On continuity of injectivity radius function

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ON CONTINUITY OF INJECTIVITY RADIUS FUNCTION

Dedicated to Prof. S. Sasaki on his 70th birthday

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0. Introduction. Let $M$ be a compact connected $C^\infty$-manifold. For a fixed riemannian structure $g$ on $M$ we define the injectivity radius $i_g(M)$ of $g$ by

$$\text{Sup}\{r > 0 : \text{Exp}_x : B_r(o_x) \to M \text{ is a diffeomorphism for every } x \text{ of } M\},$$

where $\text{Exp}_x$ denotes the exponential map at $x$. $B_r(o_x)$ an $r$-ball in $T_xM$ centered at the origin. This gives a uniform estimate of the size of domains over which normal coordinates are valid. Now we consider the space $\mathfrak{M}$ of $C^3$-riemannian structures $g$ on $M$ with the $C^2$-topology. We may consider $g \to i_g(M)$ as a function on $\mathfrak{M}$. P. Ehrlich ([E]) proved

**Theorem.** $g \to i_g(M)$ is a continuous function.

Namely when $g_n \to g_0$ (w.r.t. $C^2$-topology) we have to show $\limsup i_{g_n}(M) \leq i_g(M)$ and $\liminf i_{g_n}(M) \geq i_g(M)$. I applied the first inequality in a previous paper ([S]). Since Ehrlich's proof is rather complicated, I presented a simple proof of the first inequality in my first draft of the above paper. Then the referee of my paper pointed out that the first inequality may be proved much more simply by using the Busemann type argument. Here I will give a very simple proof of the second inequality, which is a more difficult one. We include a proof of the first inequality for completeness and I heartily appreciate the referee of my paper ([S]).

1. Preliminaries. Let $g$ be a riemannian metric of class $C^3$ on a compact connected $C^\infty$-manifold $M$. We denote by $d_g$ the distance function induced from $g$. Firstly we recall two fundamental facts about the injectivity radius $i_g(M)$:

1.1) $i_g(M)$ is given by the minimum of the shortest distance to the first conjugate points along geodesics and half the length of the shortest closed geodesic.

1.2) Take $x, y \in M$ with $d_g(x,y) = i_g(M)$ and let $c : [0, i_g(M)] \to M$ be a minimal geodesic from $x$ to $y$ parametrized by arc-length. If $y$ is not conjugate to $x$ along any minimal geodesic from $x$ to $y$, $c[[0,2i_g(M)]]$ defines a closed geodesic.
Next we recall some fundamental facts from the ordinary differential equations:

**Lemma 1.3.** Let $N$ be a $C^2$-riemmanian manifold, $X$ a $C^1$-vector field on $N$ and $x(t), 0 \leq t \leq R$ an integral curve of $X$ in a bounded domain $D \subset N$. Then there exist positive constants $C, \varepsilon_0, \delta_0$ depending on $D, X, x(0)$ and $R$ with the following property: Take any $0 < \varepsilon (\leq \varepsilon_0), 0 < \delta (\leq \delta_0)$ and a $C^1$-vector field $Y$ on $N$ such that $\sup_{x \in D} |X_x - Y_x| < \delta$. Then for any integral curve $y(t)$ of $Y$ with $d(x(0), y(0)) < \varepsilon$ we have $y(t) \in D, 0 \leq t \leq R$ and $d(x(t), y(t)) < C(\varepsilon + \delta t)$.

**Proof.** Take a local chart $(\varphi, U), D \supset \bar{U}, U \supset \bar{V}, V \supset x([0, R])$ such that $x(t), 0 \leq t \leq R$ lie on a coordinate axis and $\varphi(\bar{V})$ is a convex subset in $R^{\dim N}$ (e.g., take Fermi coordinate of $x(t), 0 \leq r \leq R$ if $X_{x(0)} \neq 0$ and normal coordinate of $x(0)$ if $X_{x(0)} = 0$). There exists a constant $A > 1$ such that

$$1/A^2 \cdot (\delta t) \leq (g_{ij}) \leq A^2(\delta t)$$

on $\bar{V}$, where $(g_{ij})$ denotes the riemmanian metric. Denoting the euclidean norm and riemmanian norm by $\| \cdot \|$ and $| \cdot |$ respectively, we have then

$$\|x(0) - y(0)\| \leq Ad(x(0), y(0))$$

if $y(0)$ is in a convex neighborhood of $x(0)$ and

$$|X_x - Y_x| \leq A|X_x - Y_x| < A\delta$$

for $x \in \bar{V}$. Since $X$ is of class $C^1$ and $\varphi(\bar{V})$ is convex, there exists a positive constant $B$ such that $\|X_x - X_y\| \leq B\|x - y\|$ on $\bar{V}$. First assuming that $y(t) \in V, 0 \leq t \leq R$ and $d(x(0), y(0)) < \varepsilon$, we have

$$\|x(t) - y(t)\| \leq \|X_{x(t)} - X_{y(t)}\| + \|X_{y(t)} - Y_{y(t)}\| \leq A\delta + B\|x(t) - y(t)\|.$$

From this we easily see that

$$\|x(t) - y(t)e^{-Bt}\| < A\delta e^{-Bt} \leq A\delta$$

as far as $x(t) \neq y(t)$.

Putting $s(t) := \text{Inf}\{s \geq 0 : x(s) \neq y(s) \text{ for } 0 \leq s \leq t\}$, we get

$$\|x(t) - y(t)e^{-Bt}\| \leq \int_{s(t)}^t \|x(t) - y(t)e^{-Bt}\| \, dt + \|x(s(t)) - y(s(t))e^{-Bst(t)}\|
\leq A\delta t + \|x(0) - y(0)\|,$$

namely

$$\|x(t) - y(t)\| \leq e^{Bt}A(\delta t + d(x(0), y(0))).$$

Setting $C := \text{Max}\{1, A^2e^{BR}\}$, we have

$$d(x(t), y(t)) \leq C(\delta t + d(x(0), y(0))).$$
as far as $|X_x - Y_x| < \delta$ and $y(t) \in V$.

Next choose $\varepsilon_0, \delta_0$ so that $C(\delta_0 R + \varepsilon_0) < a$, where $a$ is a positive number such that $B_{x(t)}(a) := \{ y : d(x(t), y) < a \} \subset V$ for all $0 \leq t \leq R$. Then we will be done if we show that $t_1 := \sup \{ t : y(s) \in V \text{ for } 0 \leq s \leq t \}$ equals $R$ for $Y$ satisfying the assumption of the lemma. Clearly $t_1 > 0$ and assume that $t_1 < R$. Then by continuity we have $d(x(t_1), y(t_1)) \leq C(\delta t_1 + \varepsilon) < a$. Then $y(t_1) \in V$ and consequently $y(t) \in V$ for $t > t_1$, $|t - t_1|$ sufficiently small, a contradiction.

Now on a $C^\infty$-manifold we mean by $C'$-topology the topology of uniform convergence on compact subsets of derivatives up to order $r$.

**Corollary 1.4.** Let $X_n (n = 1, 2, \cdots)$ be $C^1$-vector fields on a $C^\infty$-manifold $N$. Assume that $X$ is complete and $X_n \to X$ (w.r.t. $C^0$-topology). Let $\phi^n_t, \phi_t$ be flows generated by $X_n, X$ respectively. If $x_n \to x, t_n \to t$ then we have $\lim_{n \to \infty} \phi^n_{t_n}(x_n) = \phi(x)$.

**Proof.** Choose a riemannian metric on $N, R > t_n, t$. Let $C, \varepsilon_0, \delta_0$ be as in Lemma 1.3 for $X, x, R$. For any $\varepsilon_1 > 0$ take $(\varepsilon_0 \geq \varepsilon > 0, (\delta_0 \geq \delta > 0$ so that $C(\varepsilon + \delta R) < \varepsilon_1/2$. Then for sufficiently large $n$ we have $d(x_n, x) < \varepsilon, |X_n - X| < \delta$ on a compact domain $D$ whose interior contains $\phi_t(x), 0 \leq t \leq R$ and $d(\phi^n_{t_n}(x_n), \phi_t(x)) < \varepsilon_1/2$. Then from Lemma 1.3 we get for such $n$

$$d(\phi^n_{t_n}(x_n), \phi_t(x)) \leq d(\phi^n_{t_n}(x_n), \phi_{t_n}(x)) + d(\phi_{t_n}(x), \phi_t(x)) \leq C(\varepsilon + \delta t_n) + \varepsilon_1/2 < \varepsilon_1.$$

Now to prove the continuity of $g \to i_g(M)$ we take the geodesic flow view point. For a riemannian structure $g$ on $M$ we denote by $\phi^t$ the geodesic flow on the tangent bundle $\pi : TM \to M$. Namely for $v \in TM - \{0\}, t \to \pi \circ \phi^t(v)$ is a geodesic emanating from $\pi v$ with the initial direction $v$ which will be denoted by $c^g_\pi(t)$.

We denote by $U(M, g)$ the unit tangent bundle with respect to $g$.

**Lemma 1.5.** Let $(M, g_n)$ be a complete $C^2$-riemannian manifold and $g_n (n = 1, 2, \cdots)$ a sequence of $C^2$-riemannian metrics on $M$ such that $g_n \to g_0$ (w.r.t. $C^1$-topology). Then for $v_n \in U(M, g_n), v_n \to v \in U(M, g_0)$ w.r.t. the topology of $TM$ and $t_n \to t$ we have

1. $\lim_{n \to \infty} \phi^{g_n}_{t_n}(v_n) = \phi^g_t(v)$
2. $\lim_{n \to \infty} c^{g_n}_{t_n}(v_n) = c^g_t(v)$. 

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Proof. Let $S^g$ be the geodesic spray with respect to $g$ which is a $C^1$-vector field on $TM$ given by

$$S_{x,v}^g = \sum_i \xi^i \frac{\partial}{\partial x^i} - \sum_{i,j,k} \Gamma_{jk}^i \xi^j \xi^k \frac{\partial}{\partial \xi^i}.$$ 

in terms of local coordinates of $M$. Then clearly $S^{g_n} \rightarrow S^{g_0}$ (w.r.t. $C^0$-topology) when $g_n \rightarrow g_0$ (w.r.t. $C^1$-topology). Since $\phi_t^g$ is a flow generated by $S^g$ we get the first assertion from Corollary 1.4. The second assertion is trivial because $c^g_\phi(t) = \pi \circ \phi_t^g(v)$.

Lemma 1.6. Let $(M,g_0)$ be a complete $C^3$-riemannian manifold and $g_n \rightarrow g_0$ (w.r.t. $C^2$-topology) be a sequence of $C^3$-riemannian metrics on $M$. Then if $v_n \in U(M,g_n) \rightarrow v \in U(M,g_0)$ in $TM$, $w_n \in TV_n TM \rightarrow w \in TV_0 TM$ in $TTM$ and $t_n \rightarrow t$, we have

(i) $\lim_{n \to m} (\phi_t^{g_n})_* w_n = (\phi_t^{g_0})_* w$

(ii) $\lim_{n \to m} \pi_*(\phi_t^{g_n})_* w_n = \pi_*(\phi_t^{g_0})_* w$.

Proof. First note that the following easily proved fact: Let $X = \sum X^i \frac{\partial}{\partial x^i}$ be a $C^2$-vector field on a $C^\infty$-manifold $N$ and $\phi_t$ a flow generated by $X$. Then the $C^1$-vector field $X^*$ on $TN$ defined from a flow $(\phi_t)_*: TN \rightarrow TN$ is given by

$$X^*_{(x,v)} = \sum X^i(x) \frac{\partial}{\partial x^i} + \sum v^i \frac{\partial}{\partial \xi^i}$$

with respect to the adapted chart on $TN$.

Applying this to the geodesic spray on $TM$, we have with respect to the adapted local coordinate $(x^i, \xi^j, y^i, \eta^i)$ on $TTM$,

$$S_{x,v,u,y}^g = \sum_i \xi^i \frac{\partial}{\partial x^i} - \sum_{i,j,k} \Gamma_{jk}^i(x) \xi^j \xi^k \frac{\partial}{\partial \xi^i}$$

$$+ \sum_i \eta^i \frac{\partial}{\partial y^i} - \sum_{i,j,k} (2\Gamma_{jk}^i(x) \xi^j \eta^k + \frac{\partial \Gamma_{jk}^i(x)}{\partial x^l} y^l \xi^j \xi^k) \frac{\partial}{\partial \eta^i}.$$ 

Namely if $(M,g)$ is a $C^3$-riemannian manifold, $S^*$ is a $C^1$-vector field on $TTM$. Moreover $(S^g)^* \rightarrow (S^g_0)^*$ (w.r.t. $C^0$-topology) if $g_n \rightarrow g_0$ (w.r.t. $C^2$-topology) and Lemma 1.6 also follows from Corollary 1.4.

Lemma 1.7. Let $M$ be a compact manifold and $g_n, g_0$ be $C^0$-riemannian metrics such that $g_n \rightarrow g_0$ (w.r.t. $C^0$-topology). namely $(1-\varepsilon_n) g_0 \leq g_n \leq (1+\varepsilon_n) g_0$ with $\varepsilon_n \rightarrow 0$. Then we have

$$(1-\varepsilon_n) d_{g_n}(x,y) \leq d_{g_0}(x,y) \leq (1+\varepsilon_n) d_{g_n}(x,y).$$
for $x$, $y \in M$.

**Proof.** This follows easily from

$$(1 - \varepsilon_n) L_{g_0}(c) \leq L_{g_n}(c) \leq (1 + \varepsilon_n) L_{g_0}(c)$$

for any piecewise smooth curve connecting $x$ and $y$, and the definition of the distance.

**Corollary 1.8.** Under the assumption of Lemma 1.7 we have

$$\lim_{n \to \infty} d_{g_n}(M) = d_{g_0}(M).$$

where $d_{g}(M)$ denotes the diameter of $M$.

**Proof.** This is clear from the definition of the diameter, i.e., $d_{g}(M) = \max_{x, y \in M} d_{g}(x, y)$ and the lemma.

2. **Proof of Theorem.**

1°. $\limsup i_{g_n}(M) \leq i_{g_0}(M)$.

Put $R_n := i_{g_n}(M)$, $R := \limsup R_n$ and fix any $x \in M$. Then it suffices to show that for any $g_0$-geodesic $c_v$ which emanates from $x$ with initial tangent vector $v \in U_x(M, g_0)$ we have $d_{g_n}(x, c_v(R)) \geq R$. Taking a subsequence we may also assume that $R_n \to R$. Recall that $c_v(t) = \pi \circ \phi^v_{g_n}(v)$. Let $c^0_n$ be the $g_n$-geodesic emanating from $x$ with initial tangent vector $v_n := v/|v| \in U_x(M, g_n)$, namely $c^0_n(t) = \pi \circ \phi^v_{g_n}(v_n)$. Since $g_n \to g_0$ (w.r.t. $C^1$-topology), $v_n \to v$ in $TM$, $R_n \to R$ hold and we have from Lemmas 1.5 and 1.7

$$R = \lim_{n \to \infty} R_n = \lim_{n \to \infty} d_{g_n}(x, c^0_n(R_n))$$

$$\leq \lim_{n \to \infty} (d_{g_n}(x, c_v(R)) + d_{g_n}(c_v(R), c^0_n(R_n)))$$

$$\leq d_{g_0}(x, c_v(R)) + \lim_{n \to \infty} (1 + \varepsilon_n) d_{g_n}(c_v(R), c^0_n(R_n))$$

$$= d_{g_0}(x, c_v(R)).$$

**Remark 1.** For this inequality we only have to assume that $g_n \to g_0$ (w.r.t. $C^1$-topology).

2°. $\liminf i_{g_n}(M) \geq i_{g_0}(M)$.

Taking a subsequence we may assume that $R_n := i_{g_n}(M) \to R$ and we have to show that $R \geq i_{g_0}(M)$. Choose points $x_n, y_n \in M$ with $d_{g_n}(x_n, y_n)$
$= R_n$ and let $v_n$ be the $g_n$-unit initial tangent vector to a minimal $g_n$-geodesic $c_{n}^{2}$ connecting $x_n$ and $y_n$. Again we may assume that $x_n \to x, y_n \to y, v_n \to v$ by taking subsequences if necessary. Firstly we show that $R$ is positive. In fact if $R = 0$, we have $x = y$ by Lemma 1.7. Since $g_n \to g_0$ (w.r.t. $C^2$-topology) we can find a universal positive constant $K$ such that sectional curves for the metric $g_n$ satisfy $K_{s} \leq K, K_{s} \leq K$ for sufficiently large $n$. Then there appears no conjugate point to $x_n$ along $g_n$-geodesic $c_{n}^{2}$, up to the (arc-length) parameter value $\pi/\sqrt{\lambda}$. Thus from (1.2) we see that $c_{n}^{2} \in (0, R_n)$ are $g_n$-closed geodesics for sufficiently large $n$ because $R_n \to 0$. Then for any $\varepsilon > 0$, $t \to c_{n}^{\varepsilon}(t)$ are contained in an $\varepsilon$-ball $B_{\varepsilon}(x)$ (w.r.t. $g_0$) centered at $x$ for sufficiently large $n$ by Lemma 1.7. Then from Lemma 1.5 we see that $g_0$-geodesic $t \to c_{\varepsilon}(t) = \pi\phi_{k}^{(\varepsilon)}(v)$ is also contained in $B_{\varepsilon}(x)$. Taking $\varepsilon < i_{s}(M)/2$ we get a contradiction. Thus $R$ is positive. Next we consider the following two cases:

Case 1. For infinitely many $n, y_n$ is not conjugate to $x_n$ along any minimal geodesic connecting $x_n$ and $y_n$. Let $c_{n}^{2} \in (0, R_n)$ be a minimal $g_n$-geodesic from $x_n$ to $y_n$ with initial direction $v_n \in U_{x_n}(M, g_n)$. Then from (1.2) $c_{n}^{2}$ defines a closed geodesic, namely $\phi_{k}^{(\varepsilon)}(v_n) = v_n$. Again from Lemma 1.5 we have $\phi_{k}^{(\varepsilon)}(v) = v$, where $v$ is a cluster point of $v_n$ and belongs to $U_{x}(M, g_0)$. This means that there exists a closed $g_0$-geodesic of length $2R$ and we get $i_{s}(M) \leq R$ ((1.1)).

Case 2. For almost all $n, y_n$ is conjugate to $x_n$ along some minimal $g_n$-geodesic $c_{n}^{2} \in (0, R_n)$. First we recall a characterization of Jacobi fields from the geodesic flow point of view.

**Lemma 2.1.** Let $(M, g)$ be a riemannian manifold and $v, w \in T_{x}M$. Regarding $w$ as an element of $T_{v}T_{x}M \subset T_{v}TM$ via the canonical identification $T_{v}T_{x}M \cong T_{x}M, Y(t) := \pi_{*}(\phi_{k}^{t})_{*}w$ is a Jacobi field along a geodesic $t \to \pi_{*}\phi_{k}^{t}(v)$ with the initial condition $Y(0) = 0, \forall Y(0) = w$. Conversely any Jacobi field $Y$ with $Y(0) = 0$ may be expressed in the above way.

**Proof.** Put $a(s, t) := \pi_{*}\phi_{k}^{t}(sw + v)$. Then $t \to a_{0}(t) := a(s, t)$ are geodesics from $x$ for all $s$ and $Y(t) = \pi_{*}(\phi_{k}^{t})_{*}w = \frac{\partial a(s, t)}{\partial s}|_{s=0}$ is a Jacobi field. Moreover we have $Y(0) = \partial / \partial s|_{s=0}a(s, 0) = 0$ and

$$\nabla Y(0) = \nabla \frac{\partial a(s, t)}{\partial s}|_{s=0} = \nabla \frac{\partial a(s, t)}{\partial t}|_{t=0} = \nabla \frac{\partial a(s, t)}{\partial s}|_{s=0}(sw + v) = w.$$  

Converse is clear because the Jacobi field is uniquely determined by $Y(0)$ and $\nabla Y(0)$.  

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Now we return to the second case. We have non-trivial $g_0$-Jacobi fields $Y_n$ along $c\nu^\nu_n$ which vanish at 0 and $K_n$. We may assume that $Y_n$ is $g_0$-perpendicular to $c\nu^\nu_n$ and $w_n := \nabla Y_n(0) \in U_x(M,g_0)$. Then from Lemma 2.1 we have $Y_n(t) = \pi_\# (\phi_t^{g_0})_* w_n$. Since $\{(v_n,w_n)\}$ is contained in a compact subset of $TM \times TM$, we may assume that $v_n \to v$, $w_n \to w$ and $K_n \to K$ by taking subsequences if necessary. Clearly $v, w \in U_x(M,g_0)$. $g_0(v,w) = 0$. Now from Lemma 1.6 we get

$$0 = Y_n(K_n) = \pi_\# (\phi_t^{g_0})_* w_n \to \pi_\# (\phi_t^{g_0})_* w = 0.$$ 

Then $Y(t) := \pi_\# (\phi_t^{g_0})_* w$ is a $g_0$-Jacobi field along $c_\nu$ which satisfies $Y(0) = 0$, $\nabla Y(0) = w (\neq 0)$ and $Y(K) = 0$. Namely $y$ is conjugate to $x$ along $c_\nu$ and we get again $i_{\nu}(M) \leq K$ ((1.1)).

Remark 2. Define the injectivity radius $i_x(M,g)$ of $(M,g)$ at $x$ as

$$i_x(M,g) := \sup \{ r > 0 : \text{Exp}_x : B_r(o_x) \to M \text{ is a diffeomorphism} \},$$

and the diameter from $x$ as

$$d_x(M,g) := \max_{y \in M} d(x,y).$$

Then by a similar argument we can show that the functions on $\mathfrak{M}$ defined by $g \to i_x(M,g)$ and $g \to d_x(M,g)$ is continuous.

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