(x + x^2 + \cdots + x^n)^{(n)} = 0\]

YASUYUKI HIRANO, HISAO TOMINAGA and ADIL YAQUB

Throughout the present paper, \( R \) will represent a ring with center \( C \). Let \( N \) be the set of nilpotents in \( R \), \( N^* \) the subset of \( N \) consisting of all \( a \) with \( a^2 = 0 \), \( E \) the set of idempotents in \( R \), and \( D \) the commutator ideal of \( R \). For \( x \in R \), we define inductively \( x^{(1)} = x \), \( x^{(k)} = x^{(k-1)} + x \), where \( x \cdot y = x + y + xy \). We may write formally \( x^{(k)} = (1 + x)^k - 1 \).

Let \( n \) be a positive integer, and consider the following properties:

(i) \( (x + x^2 + \cdots + x^n)^{(n)} = 0 \) for all \( x \in R \).

(ii) \( N \) is commutative.

(ii)* \( N^* \) is commutative.

(iii) \( [[a,x],x] = 0 \) for all \( a \in N \) and \( x \in R \).

(iv) If \( a \in N \), \( x \in R \) and \( [a,x]^2 = 0 \) then \( [a,x] \in C \).

(*) For any \( x \), \( y \in R \), \( (x + xy)^n(y + yx) = 0 \) if and only if \( x = y \).

If \( R \) has 1, then (i) becomes

(i) \( (1 + x + x^2 + \cdots + x^n)^n = 1 \) for all \( x \in R \).

The present objective is to prove the following theorems.

**Theorem 1.** Let \( R \) be a left s-unital ring satisfying (i). If \( R \) is normal (i.e., \( E \) is central) then \( N \) is a nil ideal and \( R/N \) is commutative.

**Theorem 2.** Let \( R \) be a left s-unital ring satisfying (i).

1. If \( R \) satisfies (ii)*, then \( N \) is a nil ideal and \( R/N \) is a commutative regular ring.

2. If \( R \) satisfies (ii), then \( N \) is a commutative nil ideal and \( R/N \) is a commutative regular ring.

3. If \( R \) satisfies (ii) and (iii) (or (iv)), then \( R \) is commutative and \( R/N \) is a regular ring.

**Theorem 3.** Let \( R \) be a left s-unital ring satisfying (i). Then \( N \) is a nil ideal and \( R = R_1 \oplus R_2 \), where \( R_1 \) is either 0 or a commutative regular ring of odd characteristic, \( R_2 \cong N \) and \( R_2/N \) is a Boolean ring.

**Theorem 4.** If \( R \) is a normal, left s-unital ring satisfying (i) and...
(ii), then \( R \) is commutative and \( R = R_1 \oplus R_2 \), where \( R_1 \) is either 0 or a regular ring of odd characteristic, \( R_2 \cong N \) and \( R_2/N \) is a Boolean ring.

**Theorem 5** (cf. [4, Theorem 2]). A left \( s \)-unital ring \( R \) satisfies \((\ast)\) if and only if (a) \( R \) is commutative and \( R/N \) is a Boolean ring and (b) \( a^{2^n} = 0 \) for all \( a \in N \).

We start with the following lemma.

**Lemma 1.** Suppose that \( R \) satisfies \((i)_n\) and \( p^aR = 0 \), where \( p \) is a prime. Then there exists a positive integer \( m \) such that \( x^m = x^{2m} \) for all \( x \in R \).

**Proof.** Let \( y = x + x^2 + \cdots + x^n \), and \( n = p^a t \), where \( a \geq 0 \) and \( (p, t) = 1 \). Then there exists \( u(\lambda) \in \mathbb{Z}[\lambda] \) such that \( y^{(t)} = (1 + y)^{(t)} - 1 = tx + x^2 u(x) \). Noting here that \( (pR)^x = 0 \), we can easily see that \( (tx + x^2 u(x))^{p^a} = 0 \). Because \( (t, p) = 1 \), we readily obtain \( x^{p^a} - x^{p^a+1} f(x) = 0 \) with some \( f(\lambda) \in \mathbb{Z}[\lambda] \). Now, by making use of the argument employed in the proof of [1, Lemma], we can find a positive integer \( m \) such that \( x^m = x^{2m} \) for all \( x \in R \).

**Remark 1.** (1) Recently, Komatsu [5] proved that a ring \( R \) with 1 satisfies a polynomial identity \( x^{2m} - x^m = 0 \) for some positive integer \( m \) if and only if the addition of \( R \) is equationally definable in terms of the multiplication and the successor operation.

(2) Let \( p \) be a prime. If \( (1+n)^n \equiv 1 \pmod{p} \) and \( p-1 \mid n \), then \( GF(p) \) satisfies \((i)_n\). For instance, \( GF(3) \) satisfies \((i)_n\).

(3) Let \( R \) be a left \( s \)-unital ring satisfying \((i)_n\). Let \( x \) be an arbitrary non-zero element of \( R \), and choose \( e \in R \) such that \( ex = x \). Then, by \((i)_n\),

\[
0 = ((1+e+e^2+\cdots+e^n)^n - 1)x = ((n+1)^n - 1)x.
\]

This means that the characteristic of \( R \) is non-zero. Furthermore, \( n \) has to be even. In fact, if \( n \) is odd, then

\[
0 = ((1-e+e^2-\cdots-e^n)^n - 1)x = -x,
\]

which is a contradiction. In particular, if \( p \) is a prime and \( n = p^b \) then \( p = 2 \).

**Proof of Theorem 1.** By Remark 1 (3), the characteristic of \( R \) is non-zero. Obviously, the hypothesis \((i)_n\) and the normality are inherited by subrings, so we may assume that the additive group of \( R \) is a \( p \)-group, and
therefore $p^nR = 0$ for some prime $p$. Then, by Lemma 1, there exists a positive integer $m$ such that $x^m = x^{2m}$ for all $x \in R$, and so $R$ satisfies the polynomial identity $[x^m, y] = 0$. Since $[E_{11}, E_{11} + E_{12}] = E_{12} \neq 0$ in $(GF(q))^2$ ($q$ a prime), $D$ is a nil ideal by [3, Proposition 2].

Remark 2. For $e \in E$, the following are equivalent:

1) $e \in C$.

2) $[e, a] = 0$ for all $a \in N^*$.

3) $[e, [e, a]] = 0$ for all $a \in N^*$.

4) $[(e + a)e, e(e + a)] = 0$ for all $a \in N^*$.

5) $[e, f] = 0$ for all $f \in E$.

In fact, it is clear that 1) implies 2) – 5), and 2) does 3). In order to see that each of 3) – 5) implies 1), given $x \in R$, we set $a = ex(1 - e) \in N^*$. It is easy to see that if 3) or 4) is satisfied then $a = 0$, i.e., $ex = exe$. Similarly, we can see that $xe = exe$. Finally, if 5) is satisfied, then $e + a \in E$ and $e + a = e(e + a) = (e + a)e = e$, whence it follows again $a = 0$, i.e., $ex = exe$; similarly $xe = exe$. In particular, $R$ is normal if and only if $E$ is commutative. and (iii) implies the normality of $R$. Furthermore, if $N^*$ is central then $R$ is normal and $N$ coincides with the prime radical of $R$. In fact, if $a^n = 0$ and $a^{n-k}(Ra)^{k+1} = 0$ then $(a^{n-k-1}(Ra)^{k+1})^2 \subseteq a^{n-k-1}Ra^{n-k}(Ra)^{k+1} = 0$. Hence $a^{n-k-1}(Ra)^{k+1} \subseteq C$, and so $a^{n-k-1}(Ra)^{k+2} = Ra^{n-k}(Ra)^{k+1} = 0$. We get eventually $(Ra)^{n+1} = 0$.

In advance of proving Theorem 2, we state the next lemma.

Lemma 2. (1) If $R$ satisfies (ii)* then $R/P$ is normal, where $P$ is the prime radical of $R$.

(2) If a $\pi$-regular ring $R$ satisfies (ii)*, then $N$ coincides with the Jacobson radical of $R$ and $R/N$ is strongly regular.

Proof. (1) Let $\bar{e}$ be an arbitrary idempotent of $\bar{R} = R/P$. Since $P$ is a nil ideal, we may assume from the beginning that $e$ is in $E$. By hypothesis, $eR(1 - e) = (1 - e)R = (1 - e)R$, and so $eR(1 - e)Re = 0$. Hence $e\bar{R}(1 - e)\bar{R}e = 0$. By the semiprimeness of $\bar{R}$, we get $e\bar{R}(1 - e) = 0$, and therefore $e\bar{x} = \bar{x}e$ for all $x \in R$. Furthermore, $e\bar{R}(1 - e)\bar{R} = 0$ yields $(1 - e)\bar{R}e = 0$, and therefore $\bar{x}e = e\bar{x}$ for all $x \in R$. Thus, we have seen that $\bar{e}$ is central.

(2) Let $J$ be the Jacobson radical of the $\pi$-regular ring $R$. Obviously, $J/P$ is a nil ideal of $R/P$. Since $R/J = (R/P)/(J/P)$ and $R/P$ is normal by (1), it is easy to see that $R/J$ is normal and any nilpotent element of
$R/J$ generates a nil right ideal. Hence, $R/J$ is reduced and $N$ coincides with $J$. The reduced $\pi$-regular ring $R/N$ is strongly regular (see, e.g. [2]).

**Proof of Theorem 2.** (1) As in the proof of Theorem 1, we may assume that $p^a R = 0$ with some prime $p$. Then, by Lemma 1, there exists a positive integer $m$ such that $x^m = x^{2m}$ for all $x \in R$. Hence, by Lemma 2 (2), $N$ coincides with the Jacobson radical of $R$ and $R/N$ satisfies the identity $x - x^{m+1} = 0$. By Jacobson's commutativity theorem, $R/N$ is commutative.

(2) This is clear by (1).

(3) By the proof of (1) and [6, Theorem 1], $R$ is commutative.

**Lemma 3.** If $R$ satisfies $(i)_{2^a}$ and $2^a R = 0$, then $N$ is a nil ideal and $R/N$ is a Boolean ring.

**Proof.** Let $n = 2^k$. Obviously, $y = x + x^2 + \cdots + x^n \in N$, and so $x - x^{n+1} = y - xy \in N$. Then, noting that $x(1-x)^n - (x - x^{n+1}) \in 2R \subseteq N$, we readily see that $x - x^2 = x(1-x) \in N$. Now, we claim that for any prime $q$, $(\text{GF}(q))_2$ fails to satisfy $(i)_n$. If $q \neq 2$ then $x = E_{12}$ does not satisfy $(i)_n$. On the other hand, if $(\text{GF}(2))_2$ satisfies $(i)_n$ then, as we have seen just above, $x - x^2$ is nilpotent for all $x \in (\text{GF}(2))_2$. But this is not true for $x = E_{11} + E_{12} + E_{21}$. Thus, by [3, Proposition 2], $D$ is a nil ideal, and so $R/N$ is a Boolean ring.

**Proof of Theorem 3.** By Remark 1 (3), $hR = 0$ for some positive integer $h = 2^a' h'$, $(2, h') = 1$. Then, $R = R_1 \bigoplus R_2$, where $h'R_1 = 0$ and $2^a R_2 = 0$. If $a$ is an element of $R_1$ with $a^2 = 0$ then, by $(i)_{2^a}$, we can easily see that $2^a a = 0$, and therefore $a = 0$. This proves that $R_1$ is a reduced ring and $N \subseteq R_2$. Now, the assertion is an easy combination of Theorem 2 (1) and Lemma 3.

**Proof of Theorem 4.** In view of Theorem 3, it remains only to prove that $R_2$ is commutative. Obviously, every element of $R_2$ is of the form $e + a$, where $e \in E \cap R_2$ and $a \in N$. Since $E$ is central and $N$ is commutative, it is immediate that $R_2$ is commutative.

**Remark 3.** As the following example shows, Theorem 4 is not true if we replace $2^k$ by an arbitrary positive integer: Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \text{GF}(4) \right\}.$$
ON RINGS SATISFYING THE IDENTITY

Then $12R = 0$ and $R$ is a normal ring satisfying (i)$_2$ and (ii). But $R$ is not commutative and $R/N$ is not Boolean either.

Proof of Theorem 5. "Only if": Obviously, (5) implies (i)$_2$. We claim that $R$ is normal. To see this, given $e \in E$ and $x \in R$, we put $a = ex(1-e)$. Since

$$(a-e-e(a-e) \circ (e-e\circ (a-e)) = a \circ (a) = -a^2 = 0,$$

(*) shows that $a-e = e$, and so $a = 0$, i.e., $ex = exe$. A similar argument gives $xe = exe$, and hence $ex = xe$. It is easy to see that $8R = 0$, and therefore $R$ is a normal, left $s$-unital ring with $2^3 R = 0$ and satisfies (i)$_2$. We shall show that $R$ satisfies (ii). Let $x$ be an arbitrary element of $R$, and let $e$ be such that $ex = x$. Since $(x+x^2)^{(2)} = 0 = (-x+x^2)^{(2)}$ by (i)$_2$, we readily obtain $4(x+x^2) = 0$. i.e., $4x = 4x^3$. Replacing $x$ by $e+x$ in the last, we get

$$4x+4x^2 = 4(e+x)x = 4(e)x^3x = 4x + 4x^2 + 4x^3 - 4x^4.$$

Combining this with $4x = 4x^3$, we see that $4x = 4x^4 = 4x^3x = 4x^2$. Since $x^4 = -2x^3 - 3x^2 - 2x$ by (i)$_2$, we get

$$x^5 = -2x^4 - 3x^3 - 2x^2 = -2(-2x^3 - 3x^2 - 2x) - 3x^3 - 2x^2 = x^3 + 4x^2 - 4x = x^3.$$

This implies that $a^3 = 0$ for all $a \in N$, and therefore

$$a + a^2 \circ (a + a^2)^{(2)} \circ (-a) = -a, \text{ i.e., } a^{(2)} = 0.$$

Noting that $N$ is a nil ideal by Theorem 1, we get for any $a, b \in N$

$$a \circ b = a \circ (a \circ b)^{(2)} \circ b = b \circ a,$$

which shows that $N$ is commutative. Thus, by Theorem 4, $R$ is commutative and $R/N$ is a Boolean ring.

"If": First, we claim that every quasi-regular element of $R$ is nilpotent. In fact, if $a$ is quasi-regular then the nilpotency of $a + a^2$ yields that of $a$. Obviously, $(x + x^3)^{(2)} = 0$ for all $x \in R$. Conversely, if $(x + xy)^{(2)}(y + yx) = 0$ then $x + xy$ is nilpotent, and hence $y + xy = (x + xy)^{(2)}(y + yx) = x + xy$, whence it follows that $y = x$.

Remark 4. In order to prove the only if part of Theorem 5, we quoted Theorems 1 and 4. In case $R$ has 1, we can prove it more directly. In fact, $x^5 = x^3$ yields that $u^2 = 1$ for every unit $u$ in $R$. If $u, v$ are units in $R$ then $uv = (uv)^{-1} = v^{-1}u^{-1} = vu$. In particular, if $a$ and $b$ are in $N$ then $[a, b] = [1 + a, 1 + b] = 0$, and so $N$ is commutative. Now, let $a \in N, x \in R$. Since $(x + x^3) \circ (x^2 + x^4) = -8(x + x^3)^3 = 0$ and $(x^2 + x^4) \circ (x^2 + x^4) = 0$ by (i)$_2$ and $x^4$
is a central idempotent, we have \([a, x] = [a, x - x^4] = [a, x + x^2] - [a, x^2 + x^4] = 0\), and hence \(N \subseteq C\). Therefore, \(x = (x + x^2) - (x^2 + x^4) + x^4 \in C\) for all \(x\) in \(R\), and hence \(R\) is commutative.

REFERENCES


OKAYAMA UNIVERSITY, OKAYAMA, JAPAN
OKAYAMA UNIVERSITY, OKAYAMA, JAPAN
UNIVERSITY OF CALIFORNIA, SANTA BARBARA, U.S.A.

(Received November 1, 1982)