On the nilpotency index of the radical of a group algebra. IV

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ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. IV

Dedicated to Professor Kentaro Murata on his 60th birthday

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Throughout this paper, we shall use the following notations: Let $p$ be a fixed prime number, let $G$ be a finite group with a $p$-Sylow subgroup $P$ of order $p^n$, let $KG$ be a group algebra of $G$ over a field $K$ of characteristic $p$ and let $t(G)$ be the nilpotency index of the radical $J(KG)$ of $KG$. Further, given a finite subset $S$ of $KG$, $\hat{S}$ denotes the sum of all elements of $S$.

First, in § 1, we shall investigate $t(G)$ of a group $G$ with a split $(B,N)$-pair. Next, in § 2, we shall state some remarks concerning a theorem of Morita [5]. § 3 is devoted to studying a $p$-solvable group $G$ with $t(G) = a(p-1)+1$. Unfortunately, we have not characterized yet a group $G$ of minimal order such that $G$ is $p$-solvable, $t(G) = a(p-1)+1$ and $P$ is not elementary abelian. However, we can establish the structure of such a group under certain extra conditions (Theorem 12). § 4 contains an example of a group $H$ satisfying the following conditions:

1. $H$ possesses all the conclusions of Lemma 11.
2. $t(H) > \beta(p-1)+1$ where $p^n$ is the order of a $p$-Sylow subgroup of $H$.

Finally, by making use of results in § 3, we shall prove that there exist no finite rings satisfying certain conditions (§ 5).

1. We begin with stating a lemma which is efficient in studying the nilpotency index of the radical of a group algebra.

**Lemma 1.** Let $A$ be a ring. Let $B$, $I$ and $J$ be subsets of $A$ satisfying the following conditions:

1. $IAI = IBI$.
2. $IJ = JI$.
3. $BJ \subseteq JB$.

Then $(JIA)^n \subseteq J^n IA$. Moreover, if $J^n = 0$ then $JIA$ is contained in the radical of $A$.

**Proof.** Clearly, the result holds for $n = 1$. Assume the result for $n$. Then we conclude that $(JIA)^{n+1} \subseteq J^n IA JIA = J^n IA JIA = J^n IB JIA = J^n IB JIA \subseteq J^n JIB JIA = J^{n+1} IBIA \subseteq J^{n+1} IA$. 

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Theorem 2. Suppose that a group $G$ has finite subgroups $H$ and $U$ such that $G = U N_G(H) U$ and $H \subseteq N_G(U)$. Then $(J(KH)\bar{U}KG)^{c(H)} = 0$ and $J(KH)\bar{U} \subseteq J(KG)$.

Proof. Taking $A$ to be $KG$, $B$ to be $KN_G(H)$, $I$ to be $\{\bar{U}\}$, and $J$ to be $J(KH)$ in Lemma 1, we readily obtain the conclusion.

We suggest the effectiveness of Theorem 2 by giving the following example which played a fundamental role in [8].

Example 3. Let $q = p^s$, $r$ a divisor of $q - 1$, and $s$ a multiple of $p$ with $(s, q - 1) = 1$. Let $H$ be the Galois group of $R = GF(q^s)$ over $S = GF(q)$ and let $T$ be $\langle b^r \rangle$ where $b$ is a generator of the multiplicative group of $R$. We consider the following permutation groups on $R$: $V = \{v_a : x \rightarrow x + a | a \in R\}$ and $U = \{u_t : x \rightarrow tx | t \in T\}$, and put $G = \langle H, U, V \rangle$. Since $q - 1$ and $(q^s - 1)/(q - 1)$ are relatively prime, every element of $R$ is a product of an element of $T$ and an element of $S$. This shows that $G = U C_c(H) U$ since $C_c(H)$ contains $\{v_a | a \in S\}$ and $u_t v_a u_t^{-1} = v_{ta}$. Noting that $hu_t h^{-1} = u_{ht}$ for all $h \in H$, by Theorem 2, we obtain that $J(KH)\bar{U} \subseteq J(KG)$ and the nilpotency index of $J(KH)\bar{U}KG$ is the $p$-part of $s$.

Theorem 2 is also applicable to finite groups of Lie type (see [1]).

Proposition 4. Suppose that a finite group $G$ has a split $(B, N)$-pair of characteristic $r$ such that $B$ is a semi-direct product of a normal $r$-subgroup $U$ and an abelian $r'$-subgroup $H = B \cap N$ (see [1]). Then $J(KH)\bar{U} \subseteq J(KG)$.

Proof. Since $G = UNU$ by Bruhat decomposition and $H$ is a normal subgroup of $N$, the assertion follows from Theorem 2.

Although the next is stated for a special type of a finite group, similar observation is possible for finite groups of Lie type, too.

Proposition 5. Let $r$ be a prime number, $q = r^e$, $p$ an odd prime divisor of $q - 1$, and $G = SL(2, q)$. Then $t(G)$ is the $p$-part $p^m$ of $q - 1$.

Proof. Obviously, $G$ has a split $(B, N)$-pair, where $B$ is the subgroup of upper triangular matrices and $N$ is the subgroup of monomial matrices. Thus $H = B \cap N$ is the subgroup of diagonal matrices which is isomorphic to the multiplicative group of the finite field of $q$ elements. Thus, by
Proposition 4, we have \( p^m = t(H) \leq t(G) \). Since the order of \( G \) is \( q(q-1)(q+1) \), a \( p \)-Sylow subgroup of \( G \) is a cyclic group of order \( p^m \), and so \( t(G) = p^m \) by Dade's theorem [2] (see also [7]).

2. Throughout this section, we shall use the following notations:
Let \( N \) be a normal subgroup (not necessarily a \( p' \)-subgroup) of a finite group \( G \) and let \( e \) be a centrally primitive idempotent of \( KN \). We set \( H = \{ x \in G \mid xex^{-1} = e \} \) and \( \bar{e} = \sum a_i e a_i^{-1} \), where \( \{ a_i \} \) (\( a_i = 1 \)) is a set of representatives of the right cosets of \( H \) in \( G \).

The following interesting theorem has been proved by K. Morita.

Theorem 6 (Morita [5]). If \( K \) is algebraically closed and \( KN = e \) is simple, then \( KG\bar{e} \) is isomorphic to a complete matrix algebra over a twisted group algebra of \( H/N \).

The next is immediate from the preceding theorem.

Corollary 7. Assume that \( K \) is algebraically closed. If \( N = H \) and \( KN = e \) is simple, then \( KG\bar{e} \) is simple.

This result applies especially to the radical of a group algebra of a Frobenius group.

Corollary 8 ([6]). Let \( G \) be a Frobenius group with kernel \( N \) and complement \( W \). If \( p \) divides the order of \( W \) then \( J(KG) = J(KW)^{N} \).

Proof. We may assume, in the usual way, that \( K \) is algebraically closed. If \( e \) is not equal to \( |N|^{-1} \bar{N} \), then \( H = N \) implies that \( KG\bar{e} \) is simple (see [3, the proof of (25.4)]). Thus we obtain the assertion.

As an application of Lemma 1, we have the following

Proposition 9. \( J(KHe) = J(KH)e \subseteq J(KG) \).

Proof. Setting \( A = KG \), \( B = KH \), \( I = \{ e \} \) and \( J = J(KH) \), we have \( IAI = eKGe = \sum a_i KHe a_i^{-1} a_i = KHe = eKHe = IBI \). Also, it is easy to check other assumptions of Lemma 1. Thus we have \( J(KH)e \subseteq J(KG) \).

The next shows that the converse of Corollary 7 holds.

Corollary 10. If \( N = H \) and \( KG\bar{e} \) is simple, then \( KN = e \) is simple.
Proof. By Proposition 9, \(0 = J(KG)e \supseteq J(KN)e = J(KNe)\). Thus \(KNe\) is simple.

3. Throughout this section, let \(G\) be a group of the minimal order which satisfies the following conditions:
   1. \(G\) is a \(p\)-solvable group.
   2. \(P\) is not elementary abelian.
   3. \(t(G) = \alpha(p-1) + 1\).

We show that \(G\) possesses the properties listed in the following lemma.

Lemma 11. (1) \(O_p(G) = 1\).
(2) \(U = O_p(G) (\neq 1)\) is elementary abelian and \(G = O_{p,p,p}(G)\).
(3) \(|G| \leq p^{\alpha*(p^\alpha-1)/(p-1)}\).
(4) \(U = [U, V], C_U(V) = 1\) and \(G\) is a semi-direct product of \(U\) by \(N_C(V)\) where \(V\) is a \(p\)-subgroup such that \(O_{p,p}(G) = UV\).
(5) \(V = [W, V]\) where \(W\) is a \(p\)-Sylow subgroup of \(N_C(V)\).
(6) \(N_C(V) = W\) is isomorphic to a subgroup of \(Aut(U) = GL(U)\).
(7) \(U\) is a minimal normal subgroup of \(UV\).

Proof. (1) We have \(t(G) \geq t(G/O_p(G)) \geq (\alpha(p-1)+1)\) (see [12]). Thus \(O_p(G) = 1\) by the minimality of the order of \(G\).

(2) From the inequality \(t(G) \geq t(G/U) + t(U) - 1\) (see [12]) we see that both \(U\) and \(P/U\) are elementary abelian. Hence \(G/U\) is of \(p\)-length 1, and so \(H = O_{p,p,p}(G)\) is a normal subgroup of \(G\) whose index is a \(p\)-number. Since \(t(G) = t(H)\) (see [11]), we get \(G = H = O_{p,p,p}(G)\).

(3) It is known that there exists a group \(H\) of order \(p^{\alpha*(p^\alpha-1)/(p-1)}\) such that \(t(H) = p^2\) and a \(p\)-Sylow subgroup of \(H\) is nonabelian (see [8]). Hence, we have \(p^{\alpha*(p^\alpha-1)/(p-1)} \geq |G|\).

(4) We can see that \(G = N_G(V)U\) by Frattini argument, and \(N_U(V) = C_U(V)\). Thus, to prove the last assertion, it suffices to show that \(C_U(V)\) is trivial. In view of \(C_G(U) = U\) (see [4, Theorem 3.2, p.228]), \(V\) can be regarded as an automorphism group of \(U\) by conjugation, and consequently \(U = C_U(V) \times [V, U]\) (see [4, Theorem 2.3, p.177]). Clearly, \([V, U] = [UV, U]\) is a non-trivial normal subgroup by \(O_p(G) = 1\). If \(C_U(V)\) is non-trivial, then \(P\) must be elementary abelian since \(C_U(V) = C_U(UV)\) is normal and so \(G\) can be embedded in \((G/C_U(V)) \times (G/[V, U])\). This contradiction shows that \(C_U(V)\) is trivial.

(5) Set \(V_1 = [V, W]\). Then \(H = WV_1U\) is a normal subgroup of \(G\) whose index is a \(p\)-number. Since \(t(G) = t(H)\) (see [11]) and \(P = WU\).
we have $G = H$ and so $V = [V, W]$. 

(6) The assertion follows from $C_c(U) = U$. 

(7) Let $U_1$ be a minimal normal subgroup of $UV$ contained in $U$. We prove first that $U_1$ is normal in $G$. Assume that $U_1^\sigma \neq U_1$ for some $\sigma \in W$. Then $U_1^\sigma \cap U_1 = 1$ since $U_1^\sigma \cap U_1$ is normal in $UV$. Noting that $V$ is a $p'$-group and $U$ is elementary abelian, we can easily see that there exists a normal subgroup $U_2$ of $UV$ such that $U = U_1 \times U_1^\sigma \times U_2$. Since $J(KG)$ contains the kernel of the natural homomorphism $KG \to KG/U$, we obtain $J(KG) \supset J(KW)V + J(KU)K$ (see [8]). We set $U_3 = U_1^\sigma \times U_2$. Then, in view of $t(G) = a(p-1)+1$, we have $0 = W\tilde{V}\tilde{U}_1(1-\sigma)\tilde{V}\tilde{U}_3 = W\tilde{V}(\tilde{U}_1 - \tilde{U}_1^\sigma)\tilde{V}\tilde{U}_3 = |V|W\tilde{V}(\tilde{U}_1 - \tilde{U}_1^\sigma)\tilde{V}\tilde{U}_3 = |V|W\tilde{V}\tilde{U}_1\tilde{U}_3 = |V|\tilde{G}$, which is impossible. Hence $U_1$ is normal in $G$. Since $V$ is a $p'$-group, we have $U = U_1 \times U_2 \times \cdots \times U_t$ where every $U_i$ is a minimal normal subgroup of $UV$. Then, by the preceding argument, every $U_i$ is a normal subgroup of $G$. If $t \geq 2$, then $P$ must be elementary abelian since $G$ can be embedded in $G/U_1 \times G/U_2 \times \cdots \times G/U_t$. This contradiction shows that $t = 1$ and so $U$ is a minimal normal subgroup of $G$.

In the remainder of this section, we preserve the notations used in Lemma 11. The next is the main result of this section.

**Theorem 12.** If $V$ is abelian, then $G$ can be regarded as a permutation group on $GF(p^n)$ such that $U = \{x \to x+b \mid b \in GF(p^n)\}$, $W$ is the Galois group of $GF(p^n)$ over $GF(p)$ and that $V \subseteq \{x \to tx \mid t \in \langle \lambda^{p-1} \rangle\}$ where $\lambda$ is a generator of the multiplicative group of $GF(p^n)$.

**Proof.** By virtue of Lemma 11 (7) and the proofs of [9, Proposition 19.8 and Theorem 19.11], we can regard $G$ as a permutation group on some $GF(p^n)$ such that $U = \{x \to x+b \mid b \in GF(p^n)\}$, $W$ is a subgroup of the Galois group of $GF(p^n)$ over $GF(p)$ and that $V$ is a subgroup of $\{x \to ax \mid a \in GF(p^n)\}$. Since $W$ is elementary abelian and cyclic, it follows that $W$ is of order $p$. Thus $n = pr$ with some integer $r$, and $p^{p+1} > p^{p+1}(p^p-1)/(p-1) \geq |G| > |W| \geq p^p$, which implies $r = 1$. Now, by Lemma 11 (5), we can easily see that the order of $V$ divides $(p^p-1)/(p-1)$. This completes the proof.

In [10], Y. Tsushima stated that if $H$ is a $p$-solvable group with a regular $p$-Sylow subgroup $S$ of order $p^e$ and $t(H) = \beta(p-1)+1$, then $S$ is elementary abelian. On page 37, line 11 [10], he claimed that since $P$ is of exponent $p$, $G$ is of $p$-length 1 by Hall Higman's theorem. However,
unfortunately, Tsushima's argument is unjustifiable for Fermat primes. The next shows that Tsushima's result holds under extra assumption that $O_{p^r,p}(H)/O_{p^r,p}(H)$ is abelian.

**Corollary 13.** Let $H$ be a $p$-solvable group with a $p$-Sylow subgroup $S$ of order $p^b$ and $O_{p^r,p}(H)/O_{p^r,p}(H)$ abelian. If $t(H) = \beta(p-1)+1$ and $S$ is regular, then $S$ is elementary abelian.

**Proof.** Let $H$ be a counter example of the minimal order. Then $S$ is not regular by Theorem 12 and [8, Lemma 4]. Hence $S$ is elementary abelian.

4. In this section, for $p = 3$, we shall give an example of a group $H$ which possesses all the properties listed in Lemma 11 but satisfies $t(H) > \beta(p-1)+1$ where $p^b$ is the order of a $p$-Sylow subgroup of $H$.

We set $a = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$, and $c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in $M = SL(2, 3)$. Then $a^4 = 1, b^2 = a^2, b^a = a^{-1}ba = b^{-1}, c^3 = 1, a^c = a^3b, b^c = a$, and $Q = \langle a, b \rangle$ is a normal $3'$-subgroup of $M$. Let $H$ be a semi-direct product of $U = \langle x, y \mid x^3 = y^3 = 1, xy = yx \rangle$ by $M$ with respect to the identity map of $M$. Then we have the following relations:

$x^a = y^2, y^a = x, x^b = xy^2, y^b = x^3y^2, x^c = x, y^c = xy$.

It is easy to see that $H$ possesses all the properties listed in Lemma 11. We set $a \circ b = a + b - ab, \chi = 1 + x + x^2, \nu = 1 + y + y^2, f = a^2 - 1$ and $\tau = c(1+a \circ b)f$ in the group algebra $KH$. Then we have the following

**Lemma 14.**

(1) $f$ is a central idempotent in $KM$.

(2) $a^2f = -f, f_0f = f_0f$ and $f_1x = xf$.

(3) $\tau$ is central in $KM$. In particular, $c$ commutes with $(a \circ b)f$.

(4) $\tau^2 = c^2(1-a \circ b)f$ and $\tau^3 = f$.

(5) $J(KH) \cong J(KT)$ where $KT$ is the group algebra of a cyclic group $T = \{f, \tau, \tau^2\}$ over $K$.

**Proof.** (1)---(4) can be proved by direct verification.

(5) It follows from (3) that $KT$ is contained in the center of $KM$. Since $J(KH)$ contains the kernel of the natural homomorphism $KH \rightarrow KH/U = KM$, we obtain $J(KH) \supseteq J(KM) \supseteq J(KT)$.

**Lemma 15.** $\chi f(y + \tau y \tau^2 + \tau^2 y \tau)f = -\overline{Uf}(1 + b + ab)$.
Proof. By making use of Lemma 14, we obtain the following equations:

\[ \chi ftr^2f = \chi c(1+a \circ b)fc^2(1-a \circ b)f \]
\[ = \chi(1+a \circ b)fci^{-2}yc^2(1-a \circ b)f \]
\[ = \chi f(1-b^{-1}a^{-1})xyt(1-a \circ b) \]
\[ = \chi f(x^2y+(x^2y)a+(x^2y)b-(x^2y)ab)f(1-a \circ b) \]
\[ = f(x^2y+yab)(1-a-b+ab)f \]
\[ = \chi f(1+a+b+ab-ya-yb-yab)f. \]

Similarly,

\[ \chi fr^2yrf = \chi c^2(1-a \circ b)fyfc(1+a \circ b)f \]
\[ = \chi(1-a \circ b)fyc(1+a \circ b)f \]
\[ = \chi f(1+a^{-1}b^{-1})xyf(1+a \circ b) \]
\[ = \chi f(xy-(xy)a-(xy)b+(xy)ab)f(1+a \circ b) \]
\[ = f(xy-xy^2a-yb+x^2ab)f(1+a \circ b) \]
\[ = \chi f(y^2a-yb+ab)f(1+a \circ b) \]
\[ = \chi f(y-ya-yb+ab)(1+a-b-ab)f \]
\[ = \chi f(1-a+b+ab-ya-yb-yab)f. \]

Thus, from those above we get

\[ \chi f(y+yr^2+yr)f = \chi f(y-1)(1+b+ab)f \]
\[ = -\chi f(1+b+ab) = -\tilde{U}(1+b+ab). \]

We are now ready to establish \( t(H) > 7 = 3(3-1)+1. \)

Proposition 16. \( t(H) \geq 9. \)

Proof. It follows from Lemma 14 (5) that \( J(KH)^a \) contains an element \( \chi \tilde{T}v \tilde{T} \) where \( \tilde{T} = f+r+r^2. \) By Lemma 15, we have

\[ \chi \tilde{T}v \tilde{T} = \chi \tilde{T}(f+y+yr^2)f \tilde{T} = -\chi \tilde{T}yf \tilde{T} = -\chi f \tilde{T}yf \]
\[ = -\chi f(y+yr^2+yr^2)f \tilde{T} = \tilde{U}(1+b+ab) \tilde{T} \]
\[ = \tilde{U}(1+b+ab) = \tilde{U}(1+c+c^2)(1+b+ab)f \neq 0. \]

This completes the proof.

5. In this section, we shall prove that there exist no finite rings \( R \) satisfying the following conditions:

1) \( R \) is a finite ring of characteristic \( p. \)
2) \( R \) admits an automorphism \( \sigma \) of order \( p. \)
3) \( \text{Tr}(a) = 0 \) for every \( a \in R \) where \( \text{Tr}(a) \) means the \( \langle \sigma \rangle \)-trace of \( a. \)
4) There exists a \( p' \)-subgroup \( T \) of the unit group of \( R \) such that \( T \)
is abelian, \( \sigma(T) \subseteq T, \ T \cap R^\sigma = 1 \) and \( R = \{ tc \mid t \in T, \ c \in R^\sigma \} \), where \( R^\sigma = \{ c \in R \mid \sigma(c) = c \} \).

Suppose to the contrary that there exists such a ring, and consider a permutation group \( H = \langle U, V, W \rangle \) on \( R \) (acting on the left), where \( U = \{ u_r : x \to x + r \mid r \in R \} \), \( V = \{ v_t : x \to tx \mid t \in T \} \) and \( W = \langle \sigma \rangle \). Since the condition \( T \cap R^\sigma = 1 \) implies that \( VW \) is a Frobenius group, we get \( J(KH) = J(KW) \bar{V}K + J(KU)KH \) (see Corollary 8 and [8, Proposition 3]). On the other hand, \( R = \{ tc \mid t \in T, \ c \in R^\sigma \} \) gives \( H = VC_G(W)V \) (see Example 3). Thus \( (J(KW) \bar{V}K)^p = 0 \) by Theorem 2 and hence \( t(G) = \beta(p-1) + 1 \). The condition 3) implies that \( (\sigma^k u_a)^p = \sigma^kp^u_{R(\alpha)} = 1 \), and so \( S = WU \) is of exponent \( p \). Hence a \( p \)-Sylow subgroup \( S \) of \( H \) is regular. Then, since \( V \) is abelian, it follows from Corollary 13 that \( S \) is elementary abelian, which contradicts the fact that \( S \) is not abelian.

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