ℵ₀-continuous directly finite pprojective modules over regular rings

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\( R_0 \)-CONTINUOUS DIRECTLY FINITE PPROJECTIVE MODULES OVER REGULAR RINGS

Dedicated to Prof. Kentaro Murata on his 60th birthday

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We have defined, in [5], the notions of quasi-\( R_0 \)- and \( R_0 \)-continuous modules which can be regarded as generalized notions of right \( R_0 \)-continuous regular rings (see [2], [3]). In this paper, we study the properties of quasi-\( R_0 \)-continuous directly finite projective modules over a regular ring.

Let \( R \) be a regular ring. For a quasi-\( R_0 \)-continuous directly finite projective \( R \)-module \( M \) and essentially \( R_0 \)-generated submodules \( A \) and \( B \) of \( M \) with \( A \cong B \), it is shown that \( A^* \cong B^* \), where \( A^* \) and \( B^* \) are direct summands of \( M \) with \( A \leq \epsilon A^* \) and \( B \leq \epsilon B^* \), respectively (Theorem 5). This is a generalization of [2, Corollary 14.26]. Using this theorem, we show that if \( M \) and \( N \) are quasi-\( R_0 \)-continuous directly finite projective \( R \)-modules then \( M \oplus N \) is directly finite (Theorem 8). Finally, we shall give examples of quasi-\( R_0 \)-continuous directly finite projective modules over regular rings which are not finitely generated.

Throughout this paper, \( R \) is a ring with identity and \( R \)-modules are unitary right \( R \)-modules. If \( M \) and \( N \) are \( R \)-modules, then the notation \( N \leq M \) means that \( N \) is isomorphic to a submodule of \( M \). For a submodule \( N \) of an \( R \)-module \( M \), \( N \leq \epsilon M \) means that \( N \) is essential in \( M \), while \( N \leq \bigoplus M \) means that \( N \) is a direct summand of \( M \).

Let \( M \) be an \( R \)-module and let \( \mathcal{A}(M) \) be the family of all submodule \( A \) of \( M \) such that \( A \) contains a countably generated essential submodule. Given such an \( \mathcal{A}(M) \), we consider the following conditions:

\( (C_1) \) For any \( A \in \mathcal{A}(M) \) there exists a submodule \( A^* \) of \( M \) such that \( A \leq \epsilon A^* \) and \( A^* \leq \bigoplus M \).

\( (C_2) \) For any \( A \in \mathcal{A}(M) \) with \( A \leq \bigoplus M \), any exact sequence \( 0 \to A \to M \) splits.

\( (C_3) \) For any \( A \in \mathcal{A}(M) \) with \( A \leq \bigoplus M \), if \( N \leq \bigoplus M \) and \( A \cap N = 0 \) then \( A \oplus N \leq \bigoplus M \).

We say that \( M \) is quasi-\( R_0 \)-continuous (resp. \( R_0 \)-continuous) if \( M \) satisfies the conditions \( (C_1) \) and \( (C_3) \) (resp. \( (C_1) \) and \( (C_2) \)). According to [8], every quasi-injective module is \( R_0 \)-continuous, every \( R_0 \)-continuous module is quasi-\( R_0 \)-continuous and \( M \) is quasi-\( R_0 \)-continuous if and only if \( M \) satisfies \( (C_1) \) and the condition

\( (\ast) \) For any \( A \in \mathcal{A}(M) \) and a direct summand \( N \) of \( M \) with \( A \cap N = 0 \),

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every homomorphism from $A$ to $N$ can be extended to a homomorphism from $M$ to $N$.

Let $M$ be an $R$-module. A submodule $B$ of $M$ is said to be $\mathcal{A}$-closed if $M/B$ is nonsingular. For any submodule $A$ of $M$ there exists the smallest $\mathcal{A}$-closed submodule $C$ of $M$ containing $A$, which is called the $\mathcal{A}$-closure of $A$ in $M$.

Now, we recall the following

**Lemma 1** ([5, Lemma 1]). Let $M$ be an $R$-module, and let $A$ and $B$ be submodules of $M$ such that $A \leq_e B$. Then $B$ is contained in the $\mathcal{A}$-closure of $A$ in $M$. If, in addition, $M$ is nonsingular and $B < \bigoplus M$, then $B$ coincides with the $\mathcal{A}$-closure of $A$ in $M$.

We note that if $M$ is a nonsingular module satisfying (C$_1$) then, for any $A \in \mathcal{A}(M)$, there exists a unique $A^*$ such that $A \leq_e A^*$ and $A^* < \bigoplus M$. Moreover, if $M$ is nonsingular and quasi-$\mathfrak{N}_0$-continuous (resp. $\mathfrak{N}_0$-continuous), then so is every direct summand of $M$ (Lemma 1).

**Lemma 2.** Let $M$ be a quasi-$\mathfrak{N}_0$-continuous $R$-module, and let $A$, $B \in \mathcal{A}(M)$. If $A \cap B = 0$ and $A \cong B$, then $A^* \cong B^*$.

**Proof.** Let $f : A \to B$ be an isomorphism. Since $A \cap B^* = 0$, in view of (*), $f$ can be extended to a homomorphism $f^* : A^* \to B^*$, which is a monomorphism, because $A \leq_e A^*$. Since $A^* \cong f^*(A^*) \in \mathcal{A}(M)$, $f^*(A^*) \leq_e B^*$ and $A^* \cap f^*(A^*) = 0$, again by (*), $(f^*)^{-1} : f^*(A^*) \to A^*$ can be extended to a monomorphism $h : B^* \to A^*$. Then $h$ is an isomorphism, and thus $f^* = h^{-1}$ is an isomorphism with $f^* |_A = f$.

An $R$-module $M$ is *directly finite* provided that $M$ is not isomorphic to any proper direct summand of itself. We can use Lemma 2 to get the following generalization of [5, Theorem 2]. The proof is quite similar to the $\mathfrak{N}_0$-continuous case.

**Proposition 3.** If $M$ is a nonsingular quasi-$\mathfrak{N}_0$-continuous $R$-module, then the following are equivalent:

a) $M$ is directly finite.

b) $M$ contains no infinite direct sums of nonzero pairwise isomorphic submodules.

c) Any submodule of $M$ is directly finite.

An $R$-module $M$ is said to have the *finite exchange property* if, for any
direct decomposition \( G = M' \oplus C = \bigoplus_{i \in I} D_i \) with \( M' \cong M \) and the index set \( I \) finite, there are submodules \( D_i \leq D_i \) such that \( G = M' \oplus (\bigoplus_{i \in I} D_i) \).

**Lemma 4** ([7, Corollary 4]). *Every projective module over a regular ring has the finite exchange property.*

We are now in a position to prove the main theorem, which generalizes [2, Corollary 14.26] (cf. [6, Theorem 4] for quasi-continuous modules).

**Theorem 5.** Let \( M \) be a quasi-\( \aleph_0 \)-continuous directly finite projective module over a regular ring \( R \), and let \( A, B \in \mathcal{A}(M) \).

(a) If \( A \cong B \) then \( A^* \cong B^* \).

(b) If \( A \leq B \) then \( A^* \leq B^* \).

**Proof.** (a) Put \( D = A \cap B \) and let \( f : A \to B \) be an isomorphism. In view of [2, Lemma 14.10 (a)], we see that \( D \in \mathcal{A}(M) \), and so there exists \( D^* \) such that \( D^* < \mathcal{A} \neq B^* \) (Lemma 1). Put \( A^* = D^* \oplus X \) and \( B^* = D^* \oplus Y \) with some \( X \) and \( Y \). We claim the following facts:

1. If \( L \in \mathcal{A}(M) \) and \( N \leq \mathcal{A}(M) \), then \( N \in \mathcal{A}(M) \), and therefore both \( X \) and \( Y \) are in \( \mathcal{A}(M) \).

2. \( A^* \cap B^* = D^* \) and \( X \cap Y = X \cap B^* = A^* \cap Y = 0 \).

**Proof of (1).** Put \( L = N \oplus K \) with some \( K \), and let \( \pi : L \to N \) be the projection map. There exists a countably generated submodule \( L' \) such that \( L' \leq \epsilon L \). Then \( \pi(L') \) is countably generated and \( L' \cap N \leq \pi(L') \leq N \). Hence \( \pi(L') \leq \epsilon N \).

**Proof of (2).** Obviously, \( D^* \leq A^* \cap B^* \). On the other hand, \( D = A \cap B \leq \epsilon A^* \cap B^* \) and \( A^* \cap B^* \leq D^* \) by Lemma 1. Hence \( A^* \cap B^* = D^* \), and therefore \( X \cap Y = X \cap B^* = X \cap A^* \cap B^* = X \cap D^* = 0 \). Similarly, \( A^* \cap Y = 0 \).

We claim further that there exist decompositions

\[
X = X_1 \oplus X_1, \quad X'_n = X_{n+1} \oplus X_{n+1},
\]

\[
Y = Y_1 \oplus Y_1, \quad Y''_n = Y'_{n+1} \oplus Y'_{n+1},
\]

\[
D^* = D_1 \oplus D_1, \quad D''_n = D'_{n+1} \oplus D'_{n+1}
\]

such that

\[
X_n \cong Y_n, \quad D_{2n-1} \cong X_{2n-1}, \quad D_{2n} \cong Y''_{2n} \quad (n = 1, 2, \cdots).
\]

Since \( f(A \cap X) \) and \( f(D) \) are in \( \mathcal{A}(M) \) and \( f(A \cap X) \oplus f(D) \leq \epsilon f(A) = B \), we have \( B^* = (f(A \cap X)) \oplus f(D)^* \) by Lemma 1 and \((C_3)\). According to Lemma 4, there exist decompositions \( Y = Y_1 \oplus Y''_1 \) and \( D^* = D_1 \oplus D''_1 \).
such that $Y \oplus D^* = B^* = (f(A \cap X))^* \oplus Y_i^* \oplus D_i^*$. Then we have an isomorphism $g : Y_i^* \oplus D_i^* \rightarrow (f(D))^*$. Noting that $A \cap X \leq_e X$ and $(A \cap X) \cap f(A \cap X) \leq X \cap B^* = 0$, we see that $X \cong (f(A \cap X))^* \cong Y_i^* \oplus D_i^*$ by Lemma 2, and so there exists a decomposition $X = X_i^* \oplus X_i^*$ with $X_i^* \cong Y_i^*$ and $h : D_i \cong X_i^*$. Putting here $E = g^{-1}f(D)$ and $k = (h \oplus$ the identity map on $D_i^*) f^{-1} g |_{E} : E \rightarrow X_i^* \oplus D_i^*$. Since $k(Y_i^* \cap E) \oplus k(D_i^* \cap E) \leq_e X_i^* \oplus D_i^*$, we have

$$X_i^* \oplus D_i^* = X_i^* \oplus (D_i^* \cap E)^* = (k(Y_i^* \cap E))^* \oplus (k(D_i^* \cap E))^*,$$

where $Y_i^* \cong (k(Y_i^* \cap E))^*$. Then, by the above discussion, there exist decompositions

$$X_i^* = X_2^* \oplus X_2^*$$

$Y_i^* = Y_2^* \oplus Y_2^*$ and $D_i^* = (D_i^* \cap E)^* = D_2^* \oplus D_2^*$

with $X_i^* \oplus D_i^* = (k(Y_i^* \cap E))^* \oplus X_2^* \oplus D_2^*$ and isomorphisms $g_1 : X_i^* \oplus D_i^* \cong (k(D_i^* \cap E))^* \oplus X_2^* \cong Y_2^*$ and $h_1 : D_2 \cong Y_2^*$. Repeating this procedure successively, we obtain the desired decompositions.

Next, we claim that $\oplus_{n=0}^\infty X_n \leq_e Y$ and $\oplus_{n=0}^\infty Y_n \leq_e Y$ (cf. the proof of [3, Theorem 1.4]). Suppose that $C \cap (\oplus_{n=0}^\infty X_n) = 0$ for a cyclic submodule $C$ of $X$. Obviously, $C \subset \mathcal{A}(M)$, $C < \oplus X \leq \oplus M$ and $X_1 \oplus \cdots \oplus X_{2n-1} \subset \oplus M$. Since $C \oplus (X_1 \oplus \cdots \oplus X_{2n-1}) \subset \oplus X$ by (C3), we see that $C \leq X_{2n-1} \leq D_{2n-1}$, and so $C \oplus C \oplus \cdots \leq D_1 \oplus D_3 \oplus \cdots \leq D^* \leq \oplus M$. Thus, $C=0$ by Proposition 3. Similarly, we can show that $\oplus_{n=0}^\infty Y_n \leq_e Y$.

Now, noting that $\oplus_{n=0}^\infty X_n \cap \oplus_{n=0}^\infty Y_n = 0$ by (2) and that $\oplus_{n=0}^\infty X_n \cong \oplus_{n=0}^\infty Y_n \in \mathcal{A}(M)$ by (1), we see that $X \cong Y$ by Lemma 2, and so $A^* = D^* \oplus X \cong D^* \oplus Y = B^*$.

(b) If $f : A \rightarrow B$ is a monomorphism, then $A \cong f(A) \in \mathcal{A}(M)$ and $f(A) \leq B$. Hence $A^* \cong (f(A))^* \leq B^*$ by (a) and Lemma 1.

**Corollary.** Let $M$ be a quasi-$\aleph_0$-continuous directly finite projective module over a regular ring $R$ such that $M \in \mathcal{A}(M)$. Then the following are equivalent:

a) $M$ is $\aleph_0$-continuous.

b) Every injective endomorphism of $M$ is an automorphism.

An $R$-module $M$ is said to have the cancellation property if $M \oplus H \cong M \oplus K$ implies always $H \cong K$, or equivalently if $M \oplus H = N \oplus K$ with $M \cong N$ implies $H \cong K$ (cf. [1]).

From the proof of [1, Theorem 2], we see the next

**Lemma 6.** Let $M$ be an $R$-module with the finite exchange property.
Then \( M \) has the cancellation property if and only if isomorphic direct summands of \( M \) have isomorphic complements.

**Theorem 7.** Let \( M \) be a quasi-\( \aleph_0 \)-continuous directly finite projective module over a regular ring \( R \). If \( M \in \mathcal{A}(M) \), then \( M \) has the cancellation property.

**Proof.** Let \( M = A \oplus C = B \oplus D \) with \( C \cong D \) for some \( R \)-modules \( A \), \( B \), \( C \) and \( D \). Since \( M \in \mathcal{A}(M) \), \( A \), \( B \), \( C \) and \( D \) are also in \( \mathcal{A}(M) \) by the statement (1) in the proof of Theorem 5. Then \( A \cong B \) by Proposition 3, Lemma 4, Theorem 5 and the proof of [3, Lemmas 1.2, 1.3 and Theorem 1.4]. Hence, by Lemmas 4 and 6, \( M \) has the cancellation property.

**Theorem 8.** Let \( R \) be a regular ring. If \( M \) and \( N \) are quasi-\( \aleph_0 \)-continuous directly finite projective \( R \)-modules, then \( M \oplus N \) is directly finite.

**Proof.** Let \( M \oplus N = X \oplus Y \) for some submodules \( X \) and \( Y \) with an isomorphism \( f : M \oplus N \rightarrow X \). Take a cyclic submodule \( yR \) of \( Y \). Then \( \{ yR, f(yR), f^2(yR), \cdots \} \) is an infinite independent sequence of pairwise isomorphic cyclic submodules of \( M \oplus N \). Noting that \( yR \perp M \oplus N \), we have decompositions \( M = M_1 \oplus M_1' \) and \( N = N_1 \oplus N_1' \) such that \( M \oplus N = yR \oplus M_1 \oplus N_1 \) and \( yR \cong M_1 \oplus N_1 \) (Lemma 4). Next, noting that \( yR \perp f(yR) \perp M_1 \oplus N_1 ' \), again by Lemma 4 we have decompositions \( M_1' = M_2 \oplus M_2' \) and \( N_1' = N_2 \oplus N_2' \) such that \( M \oplus N = yR \oplus f(yR) \oplus M_2 \oplus N_2 \) and \( f(yR) \cong M_2 \oplus N_2 \). Continuing this procedure, we have decompositions

\[
M = M_1 \oplus M_1', \quad M_2 = M_{n+1} \oplus M_{n+1}'
\]
\[
N = N_1 \oplus N_1', \quad N_2 = N_{n+1} \oplus N_{n+1}'
\]

such that \( yR \cong M_i \oplus N_i \) and \( f^n(yR) \cong M_{n+1} \oplus N_{n+1} \) (\( n = 1, 2, \ldots \)), where \( M_n \) and \( N_n \) are cyclic \( R \)-modules.

Put \( A = M_1 \oplus M_2 \oplus \cdots, B = N_1 \oplus N_2 \oplus \cdots, C = M_2 \oplus M_3 \oplus \cdots, \) and \( D = N_2 \oplus N_3 \oplus \cdots \). Then \( A = M_1 \oplus C, B = N_1 \oplus D, \) and there exists an isomorphism \( g : A \oplus B \rightarrow C \oplus D \). Since \( g(A) \) is projective, there exist decompositions \( C = C' \oplus C'' \) and \( D = D' \oplus D'' \) such that \( C \oplus D = g(A) \oplus C'' \oplus D'' \) (Lemma 4). Then \( A \cong g(A) \cong C' \oplus D' \) and \( B \cong g(B) \cong C'' \oplus D'' \), and so there exist decompositions \( A = A' \oplus A'' \) and \( B = B' \oplus B'' \) with \( A' \cong C', \) \( m : A'' \cong D', l : B' \cong C'' \) and \( B'' \cong D'' \). Noting that \( A' \oplus A'' \cong M_1 \oplus C' \oplus C'' \) and \( A' \cong C' \), by Theorem 5 (a) we see that \( A^* \oplus A'^* = M_1 \oplus C^* \oplus C'^* \) and \( A^* \cong C^* \) in \( M \). Since \( A^* \) is a quasi-\( \aleph_0 \)-continuous directly finite projective module belonging to \( \mathcal{A}(A^*) \), we have an isomorphism \( h : A'^* \cong
$M_i \oplus C''$ by Theorem 7. Similarly, we get an isomorphism $k : B^* \cong N_i \oplus D^*$. Now, we put $E = l(k^{-1}(k(B') \cap D'))$. Then $m^{-1}(k(B') \cap D') \cong E \leq C''$ and $m^{-1}(k(B') \cap D')(\subseteq A(M))$ is an essential submodule of $A''$, and so $A'' = E^* \ll \oplus C''$ in $M$ by Theorem 5 (a), and $A'' = h^{-1}(M_i) \oplus h^{-1}(C'') \oplus h^{-1}(M_i) \oplus h^{-1}(E^*)$. Since $A''$ is directly finite, we have $h^{-1}(M_i) = 0$, and so $M_i = 0$; likewise $N_i = 0$. Hence $yR \cong M_i \oplus N_i = 0$, and so $y = 0$, which means that $M \oplus N$ is directly finite.

A regular ring $R$ is said to satisfy the comparability axiom provided that, for any $x, y \in R$, either $xR \leq yR$ or $yR \leq xR$ ([2, p. 80]). It is known that if $R$ is a regular ring satisfying the comparability axiom then, for any finitely generated projective $R$-modules $A$ and $B$, either $A \leq B$ or $B \leq A$ ([2, Proposition 8.2]).

**Theorem 9.** Let $R$ be a regular ring satisfying the comparability axiom. Then every quasi-$\aleph_0$-continuous directly finite projective $R$-module $P$ is finitely generated, and hence $\aleph_0$-continuous.

**Proof.** Suppose, to the contrary, that $P$ is not finitely generated. Then, by [4], $P = \bigoplus_{i \in I} x_i R$ for some infinite index set $I$. Given $j \in I$, we set $N_j = \{i \in I \mid x_j R \leq x_i R\}$. Then, $N_j$ is a finite set by Proposition 3. We can therefore find an infinite sequence $\{i_1, i_2, \cdots\}$ in $I$ (and (non-isomorphic) monomorphisms $f_{i_n} : x_{i_n} R \to x_{i_n} R (n = 1, 2, \cdots)$. Then there exist nonzero $a_{i_n} R$ such that $x_{i_n} R = f_{i_n} (x_{i_n} R) \oplus a_{i_n} R$. Since $\bigoplus_{n=1}^\infty x_{i_n} R = (\bigoplus_{n=1}^\infty f_{i_n} (x_{i_n} R)) \oplus (\bigoplus_{n=2}^\infty a_{i_n} R)$ and $\bigoplus_{n=2}^\infty a_{i_n} R$ are quasi-$\aleph_0$-continuous directly finite countably generated projective $R$-modules, Theorem 7 shows that $x_{i_n} R \cong \bigoplus_{n=1}^\infty a_{i_n} R$, which is a contradiction.

A regular ring is called abelian if all idempotents in $R$ are central ([2, p. 25]). As the following examples show, the assumption “the comparability axiom” cannot be deleted in Theorem 9.

**Example 1.** Let $R$ be an abelian right $\aleph_0$-continuous regular ring. Then, it is easy to see that every right ideal of $R$ is an $\aleph_0$-continuous directly finite $R$-module.

**Example 2.** Let $I$ be an index set. For each $i \in I$, consider a ring $R_i$ and an $R_i$-module $M_i$. Put $R = \prod_{i \in I} R_i$ and $M = \bigoplus_{i \in I} M_i$.

(a) If every $R_i$-module $M_i$ is quasi-$\aleph_0$-continuous (resp. $\aleph_0$-continuous), then so is the $R$-module $M$. 

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(b) If every $R_i$-module $M_i$ is directly finite, then so is the $R$-module $M$.

(c) In particular, if every $R_i$ is a right $\aleph_0$-continuous directly finite regular ring, then the right $R$-module $R$ is $\aleph_0$-continuous and directly finite.

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