On generalized p.p. rings

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A commutative ring $R$ is called a generalized p.p. ring or for short, a g.p.p. ring if for each $a$ in $R$ there exists a positive integer $n$ (depending on $a$) such that $a^n R$ is projective. In this paper we shall generalize the works of [1], [2], [4]. For instance, we prove that a commutative ring $R$ is a g.p.p. ring if and only if $R$ has a $\pi$-regular classical quotient ring $Q$ and all idempotents in $Q$ belong to $R$ (Theorem 2); $R$ is a g.p.p. ring if and only if $R$ has a $\pi$-regular classical quotient ring and for each maximal ideal $M$ of $R$, $R M$ is primary (Theorem 5). Moreover, we shall treat with formal power series rings over $R$ and their classical quotient rings, and prove that $R$ is a g.p.p. ring (resp. p.p. ring) if and only if some (and every) subring of the classical quotient ring of $R[[X_1, \ldots, X_n]]$ containing $R$ is a g.p.p. ring (resp. p.p. ring) (Theorem 8).

Before stating our results we introduce the notion and terminology used in this paper. Throughout this paper $R$ will denote a commutative ring with 1. $R$ is called $\pi$-regular if for each $a$ in $R$ there exists a positive integer $n$ and an element $x$ in $R$ such that $a^n = a^{2n} x$. By $Q(R)$ we denote the classical quotient ring of $R$. A ring $R$ is said to be quasi-regular (resp. quasi $\pi$-regular) provided $Q(R)$ is regular (resp. $\pi$-regular). If $K$ is an ideal of $R$, the radical of $K$, denoted by $\sqrt{K}$, consists of all elements $a$ of $R$ such that $a^t \in K$ for some positive integer $t$. Then $K$ is called primary if $xy \in K$, $x \in K$ implies $y \in \sqrt{K}$, and $R$ is said to be primary if $(0)$ is primary. By $N(R)$ we denote the prime radical of $R$ (i.e., $N(R) = \sqrt{(0)}$), and by $E(R)$ the set of all idempotents in $R$. Given a subset $S$ of the ring $R$, $\text{ann}_R(S)$ denotes the annihilator of $S$ in $R$.

We first consider the conditions for $R$ to have a $\pi$-regular classical quotient ring.

**Theorem 1.** The following are equivalent:

1) $R$ is a quasi $\pi$-regular ring.

2) For each zero-divisor $x \in R$, there exists a positive integer $n$ such that $\text{ann}_R(x^n) = \text{ann}_R(x^{n+1})$ and the ring $\text{ann}_R(x^n)$ contains a non-zero-divisor.

3) For each $x \in R$, there exists a positive integer $n$ and a non-zero-divisor $d \in R$ such that $x^n d = x^{2n}$. 7
Proof. 1) \( \Rightarrow 2 \). Let \( x \) be an arbitrary zero-divisor in \( R \). Since \( Q(R) \) is \( \pi \)-regular, \( x^nQ(R) = x^{n+1}Q(R) \) for some positive integer \( n \). Then \( \text{ann}_{R}(x^n) = \text{ann}_{Q(R)}(x^nQ(R)) \cap R = \text{ann}_{Q(R)}(x^{n+1}Q(R)) \cap R = \text{ann}_{R}(x^{n+1}) \). By the above, there is an element \( y \in Q(R) \) such that \( x^n = x^{2^n}y \). Then \( e = 1 - x^{2^n}y \) is a non-zero idempotent and \( \text{ann}_{Q(R)}(x^n) = eQ(R) \). Let \( e = cd^{-1} \), \( c, d \in R \). Then \( c \) is a non-zero-divisor of the ring \( \text{ann}_{R}(x^n) \).

2) \( \Rightarrow 3 \). If \( x \) is a non-zero-divisor in \( R \) then we can take \( x^n \) as \( d \) in 3), and so we assume that \( x \) is a zero-divisor. Choose a non-zero-divisor \( z \) of \( \text{ann}_{R}(x^n) = \text{ann}_{R}(x^{n+1}) \). We shall show that \( x^n + z \) is a non-zero-divisor in \( R \). Let \( a \in \text{ann}_{R}(x^n + z) \). Then \( ax^{2^n} = a(x^n + z)x^n = 0 \). Since \( \text{ann}_{R}(x^n) = \text{ann}_{R}(x^{2^n}) \), we see that \( a \in \text{ann}_{R}(x^n) \) and hence \( az = 0 \). But \( z \) is a non-zero-divisor of \( \text{ann}_{R}(x^n) \), and so \( a = 0 \).

3) \( \Rightarrow 1 \). Since \( d \) is invertible in \( Q(R) \), it holds that \( x^nQ(R) = x^{2^n}Q(R) \). This implies that \( Q(R) \) is \( \pi \)-regular.

Next we shall generalize [2, Theorem 3.4] and [4, Theorem 1.3].

**Theorem 2.** The following are equivalent :

1) \( R \) is a g.p.p. ring.
2) \( R \) is quasi \( \pi \)-regular and \( E(Q(R)) = E(R) \).

Proof. 1) \( \Rightarrow 2 \). Let \( x \) be an arbitrary zero-divisor in \( R \). Then \( x^nR \) is projective for some positive integer \( n \). It is easy to see that \( x^nR \) is projective if and only if \( \text{ann}_{R}(x^n) = eR \) for some \( e \in E(R) \). We show \( \text{ann}_{R}(x^{n+1}) \). If \( a \in \text{ann}_{R}(x^{n+1}) \), then \( ax \in \text{ann}_{R}(x^n) = eR \), and so \( ax = axe \). Thus \( x^nax = x^{n-1}xae = x^nea = 0 \). Therefore by Theorem 1 \( R \) is quasi \( \pi \)-regular. To prove \( E(Q(R)) = E(R) \), let \( f \in E(Q(R)) \). Then We can write \( f = cd^{-1} \) for some \( c, d \in R \). By hypothesis, \( \text{ann}_{R}(c^m) = gR \) for some \( m \) and for some \( g \in E(R) \). Since \( fQ(R) = c^kQ(R) \) for any positive integer \( k \), we can easily see \( f = 1 - g \in E(R) \).

2) \( \Rightarrow 1 \). Let \( x \in R \). Since \( Q(R) \) is \( \pi \)-regular, there is an element \( y \in Q(R) \) and a positive integer \( n \) such that \( x^n = x^{2^n}y \). Then by hypothesis the idempotent \( e = x^n y \) is in \( R \) and hence \( \text{ann}_{R}(x^n) = \text{ann}_{Q(R)}(x^n) \cap R = (1 - e)Q(R) \cap R = (1 - e)R \).

**Corollary 3.** The following are equivalent :

1) \( R \) is a g.p.p. ring which contains no infinite set of orthogonal idempotents.
2) \( R \) is a finite direct sum of primary rings.
We now consider the relationship between g.p.p. rings and p.p. rings. It is not difficult to see that \( R \) is a p.p. ring if and only if \( R \) is a reduced g.p.p. ring. More generally we have the following.

**Proposition 4.** If \( R \) is a g.p.p. ring then \( R/N(R) \) is a p.p. ring.

**Proof.** Let \( x \) be an arbitrary non-nilpotent element in \( R \). By hypothesis there exists a positive integer \( n \) and a non-zero-divisor of \((1-e)R\). Let us set \( \bar{R} = R/N(R) \). We shall show that \( \bar{x}^n = x^n + N(R) \) is a non-zero-divisor of \((1-e)\bar{R}\). If \( d \in (1-e)R \) and \( dx^n \in N(R) \), then \((dx^n)^m = 0 \) for some positive integer \( m \). Since \( x^n \) is a non-zero-divisor of \((1-e)R\), we see \( d^m = 0 \), that is \( d \in N(R) \). Thus \( \bar{x}^n \) is a non-zero-divisor of \((1-e)\bar{R}\), which implies that \( \text{ann}_R(\bar{x}^n) = \bar{e}\bar{R} \). Since \( \bar{R} \) is reduced, we can easily see that \( \text{ann}_R(\bar{x}) = \text{ann}_R(\bar{x}^n) \). In consequence, we have proved \( \text{ann}_R(\bar{x}) = \bar{e}\bar{R} \).

**Remark.** Suppose \( R \) is quasi \( \pi \)-regular. Then, using Theorem 1, we can also prove that \( R/N(R) \) is quasi-regular.

The next corresponds to [1, Proposition 1].

**Theorem 5.** The following are equivalent:

1) \( R \) is a g.p.p. ring.

2) \( R \) is quasi \( \pi \)-regular and for each maximal ideal \( M \) of \( R \), \( R_M \) is a primary ring.

**Proof.** 1) \( \Rightarrow \) 2). By Theorem 2, \( R \) is quasi \( \pi \)-regular and \( E(Q(R)) = E(R) \). Let \( M \) be a maximal ideal of \( R \), and set \( K = \{ a \in R \mid sa = 0 \text{ for some } s \in R-M \} \). For each \( e \in E(Q(R)) \) \((= E(R))\), either \( e \in R-M \) or \( 1-e \in R-M \). Thus either \( 1-e \in K \) or \( e \in K \). Since \( Q(R) \) is \( \pi \)-regular, we can easily see that \( KQ(R) \) is a primary ideal of \( Q(R) \). Combining this with \( KQ(R) \cap R = K \), we also see that \( K \) is primary. If \( S \) denotes the canonical image of \( R-M \) in \( \bar{R} = R/K \), each element of \( S \) is a non-zero-divisor and \( R_M \) is isomorphic to the localization of \( \bar{R} \) by \( S \). Therefore, since \( \bar{R} \) is primary, \( R_M \) \((\simeq \bar{R}_S)\) is primary.

2) \( \Rightarrow \) 1). Let \( M \) be a maximal ideal of \( R \), and define \( K \) in the same way as above. We show that \( K \) is a primary ideal of \( R \). Given \( a, b \in R \) such that \( ab \in K \). Then, by the definition of \( K \), we see \( \bar{a}\bar{b} = 0 \) in \( \bar{R}_M \). Since \( R_M \) is primary, either \( \bar{a} = 0 \) or \( \bar{b} \in N(R_M) \), and so either \( a \in K \) or \( b \in \sqrt{K} \). Thus we have shown that \( K \) is primary. Since \( KQ(R) \cap R = K \), we can easily see that \( KQ(R) \) is a primary ideal of \( Q(R) \). Therefore,
for each \( e \in E(Q(R)) \), either \( e \in KQ \) or \( 1-e \in KQ \). If \( e \in KQ \), then \( se = 0 \) for some \( s \in R-M \). On the other hand, if \( 1-e \in KQ \) then \( s'(1-e) = 0 \) for some \( s' \in R-M \), that is, \( s'e = s' \). Now we show that \( e \) is in \( R \). Let \( T = \{ a \in R \mid ae \in R \} \). As we have just seen above, there is no maximal ideal which contains \( T \). Thus \( T = R \), and hence \( e \in R \), proving our assertion. Therefore, by Thorem 2, \( R \) is a g.p.p. ring.

Finally, we shall investigate formal power series rings and their classical quotient rings. We begin with some preliminary results.

**Lemma 6.** Let \( R((X)) = \{ \sum_{n=r}^{\infty} a_n X^n \mid a_n \in R, r \in \mathbb{Z} \} \). Then it holds that \( E(R((X))) = E(R) \).

**Proof.** We first show that if \( e = a_0 + a_1 X + \cdots \) is an idempotent then \( e \in R \). Suppose to the contrary \( e \notin R \), and let \( n \) be the smallest positive integer such that \( a_n \neq 0 \). Then we obtain \( a_0 = a_0 \) and \( a_n = 2a_0a_n \). From these we see that \( 2a_0a_n = (2a_0)^2a_n = 4a_0^2a_n \), and hence \( a_n = 2a_0a_n = 0 \), a contradiction.

Next we shall prove that if \( e = a_0X^m + \cdots + a_0 + a_1X + \cdots (m \leq 0) \) is an idempotent then \( e \) is in \( R \). We proceed by induction on \( m \). As we have done, our assertion is true for \( m = 0 \). So we may assume that our assertion is true for \( m \geq k+1 \). In case \( m = k \), we consider the ring \( (R/(a_0))(X) \) and the canonical image \( \hat{e} \) of \( e \). Then, by induction hypothesis, we conclude that \( a_i \in (a_k) \) for all \( i \neq 0 \). Since \( e \) is an idempotent, we get \( a_k = 0 \), and hence \( a_k = \sum_{i=k}^{0} a_{k-i} a_i = 2a_0 a_0 \) and \( a_0 = \sum_{i=k}^{0} a_0 a_{k-i} = a_0 \). Therefore we have \( a_k = 2a_k \), namely \( a_k = 0 \), and hence \( a_i = 0 \) for all \( i \neq 0 \).

**Lemma 7.** If \( R \) is quasi \( \pi \)-regular (resp. quasi-regular), then so is every intermediate ring containing \( E(Q(R)) \) between \( R \) and \( Q(R)((X)) \).

**Proof.** First we show that \( Q = Q(R)((X)) \) is \( \pi \)-regular. Let \( P \) be an arbitrary proper prime ideal of \( Q \). Then \( P' = P \cap Q(R) \) is a prime ideal of \( Q(R) \) and \( Q(R)/P' \) is a field. Hence, \( Q/P'Q \cong (Q(R)/P')(X) \) is a field, and so \( P \) coincides with the maximal ideal \( P'Q \). Thus, \( Q \) is \( \pi \)-regular (see, e.g., [3, Corollary 4]).

Next, let \( S \) be an intermediate ring between \( R \) and \( Q \). Then for each \( s \in S \) there exists a positive integer \( n \) and \( d \in Q \) such that \( s^{2n}d = s^n \). By Lemma 6, \( e = s^n d \in Q(R) \). Then, \( s^n + 1 - e \) is a non-zero-divisor of \( S \) and \( s^n (s^n + 1 - e) = s^{2n} \). Hence, \( S \) is quasi \( \pi \)-regular by Theorem 1.
We can now prove the following

**Theorem 8.** Let $m$ be a positive integer. A commutative ring $R$ is a g.p.p. ring (resp. p.p. ring) if and only if $E(Q(R)) = E(R)$ and some (and every) intermediate ring between $R$ and $Q(R)((X_1, \cdots, X_m))$ is a g.p.p. ring (resp. p.p. ring).

**Proof.** Suppose $R$ is a g.p.p. ring, and let $S$ be an intermediate ring between $R$ and $Q = Q(R)((X_1, \cdots, X_m))$. Then $S$ is quasi-regular by (Theorem 2 and) Lemma 7, and $E(Q(S)) = E(S) (= E(R))$ by Lemma 6. Therefore $S$ is a g.p.p. ring by Theorem 2.

Conversely, assume that a subring $S$ of $Q$ containing $R$ is a g.p.p. ring. Let $r$ be an arbitrary element of $R$. Then there exists a positive integer $n$ and $e \in E(S)$ such that $\text{ann}_S(r^n) = eS$. Since $e \in E(R)$ by Lemma 6, we have $\text{ann}_R(r^n) = \text{ann}_S(r^n) \cap R = eS \cap R = eR$. Therefore $R$ is a g.p.p. ring.

**Acknowledgement.** The author wishes to express his thanks to Professor M. Ōhori for his valuable comments in preparing this paper.

**References**


