Note on strongly M-injective modules

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NOTE ON STRONGLY M-INJECTIVE MODULES

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Given a unitary left module $M$ over a ring $R$ with identity, G. Azumaya [1] introduced the useful notion of $M$-projective and $M$-injective modules; the notion of a strongly $M$-projective module has subsequently been given by K. Varadarajan [8]. In this note, we investigate strongly $M$-injective modules. Let $t$ be the left exact preradical (for $R$-mod) corresponding to the left linear topology which has the smallest element $l_R(M)$, the left annihilator of $M$. We prove first that a strongly $M$-injective module is nothing but a $t$-weakly divisible module (Theorem 1.3); in particular, if $M$ is faithful then every strongly $M$-injective module is injective which corresponds to [1, Theorem 14]. Next, we give the conditions for $t$ to be a radical and to be stable (Theorems 2.4 and 2.7).

Throughout this note, $R$ denotes a ring with identity, and modules mean unitary left $R$-modules, unless otherwise specified. The category of all modules is denoted by $R$-mod, and the injective hull of $A \in R$-mod by $E(A)$. As for terminologies and basic properties concerning preradicals and torsion theories, we refer to [6]. If $r$ is a left exact preradical, then $L(r) = \{ s \in R \mid r(s) = r(s) \}$ is a left linear topology on $R$. As is well-known, $L(r)$ has the smallest element if and only if $T(r)$ is closed under direct products (see, [3, III. 2. E4]).

In what follows, we fix a module $M$ and denote by $t$ the left exact preradical corresponding to the left linear topology which has the smallest element $T = l_R(M)$. As is easily seen, $t(A) = r_A(T) = \{ a \in A \mid Ta = 0 \}$ ($A \in R$-mod), and so $A$ is $t$-torsion if and only if $TA = 0$.

1. A module $Q$ is called strongly $M$-injective if every homomorphism of any submodule of $M$ into $Q$ can be extended to a homomorphism of $M'$ into $Q$ for any index set $I$. Clearly, every injective module is strongly $M$-injective and every strongly $M$-injective module is $M$-injective. But the converse is not necessarily true (see Examples 1.6 and 1.8). Also, a direct product of modules is strongly $M$-injective if and only if so are all its factors.

Now, let $r$ be a preradical. A module $Q$ is said to be $r$-weakly divisible (resp. $r$-divisible) if for any exact sequence in $R$-mod $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $B$ (resp. $C$) $r$-torsion, $\text{Hom}_R(-, Q)$ preserves the exactness. Since $M$ is $t$-torsion, every $t$-weakly divisible module is strongly $M$-injective.
Lemma 1.1. Let $Q$ be a module. Then there holds the following:

1) $Q$ is strongly $M$-injective if and only if so is $t(Q)$.
2) $Q$ is $t$-weakly divisible if and only if so is $t(Q)$.

Proof. As the proofs of the both are quite similar, we prove only (1). Suppose first that $Q$ is strongly $M$-injective, and consider a row-exact diagram in $R$-mod

$$
\begin{array}{c}
0 \rightarrow N \overset{i}{\rightarrow} M' \\
\downarrow f \\
t(Q) \\
\downarrow j \\
Q
\end{array}
$$

where $j$ is the inclusion. Then there exists $g : M' \rightarrow Q$ such that $g \circ i = j \circ f$, and $g(M') = g(t(M')) \subseteq t(Q)$. Thus $t(Q)$ is strongly $M$-injective. Conversely, suppose that $t(Q)$ is strongly $M$-injective and consider a row-exact diagram in $R$-mod

$$
\begin{array}{c}
0 \rightarrow N \overset{i}{\rightarrow} M' \\
\downarrow f \\
Q
\end{array}
$$

Since $N$ is $t$-torsion, $f(N) \subseteq t(Q)$, and therefore, by assumption, $g \circ i = f$ with some $g : M' \rightarrow t(Q)$. Thus $Q$ is strongly $M$-injective.

**Lemma 1.2.** The following are equivalent for a module $Q$:

1) $t(Q)$ is strongly $M$-injective.
2) $t(Q)$ is $t$-weakly divisible.
3) $t(Q)$ is injective as a left $R/T$-module.

Proof. Obviously, 2) $\Rightarrow$ 1).

1) $\Rightarrow$ 3). Consider a row-exact diagram in $R/T$-mod

$$
\begin{array}{c}
0 \rightarrow I/T \overset{i}{\rightarrow} R/T \\
\downarrow f \\
t(Q)
\end{array}
$$

Since $R/T$ can be embedded into a direct product of copies of $M$ and $t(Q)$ is strongly $M$-injective by assumption, there exists an $R$-homomorphism $g : R/T \rightarrow t(Q)$ such that $g \circ i = f$.

3) $\Rightarrow$ 2). Consider a row-exact diagram in $R$-mod

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\[ \begin{array}{c}
0 \longrightarrow A \overset{i}{\longrightarrow} B \\
\downarrow f \\
\downarrow \tau(Q)
\end{array} \]

where $B$ is $t$-torsion. Then, regarding the above diagram as in $R/T$-mod., we find an $R/T$-homomorphism $g : B \twoheadrightarrow \tau(Q)$ such that $g \circ i = f$, and hence $\tau(Q)$ is $t$-weakly divisible.

Now, combining Lemmas 1.1 and 1.2, we readily obtain

**Theorem 1.3.** The following conditions are equivalent for a module $Q$:
1) $Q$ is strongly $M$-injective.
2) $Q$ is $t$-weakly divisible.
3) $\tau(Q)$ is injective as a left $R/T$-module.

*In particular, if every $t$-torsion module is strongly $M$-injective, then every $R/T$-module is injective, and hence $R/T$ is semisimple artinian.*

As is easily seen, $M$ is faithful if and only if $t=1$. Thus we have

**Corollary 1.4.** If $M$ is faithful, the following are equivalent:
1) $Q$ is strongly $M$-injective.
2) $Q$ is $t$-divisible.
3) $Q$ is injective.

**Corollary 1.5.** Let $Q$ be a module with $E(Q)$ $t$-torsion. Then the following are equivalent:
1) $Q$ is strongly $M$-injective.
2) $Q$ is $t$-divisible.
3) $Q$ is injective.

**Proof:** It suffices to show that 1) $\Rightarrow$ 3). Since $Q$ is $t$-weakly divisible and $E(Q)$ is $t$-torsion, $Q$ is a direct summand of $E(Q)$. Hence $Q = E(Q)$.

We give here an example of a strongly $M$-injective module which is not injective.

**Example 1.6.** Let $K$ be a field, and $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Then $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ is an idempotent two-sided ideal of $R$. Now, we let $(T_1, T_2, T_3)$ be the 3-fold torsion theory corresponding to $M$ (see, e.g. [4]). Since $R/M$ is a flat right $R$-module, $T_1$ is a TTF-class. Let $t$ be the torsion functor
corresponding to \((T_1, T_2)\). Then, as is well-known, the left Gabriel topology corresponding to \(t\) has the smallest element \(l_t(M)\). Since \(E(R) = \begin{pmatrix} K & K \\ K & K \end{pmatrix}\), \(M\) coincides with \((E(R))\) and is strongly \(M\)-injective. However, \(M\) is not injective, since it is not a direct summand of \( _RR \).

Combining Lemma 1.1 and [7, Theorem 2.8] with the latter part of Theorem 1.3, we readily obtain

**Corollary 1.7.** The following conditions are equivalent:

1) Every module is strongly \(M\)-injective.

2) Every \(t\)-torsion module is strongly \(M\)-injective.

3) \(R/T\) is semisimple artinian.

4) \(M^J\) is completely reducible for any index set \(J\).

5) Every cyclic module is strongly \(M\)-projective.

6) Every module is strongly \(M\)-projective.

**Example 1.8.** Let \(M\) be a field, and \(R = M^J\), where \(J\) is an arbitrary infinite set. Then \(M\) is a simple \(R\)-module. Clearly, \(M^J\) is not completely reducible. In view of Corollary 1.7, this shows that there exists an \(M\)-injective module which is not strongly \(M\)-injective.

2. Given a left exact preradical \(r\), we define the following left exact preradicals \(r'\) and \(r_2\) (for \(R\)-mod):

\[
 r'(A) = \{a \in A \mid IJa = 0 \text{ for some } I \text{ and } J \text{ in } L(r)\};
\]

\[
 r_2(A)/r(A) = r(A/r(A)).
\]

**Lemma 2.1.**

1) \(r(A) \subseteq r'(A) \subseteq r_2(A)\).

2) If \(a \in r_2(A)\), then \(SLa = 0\) for some \(I \in L(r)\), where \(S = \bigcap_{I \in L(r)} I\).

3) \(L(r') = \{rK \subseteq _RR \mid K \supseteq IJ \text{ for some } I \text{ and } J \text{ in } L(r')\}\).

4) \(r = r'\) if and only if \(L(r)\) is closed under finite products.

5) If \(L(r)\) has the smallest element \(S\), then \(L(r_2)\) has the smallest element \(S^2\).

6) \(r = r_2\) if and only if \(r\) is a radical.

7) If \(L(r)\) has the smallest element, then \(r' = r_2\).

**Proof.** We can easily see (2)—(4) and (6). Moreover, (5) and (7) follow from (2). It suffices therefore to prove (1). Clearly, \(r(A) \subseteq r'(A)\) for any \(A \in R\)-mod. If \(x \in r'(A)\), then there exist \(I\) and \(J\) in \(L(r)\) such
that $\mathcal{U}x=0$. Since $r(A) = \{ y \in A \mid l_y(y) \in L(r) \}$, we have $f x \subseteq r(A)$. Thus, $f \tilde{x} = 0$, where $\tilde{x} = x + r(A) \subseteq r'(A)/r(A)$. This means $\tilde{x} \subseteq r'(A)/r(A) \subseteq r(A/r(A))$, and therefore $x \subseteq r_2(A)$.

The next shows that (1), (2) and (3) in [5, Theorem 6] are still equivalent for $n=1$.

**Corollary 2.2** ([3, III. 2. E5]). $t$ is a radical if and only if $T = T^2$.

**Proposition 2.3.** If $r$ is a left exact preradical, then the following are equivalent:

1) $r$ is a radical.
2) Every $r$-weakly divisible module is $r_2$-weakly divisible.
3) Every $r$-torsion $r$-weakly divisible module is $r$-divisible.

**Proof.** Obviously, 1) implies 2).

2) $\Rightarrow$ 3). Let $A$ be an $r$-torsion $r$-weakly divisible module. Then $A = r(A) = r(E(A))$. Since $A$ is $r_2$-weakly divisible by assumption, it is a direct summand of $r_2(E(A))$. On the other hand, it is essential in $r_2(E(A))$, and so $A = r_2(E(A))$. Hence, $r(E(A)/A) = r(E(A)/r(E(A))) = r_2(E(A))/r(E(A)) = 0$, proving that $A$ is $r$-divisible.

3) $\Rightarrow$ 1). Let $A$ be a module. Since $r(E(A))$ is $r$-divisible by assumption, [7, Proposition 1.2] shows that

$$0 = (r(E(A)) + r(E(A)))/r(E(A)) = r(E(A)/r(E(A))) \supseteq r(A + r(E(A))/r(E(A))) = r(A/r(A)),$$

proving that $r$ is a radical.

Combining Proposition 2.3 with Corollary 2.2, we get at once

**Theorem 2.4.** The following are equivalent:

1) Every $t$-weakly divisible module is $t_2$-weakly divisible.
2) Every $t$-torsion $t$-weakly divisible module is $t$-divisible.
3) $t$ is a radical.
4) $T = T^2$.

A preradical $r$ is said to be stable if $T(r)$ is closed under injective hulls. As another application of Proposition 2.3, we have

**Lemma 2.5** ([3, Proposition I.3.2]). Every stable left exact preradical $r$ is a radical.
Proof. If $A$ is an $r$-torsion $r$-weakly divisible module, then $A = r(A) = r(E(A)) = E(A)$. Hence $A$ is $r$-divisible, and therefore $r$ is a radical by Proposition 2.3.

**Proposition 2.6.** Let $r$ be an idempotent preradical. If every $r$-torsion $r$-weakly divisible module is injective, then $r$ is stable, and conversely.

Proof. Let $A \in T(r)$. Since $r(E(A))$ is injective by assumption, it is a direct summand of $E(A)$. Furthermore, it is essential in $E(A)$, and therefore $r(E(A)) = E(A)$. The converse is clear.

Now, we can prove the next

**Theorem 2.7.** The following conditions are equivalent:
1) Every $t$-torsion strongly $M$-injective module is injective.
2) Every injective left $R/T$-module is injective.
3) $t$ is stable.
4) $t$ is a radical and $R/T$ is flat as a right $R$-module.
5) $T = T^2$ and $R/T$ is flat as a right $R$-module.
6) $(T(t)^t, T(t), F(t))$ is a hereditary 3-fold torsion theory, where $T(t)^t = \{ A \in R \text{-mod} | \text{Hom}_R(A, X) = 0 \text{ for all } X \in T(t) \}$.

Proof. Obviously, 4) $\Rightarrow$ 5), and it is well-known that 5) $\iff$ 6) $\Rightarrow$ 3). The equivalence of 1) $\iff$ 3) is immediate by Lemma 1.2 and Proposition 2.6. Furthermore, 3) and 4) are equivalent by Lemma 2.5 and [2, Theorem 6].

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