Self-homotopy of the Double Suspension of the Real 7-projective Space

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Abstract

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KEYWORDS: self-homotopy, real projective space
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1. Introduction

In this paper, all spaces, maps and homotopies are based. We use the same
notation as [10] and [5]. Let $\Sigma^n X$ be an $n$-fold suspension of a space $X$ and
$P^n$ be the $n$-dimensional real projective space. The purpose of the present
paper is to determine the group structure of the homotopy set $[\Sigma^2 P^7, \Sigma^2 P^7]$. We denote by $\gamma_n : S^n \to P^n$ the covering map. According to [9], $\Sigma^2 \gamma_6 = 0$, $\Sigma^2 P^7 = \Sigma^2 P^6 \vee S^9$, and so

$[\Sigma^2 P^7, \Sigma^2 P^7] \cong [\Sigma^2 P^6, \Sigma^2 P^6] \oplus \pi_9(\Sigma^2 P^6) \oplus \pi_9(S^9)$.

Let $\mathbb{Z}$ be the group of integers and set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. The notation $(\mathbb{Z}_n)^m$
means a direct sum of $m$-copies of $\mathbb{Z}_n$. Our result is stated as follows.

Theorem 1.1. $[\Sigma^2 P^7, \Sigma^2 P^7] \cong \mathbb{Z} \oplus (\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^7$.

In this paper we sometimes identify a map with its homotopy class. For
$m < n$, let $i_{m,n} : P^m \to P^n$ and $p_{n,m} : P^n \to P^n/P^m$ be the inclusion
and collapsing maps, respectively. Especially, we write $M^n = \Sigma^{n-2} P^2$, $i_n = \Sigma^{n-2} i_{1,2} : S^{n-1} \to M^n$ and $p_n = \Sigma^{n-2} p_{2,1} : M^n \to S^n$ for $n \geq 2$. We denote by $[\alpha, \beta]$ the Whitehead product of homotopy classes $\alpha$ and $\beta$. To determine the group structure of $\pi_9(\Sigma^2 P^6)$, we use the following.

Theorem 1.2. $[\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}] = 0 \in \pi_9(\Sigma^2 P^5)$.

2. Some homotopy groups

We denote by $\iota_X \in [X, X]$ the identity class of a space $X$ and let $\iota_n = \iota_{S^n}$. For the Hopf maps $\eta_2 \in \pi_3(S^2)$ and $\nu_4 \in \pi_7(S^4)$, we set $\eta_n = \Sigma^{n-2} \eta_2$, $\eta_n^2 = \eta_n \eta_{n-1}$, $\nu_n^4 = \eta_n \eta_{n-1} \eta_{n-2}$ for $n \geq 2$ and $\nu_n = \Sigma^{n-4} \nu_4$ for $n \geq 4$. We recall from [7] that there is an element $\widetilde{\eta}_2 \in \pi_4(M^3)$ such that $p_3 \widetilde{\eta}_2 = \eta_3$ and $\Sigma \widetilde{\eta}_2 = \eta_3$, where $\eta_3$ is a coextension of $\eta_3$. Let $\eta_3 \in [M^5, S^3]$ be an extension of $\eta_3$ and set $\eta_n = \Sigma^{n-2} \eta_n^2$ for $n \geq 2$ and $\eta_n = \Sigma^{n-3} \eta_3$ for $n \geq 3$.

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Lemma 2.1. \( \pi_9(M^4) = Z_2\{\lambda_2 \nu_6\} \oplus Z_2\{[\lambda_2, i_4] \eta_8\} \oplus Z_2\{\tilde{\eta}_3 \nu_5 \eta_8\} \).

Let \( s : S^5 \rightarrow \Sigma^2 P^3 = M^4 \vee S^5 \) be the inclusion to the second factor. Then, we recall
\[
(2.1) \quad \Sigma^2 \gamma_3 = 2s \pm (\Sigma^2 i_{2,3}) \tilde{\nu}_3.
\]

By the Hilton-Milnor theorem, we obtain
\[
(2.2) \quad \pi_i(\Sigma^2 P^3) \cong \pi_i(M^4) \oplus \pi_i(S^5) \oplus \pi_i(\Sigma(M^7 \wedge M^3)),
\]
for \( i \leq 9 \). By Lemma 2.1 and the facts that \( \pi_8(M^4) = Z_2\{\lambda_2 \eta_6^2\} \oplus Z_2\{[i_4, \lambda_2]\} \oplus Z_2\{\tilde{\eta}_3 \nu_5\} \) [8, Lemma 2.4], \( \pi_8(S^5) = Z_{24}\{\nu_5\}, \pi_8(M^8) = Z_2\{i_8 \eta_7\}, \pi_9(S^5) = Z_2\{\nu_5 \eta_8\}, \pi_9(M^8) = Z_4\{\tilde{\eta}_7\} \) and \( \pi_9(\Sigma(M^7 \wedge M^3)) = Z_2\{\Sigma(i_7 \wedge i_3)\} \), we have the following.

Lemma 2.2.

(1) \( \pi_8(\Sigma^2 P^3) = Z_2\{((\Sigma^2 i_{2,3}) \lambda_2 \eta_6^2) \} \oplus Z_2\{((\Sigma^2 i_{2,3})[i_4, \lambda_2]\} \oplus Z_2\{\tilde{\eta}_3 \nu_5\} \oplus Z_{24}\{s \nu_5\} \oplus Z_2\{[\Sigma^2 i_{1,3}, s] \eta_7\}, \)

(2) \( \pi_9(\Sigma^2 P^3) = Z_2\{((\Sigma^2 i_{2,3}) \lambda_2 \nu_6\} \oplus Z_2\{((\Sigma^2 i_{2,3})[i_4, \lambda_2]\} \oplus Z_2\{\tilde{\eta}_3 \nu_5 \eta_8\} \oplus Z_2\{s \nu_5 \eta_8\} \oplus Z_4\{[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}\} \).

Let \( X \) be a connected finite CW-complex and \( X^* = X \cup_0 e^n \) for \( \theta : S^{n-1} \rightarrow X \) a complex formed by attaching an \( n \)-cell. We denote by
\[
\omega_n(X^*, X) \in \pi_n(X^*, X)
\]
the characteristic map of the \( n \)-cell \( e^n \) of \( X^* \). Let \( CY \) be a cone of a space \( Y \). For an element \( \alpha \in \pi_m(Y) \), we denote by \( \tilde{\alpha} \in \pi_{m+1}(CY, Y) \) an element satisfying \( \partial'(\tilde{\alpha}') = \alpha \), where \( \partial' : \pi_{m+1}(CY, Y) \rightarrow \pi_m(Y) \) is the connecting bijection. For \( \alpha \in \pi_m(S^{n-1}) \), we set
\[
\tilde{\alpha} = \omega_n(X^*, X) \circ \tilde{\alpha}' \in \pi_{m+1}(X^*, X).
\]

We note the following:
\[
\partial(\tilde{\alpha}) = \theta \circ \alpha \quad \text{and} \quad p_* \tilde{\alpha} = \Sigma \alpha,
\]
where \( \partial : \pi_{m+1}(X^*, X) \rightarrow \pi_m(X) \) is the boundary map and \( p : (X^*, X) \rightarrow (S^n, *) \) is the collapsing map. Now we show the following.

Lemma 2.3. \( 2(\Sigma^2 i_{3,4}) \pi_9(\Sigma^2 P^3) = 0 \).
Proof. We consider the homotopy exact sequence of a pair \((\Sigma^2P^4, \Sigma^2P^3)\):

\[
\pi_{10}(\Sigma^2P^4, \Sigma^2P^3) \xrightarrow{\partial_{10}} \pi_9(\Sigma^2P^3) \xrightarrow{(\Sigma^2i_{3,4})_*} \pi_9(\Sigma^2P^4).
\]

There exists an element \([\omega_6, s] \in \pi_{10}(\Sigma^2P^4, \Sigma^2P^3)\) for \(\omega_6 = \omega_6^{(\Sigma^2P^4, \Sigma^2P^3)}\). By the relations \([1, (3.5)]\) and \((2.1)\), we have

\[
\partial_{10}([\omega_6, s]) = -[\Sigma^2\gamma_3, s] = \pm((\Sigma^2i_{2,3})\eta_3, s) = \pm[\Sigma^2i_{2,3}, s]\eta_7.
\]

Hence, by Lemma 2.2 \((2)\), we obtain \(2((\Sigma^2i_{3,4})_*\pi_9(\Sigma^2P^3)) = 0\). This completes the proof.

Since \(\Sigma^2\gamma_3 \circ \eta_8^3 = (\Sigma^2i_{2,3})\eta_3 \circ 4\nu_5 = 0\) by \((2.1)\), there exists an element \(\eta_8^3 \in \{\Sigma^2i_{3,4}, \Sigma^2\gamma_3, \eta_8^3\} \subset \pi_9(\Sigma^2P^4)\) such that \((\Sigma^2p_{4,3})\eta_8^3 = \eta_8^3\). For this element, we show the following.

Lemma 2.4. \(\{\eta_3\eta_8^2, \eta_7, 2t_8\} = \eta_3\nu_5\eta_8\) and the order of \(\eta_8^3\) is two.

Proof. By the properties of Toda brackets and by \([3, \text{Lemma 4.1}]\), we have

\[
\{\eta_3\eta_8^2, \eta_7, 2t_8\} \circ p_9 = -(\eta_3\eta_8^2 \circ \eta_7, 2t_8, p_8) = \eta_3\eta_8^2 \eta_7 = \eta_3\nu_5\eta_8 p_9.
\]

Since \(p_9^* : \pi_9(M^4) \rightarrow [M^9, M^4]\) is a monomorphism by Lemma 2.1, we obtain the first. By \((2.1)\), the relation \((\Sigma^2i_{2,4})\eta_3\nu_5\eta_8 = \pm 2((\Sigma^2i_{3,4})s) \circ \eta_3\nu_5\eta_8 = 0\) holds. So, by the first and Lemma 2.3, we have

\[
2\eta_8^3 \in \{\Sigma^2i_{3,4}, \Sigma^2\gamma_3, \eta_8^3\} \circ 2t_9
\]

\[
= -(\Sigma^2i_{3,4} \circ \{\Sigma^2\gamma_3, \eta_8^3, 2t_8\})
\]

\[
\supset -(\Sigma^2i_{2,4} \circ \{\eta_3\eta_8^2, \eta_7, 2t_8\})
\]

\[
= (\Sigma^2i_{2,4})\eta_3\nu_5\eta_8 = 0 \mod 2((\Sigma^2i_{3,4})_*\pi_9(\Sigma^2P^3)) = 0.
\]

This leads to the second and completes the proof.

Next we compute the homotopy groups of the homotopy fibre of \(\Sigma^2p_{4,3} : \Sigma^2P^4 \rightarrow S^6\) to determine \(\pi_9(\Sigma^2P^4)\). Let \(K\) be the homotopy fibre of \(\Sigma^2p_{4,3}\). By \([2, \text{Corollary 5.8}]\), the 10-skeleton of \(K\) has a cellular decomposition

\[
K^{(10)} = \Sigma^2P^3 \cup_{[\Sigma^2i_{2,3}, \Sigma^2\gamma_3]} C\Sigma^6P^3.
\]

For \(m < n\), we denote by \(i^K_{m,n} : K^{(m)} \rightarrow K^{(n)}\) and \(i^K_{n,m} : K^{(m)} \rightarrow K\) the inclusion maps and \(p^K_{n,m} : K^{(n)} \rightarrow K^{(n)/K^{(n)}}\) the collapsing map.

Lemma 2.5.

(1) \(\pi_8(K) = \mathbb{Z}_2\{i^K_{4, i_4, \lambda_2}\} \oplus \mathbb{Z}_2\{i^K_{4, \eta_3\nu_5}\} \oplus \mathbb{Z}_2\{i^K_{5, s\nu_5}\} \oplus \mathbb{Z}_2\{i^K_{5, \Sigma^2i_{1,3}, s}\}\eta_7\),

(2) \(\pi_9(K) = \mathbb{Z}_2\{i^K_{4, \lambda_2\nu_6}\} \oplus \mathbb{Z}_2\{i^K_{5, s\nu_5\eta_8}\} \oplus \mathbb{Z}_2\{i^K_{5, [\Sigma^2i_{1,3}, s]}, \Sigma^2i_{1,3}\}\).
Proof. We consider the homotopy exact sequence of a pair \((K^{(8)}, \Sigma^2P^3)\):

\[
\pi_{10}(K^{(8)}, \Sigma^2P^3) \xrightarrow{\partial_{10}} \pi_9(\Sigma^2P^3) \xrightarrow{i_{8,5}^K} \pi_9(K^{(8)}) \xrightarrow{\partial_{8}} \pi_8(K^{(8)}, \Sigma^2P^3) \]

The group structures \(\pi_8(K^{(8)}, \Sigma^2P^3) = \mathbb{Z}\{\omega_8\}\) and \(\pi_9(K^{(8)}, \Sigma^2P^3) = \mathbb{Z}_2\{\tilde{\eta}_7\}\) are obtained by the Blakers-Massey theorem, where \(\omega_8 = \omega^{(K^{(8)}, \Sigma^2P^3)}\). By (2.1) and the relation \([i_4, \nu_M] = \lambda_2\nu_6\) [8, Lemma 1.5], the attaching map of the 8-cell of \(K^{(8)}\) is \([i_{\Sigma^2P^3}, \Sigma^2\gamma_3] \circ \Sigma^6i_{1,3} = (\Sigma^2i_{2,3})[i_4, \nu_M]\tilde{\eta}_5 = (\Sigma^2i_{2,3})\lambda_2\eta_6\). So we have \(\partial_8(\omega_8) = (\Sigma^2i_{2,3})\lambda_2\eta_6\) and \(\partial_9(\tilde{\eta}_7) = (\Sigma^2i_{2,3})\lambda_2\eta_6^2\). By (2.2), the order of these elements are two. Therefore, there exists an element \(\varphi \in \pi_8(K^{(8)})\) such that \(p_8^K\varphi = 2t_8\). Here we note that \(\varphi\) is taken as a representative of the Toda bracket

\[
\varphi \in \{i_{8,5}^K[\Sigma^2i_{2,3}, \Sigma^2\gamma_3], i_8, 2t_8\}.
\]

So, by Lemma 2.2 (1), we have

\[
(2.3) \quad \pi_8(K^{(8)}) = \mathbb{Z}_2\{i_{4,8}^K[i_4, \lambda_2]\} \oplus \mathbb{Z}_2\{i_{4,8}^K\tilde{\eta}_3\nu_5\} \oplus \mathbb{Z}_2\{i_{3,5,8}^K\nu_5\}
\]

We have \(\pi_{10}(K^{(8)}, \Sigma^2P^3) = \mathbb{Z}_2\{\bar{\eta}_2^{(8)}\} \oplus \mathbb{Z}_2\{[\omega_8, \Sigma^2i_{1,3}]\}\) by the James exact sequence [4, Theorem 2.1]. Since \(\partial_{10}(\bar{\eta}_2) = (\Sigma^2i_{2,3})\lambda_2\eta_6^2 = 0\) and

\[
\partial_{10}([\omega_8, \Sigma^2i_{1,3}]) = [(\Sigma^2i_{2,3})\lambda_2\eta_6, \Sigma^2i_{1,3}] = (\Sigma^2i_{2,3})[\lambda_2, i_4]\eta_8,
\]

we obtain

\[
(2.4) \quad \pi_9(K^{(8)}) = \mathbb{Z}_2\{i_{4,8}^K\lambda_2\nu_6\} \oplus \mathbb{Z}_2\{i_{4,8}^K\tilde{\eta}_3\nu_5\eta_8\} \oplus \mathbb{Z}_2\{i_{3,5,8}^K\nu_5\eta_8\}
\]

Note that \(\varphi\) is obtained in the following diagram between the cofiber sequences:

\[
\begin{array}{ccccccccc}
S^7 & \xrightarrow{[\Sigma^2i_{1,3}, \Sigma^2\gamma_3]} & \Sigma^2P^3 & \xrightarrow{i_{8,5}^K} & K^{(8)} & \xrightarrow{p_8^K} & S^8 \\
\downarrow & & \downarrow & & \varphi & & = \\
S^7 & \xrightarrow{i_8} & M^8 & \xrightarrow{p_8} & S^8 & \xrightarrow{2t_8} & S^8.
\end{array}
\]

Write now the homotopy exact sequence of a pair \((K^{(9)}, K^{(8)})\):

\[
\pi_{10}(K^{(9)}, K^{(8)}) \xrightarrow{\partial_{10}} \pi_9(K^{(8)}) \xrightarrow{i_{8,9}^K} \pi_9(K^{(9)}) \xrightarrow{\partial_{9}} \pi_8(K^{(9)}, K^{(8)})
\]

\[
\Rightarrow \pi_8(K^{(8)}) \xrightarrow{i_{8,9}^K} \pi_8(K^{(9)}) \Rightarrow 0.
\]
The group structures \( \pi_9(K^{(9)}, K^{(8)}) = \mathbb{Z}\{\omega_9\} \) and \( \pi_{10}(K^{(9)}, K^{(8)}) = \mathbb{Z}_2\{\tilde{\eta}_8\} \) are obtained by the Blakers-Massey theorem, where \( \omega_9 = \omega_9^{(9)}, \omega_8^{(8)} \). By use of the exact sequence of a triple \((K^{(9)}, K^{(8)}, \Sigma^2P^3)\),

\[
\partial' : \pi_9(K^{(9)}, K^{(8)}) \rightarrow \pi_8(K^{(8)}, \Sigma^2P^3)
\]

is the map of degree 2. So, by the commutative diagram

\[
\begin{array}{ccc}
\pi_9(K^{(9)}, K^{(8)}) & \xrightarrow{\partial'} & \pi_8(K^{(8)}, \Sigma^2P^3) \\
\downarrow{\partial_0} & & \downarrow{j_*} \\
\pi_8(K^{(8)}) & & \\
\end{array}
\]

\( \varphi \) is taken as the attaching map of 9-cell of \( K^{(9)} \). Hence, by (2.3) and \( \pi_8(K^{(9)}) \cong \pi_8(K) \), we obtain (1) and \( j_* = 0 \). We see that

\[
\partial_1(\tilde{\eta}_8) = \varphi \circ \eta_8 \in \{ \iota_{5,8}^{K}\Sigma^2i_{2,3}, \Sigma^2\gamma_3, i_8, 2\tau_7 \} \circ \eta_8 = \{ \iota_{5,8}^{K}\Sigma^2i_{2,3}, \Sigma^2\gamma_3 \circ \{ i_8, 2\tau_7, \eta_7 \} \\
\quad \mod \iota_{5,8}^{K}\Sigma^2i_{2,3}, \Sigma^2\gamma_3 \circ \{ \iota_8(M^8) \circ \eta_8 + i_8 \circ \pi_9(S^7) \} = 0.
\]

Here we used \( [\Sigma^2i_{2,3}, \Sigma^2\gamma_3]i_8\eta_7^2 = [\Sigma^2i_{1,3}, \Sigma^2\gamma_3] \eta_7^2 = (\Sigma^2i_{2,3})\lambda_2\eta_6^3 = 0 \). By the fact that \( [\Sigma^2i_{2,3}, \Sigma^2\gamma_3] = 2[\Sigma^2i_{2,3}, s] + (\Sigma^2i_{2,3})[\iota_{M^4}, \tilde{\eta}_3] \) and \( [\iota_{M^4}, \tilde{\eta}_3] = \tilde{\eta}_3\nu_5s\eta_8 \pm \lambda_2\eta_6 \), we obtain

\[
\partial_1(\tilde{\eta}_8) = 2\iota_{5,8}^{K}\Sigma^2i_{2,3}, s] \tilde{\eta}_7 + i_4\iota_{4,8}\tilde{\eta}_3\nu_5\eta_8,
\]

and hence

\[
\pi_9(K^{(9)}) = \mathbb{Z}_2\{\iota_{4,9}^{K}\lambda_2\nu_6\} \oplus \mathbb{Z}_2\{i_{5,9}^{K}s\nu_5\eta_8\} \oplus \mathbb{Z}_4\{i_{5,9}^{K}\Sigma^2i_{2,3}, s]\tilde{\eta}_7\} \\
\oplus \mathbb{Z}_2\{i_{5,9}^{K}\Sigma^2i_{1,3}, s], \Sigma^2i_{1,3}\}.
\]

Let \( p_M : \Sigma^2P^3 \rightarrow M^4 \) be the projection. Then,

\[
\iota_{\Sigma^2P^3} = s\Sigma^2p_{3,2} + (\Sigma^2i_{2,3})p_M,
\]

(2.6)

\[
\Sigma^2p_{3,2} \circ s = \iota_5, \text{ } p_M \circ \Sigma^2i_{2,3} = \iota_{M^4} \text{ and } p_M \circ s = 0.
\]

By (2.1) and (2.6), we have

\[
[t_{\Sigma^2P^3}, \Sigma^2\gamma_3] = [s\Sigma^2p_{3,2}, \Sigma^2\gamma_3] + [(\Sigma^2i_{2,3})p_M, \Sigma^2\gamma_3] \\
= [s, (\Sigma^2i_{2,3})\tilde{\eta}_3] \circ \Sigma^6p_{3,2} + (\Sigma^2i_{2,3})[p_M, \tilde{\eta}_3] + 2[(\Sigma^2i_{2,3})p_M, s].
\]
By (2.7), we have \( [p_M, \tilde{\eta}_3] \circ \sigma^6i_{2,3} = [\iota_{M^4}, \tilde{\eta}_3] \). So, by use of the cofiber sequence \( M^8 \xrightarrow{\sigma^6i_{2,3}} \Sigma^6p^3 \xrightarrow{\sigma^6p_{3,2}} S^9 \),
\[
[p_M, \tilde{\eta}_3] \equiv [\iota_{M^4}, \tilde{\eta}_3] \circ \sigma^4p_M \mod \pi_9(M^4) \circ \sigma^6p_{3,2}.
\]
By the same reason,
\[
[(\Sigma^2i_{2,3})p_M, s] \equiv [\Sigma^2i_{2,3}, s] \circ \sigma^4p_M \mod \pi_9(\Sigma^2p^3) \circ \sigma^6p_{3,2}.
\]
Hence, by Lemma 2.2 (2) and (2.7), we conclude that
\[
[i_{\Sigma^2p^3}, \Sigma^2\gamma_3] \circ \sigma^4s \equiv \pm [s, (\Sigma^2i_{2,3})\tilde{\eta}_3] \mod (\Sigma^2i_{2,3}) \circ \pi_9(M^4) \circ \sigma^6p_{3,2}.
\]
The attaching map of the 10-cell of \( K(10) \) is \( i_{5,9}^K[i_{\Sigma^2p^3}, \Sigma^2\gamma_3]\Sigma^4s \). By (2.1), we have
\[
i_{5,9}^K[i_{\Sigma^2p^3}, \Sigma^2\gamma_3]\Sigma^4s \equiv \pm i_{5,9}^K[s, \Sigma^2i_{2,3}]\tilde{\eta}_7 \mod i_{4,9}^K\lambda_2\nu_6.
\]
So, by the homotopy exact sequence of a pair \( (K^{(10)}, K^{(9)}) \) and (2.5), the group structure of \( \pi_9(K) \) is obtained. This completes the proof. \( \square \)

**Lemma 2.6.**
\[
\pi_9(\Sigma^2p^4) = \mathbb{Z}_2\{\eta_6^3\} \oplus \mathbb{Z}_2\{(\Sigma^2i_{2,4})\lambda_2\nu_6\} \oplus \mathbb{Z}_2\{(\Sigma^2i_{3,4})\nu_5\eta_8\} \oplus \mathbb{Z}_2\{(\Sigma^2\gamma_3)[s, \Sigma^2i_{1,3}, \Sigma^2i_{1,3}]\}.
\]

**Proof.** We consider the exact sequence induced from the fibration \( \Sigma^2p_{4,3} : \Sigma^2p^4 \to S^6 \):
\[
\pi_{10}(S^6) = 0 \to \pi_9(K) \to \pi_9(\Sigma^2p^4) \to \pi_9(\Sigma^6) \xrightarrow{\Delta_9} \pi_8(K) \to \cdots.
\]
By [8, Lemma 1.2], we obtain the relations \( \Delta_6(\iota_6) = \pm i_{4,9}^K\tilde{\eta}_3 + 2i_5^Ks \) and 
\[
\Delta_9(\nu_6) = \Delta_6(\iota_6) \circ \nu_5 = \pm i_{4,9}^K\tilde{\eta}_3\nu_5 + 2i_5^K\nu_5v_5.
\]
Using the second relation and Lemma 2.5 (1), we obtain \( \ker \Delta_9 = \mathbb{Z}_2\{\eta_6^3\} \).
Therefore, by Lemma 2.4 and 2.5 (2) and by the fact that \( i \circ i_5^K = \Sigma^2i_{3,4} \)
(\( i : K \to \Sigma^2p^4 \) is the inclusion), we obtain the result. This completes the proof. \( \square \)

Now we consider the homotopy exact sequence of a pair \( (\Sigma^2p^5, \Sigma^2p^4) \):
\[
\pi_{10}(\Sigma^2p^5, \Sigma^2p^4) \xrightarrow{\partial_{10}} \pi_9(\Sigma^2p^4) \xrightarrow{i_*} \pi_9(\Sigma^2p^5)
\]
\[
i_* : \pi_9(\Sigma^2p^5, \Sigma^2p^4) \xrightarrow{\partial_9} \pi_8(\Sigma^2p^4),
\]
where \( i = \Sigma^2i_{4,5} : \Sigma^2p^4 \to \Sigma^2p^5 \). By the James exact sequence, the group structures \( \pi_9(\Sigma^2p^5, \Sigma^2p^4) = \mathbb{Z}_2(\eta_6^2) \oplus \mathbb{Z}_2[\omega_7, \Sigma^2i_{1,4}] \) and \( \pi_{10}(\Sigma^2p^5, \Sigma^2p^4) = \mathbb{Z}_{24}(\tilde{\eta}_6) \oplus \mathbb{Z}_2[\omega_7(\Sigma^2i_{1,4})\eta_3] \) are settled, where \( \omega_7 = \omega_7(\Sigma^2p^5, \Sigma^2p^4) \). We recall, from [8, Lemma 1.3], the relation
\[
\Sigma^2\gamma_4 = (\Sigma^2i_{3,4})s\eta_5 + 2(\Sigma^2i_{2,4})\lambda_2.
\]
By Lemma 2.6 and by the relation \( \eta_5 \nu_6 = 0 \), we obtain

\[
\partial_{10}(\widehat{\eta}_6) = (\Sigma^2 \gamma_4) \nu_6 = ((\Sigma^2 \iota_{3,4}) s \eta_5 + 2(\Sigma^2 \iota_{1,4}) \lambda_2) \nu_6 = 0.
\]

The equation \( (\Sigma^2 \iota_{3,4})[\Sigma^2 \iota_{2,3}, s] \overline{\eta}_7 = 0 \) is shown in the proof of Lemma 2.6. Then

\[
\partial_{10}(\omega_7,(\Sigma^2 \iota_{1,4}) \eta_3) = [(\Sigma^2 \gamma_4, (\Sigma^2 \iota_{1,4}) \eta_3)] = [(\Sigma^2 \iota_{3,4}) s \eta_5, (\Sigma^2 \iota_{1,4}) \eta_3] = (\Sigma^2 \iota_{3,4})[s, \Sigma^2 \iota_{1,3}] \eta_7^2 = 2(\Sigma^2 \iota_{3,4})[s, \Sigma^2 \iota_{2,3}] \eta_7 = 0.
\]

Therefore \( (\Sigma^2 \iota_{4,5})_* : \pi_9(\Sigma^2 P^4) \to \pi_9(\Sigma^2 P^5) \) is a monomorphism.

By the fact that \( \pi_8(\Sigma^2 P^4) = Z_4\{[(\Sigma^2 \iota_{3,4}) s \nu_5] \} \oplus Z_2\{[(\Sigma^2 \iota_{3,4}) s, \Sigma^2 \iota_{1,4}] \eta_7]\} \oplus Z_2\{[(\Sigma^2 \iota_{2,4})[\iota_4, \lambda_2]\} \) [8, Lemma 2.5] and by (2.8), we obtain

\[
\partial_9(\overline{\eta}_6^2) = (\Sigma^2 \gamma_4) \eta_6^2 = (\Sigma^2 \iota_{3,4}) s \eta_5^2 = 4(\Sigma^2 \iota_{3,4}) s \nu_5 = 0
\]

and

\[
\partial_9(\omega_7, \Sigma^2 \iota_{1,4}) = [(\Sigma^2 \gamma_4, \Sigma^2 \iota_{1,4})] = [(\Sigma^2 \iota_{3,4}) s \eta_5, \Sigma^2 \iota_{1,4}] = [(\Sigma^2 \iota_{3,4}) s, \Sigma^2 \iota_{1,4}] \eta_7.
\]

Then there exists an element \( \overline{\eta}_6^2 \in \{\Sigma^2 \iota_{4,5}, \Sigma^2 \gamma_4, \eta_6^2\} \subset \pi_9(\Sigma^2 P^5) \) such that \( (\Sigma^2 p_{5,4}) \eta_6^2 = \eta_6^2 \). We obtain

\[
2\eta_6^2 \in \{\Sigma^2 \iota_{4,5}, \Sigma^2 \gamma_4, \eta_6^2\} \circ 2\iota_9 = -(\Sigma^2 \iota_{4,5} \circ \{\Sigma^2 \gamma_4, \eta_6^2, 2\iota_8\})
\]

and

\[
\{\Sigma^2 \gamma_4, \eta_6^2, 2\iota_8\} \subset \{\Sigma^2 \iota_{3,4}, (s \eta_5 + 2(\Sigma^2 \iota_{2,3}) \lambda_2) \eta_6^2, 2\iota_8\} = \{\Sigma^2 \iota_{3,4}, s \eta_5^3, 2\iota_8\} = \{\Sigma^2 \iota_{3,4}, \Sigma^2 \gamma_3 \circ 2\nu_5, 2\iota_8\} \supset \{\Sigma^2 \iota_{3,4}, \Sigma^2 \gamma_3, \eta_5^3\} \equiv \eta_5^2 \mod 2\pi_9(\Sigma^2 P^4) + (\Sigma^2 \iota_{3,4})_* \pi_9(\Sigma^2 P^3) = (\Sigma^2 \iota_{3,4})_* \pi_9(\Sigma^2 P^3),
\]

and hence we conclude that \( 2\overline{\eta}_6^2 \equiv (\Sigma^2 \iota_{4,5}) \eta_6^3 \mod (\Sigma^2 \iota_{3,5})_* \pi_9(\Sigma^2 P^3) \). Thus, by Lemma 2.6, we have the following.

**Lemma 2.7.**

\[
\pi_9(\Sigma^2 P^5) = Z_4\{\overline{\eta}_6^2\} \oplus Z_2\{(\Sigma^2 \iota_{2,5}) \lambda_2 \nu_6\} \oplus Z_2\{(\Sigma^2 \iota_{3,5}) s \nu_5 \eta_8\} \oplus Z_2\{(\Sigma^2 \iota_{3,5})[\Sigma^2 \iota_{1,3}, s], \Sigma^2 \iota_{1,3}]\},
\]

where \( 2\overline{\eta}_6^2 = (\Sigma^2 \iota_{4,5}) \eta_6^3 \) for a suitable choice of \( \eta_5^3 \).
3. Proofs of main theorems

First, we show Theorem 1.2.

From the fact that $\Sigma^2 \gamma_5 \in \{ \Sigma^2 i_{4,5}, \Sigma^2 \gamma_4, 2\iota_6 \}$, $\Sigma^3 \gamma_4 = (\Sigma^3 i_{3,4})(\Sigma s)\eta_6$, we see that

$[\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}]p_9 = [\iota \Sigma^2 p_5, \Sigma^2 i_{1,5}] \circ \Sigma^4 \gamma_5 \circ p_9$

$\in [\iota \Sigma^2 p_5, \Sigma^2 i_{1,5}] \circ \{ \Sigma^4 i_{4,5}, \Sigma^4 \gamma_4, 2\iota_8 \} \circ p_9$

$\ni [\iota \Sigma^2 p_5, \Sigma^2 i_{1,5}] \{ (\Sigma^4 i_{3,5})\Sigma^2 s, \eta_7, 2\iota_8 \} \circ p_9$

$\equiv -([\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}]\Sigma^2 s \circ \{ \eta_7, 2\iota_8, p_8 \})$

$\ni [\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] (\Sigma^2 s)\eta_7$

$\text{mod} \ [\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ \pi_9 (\Sigma^4 P^4) \circ p_9$.

It is easily seen that $\pi_9 (\Sigma^4 P^4) = \mathbf{Z}_2 \{ (\Sigma^4 i_{1,4})\nu_5 \eta_8 \} \oplus \mathbf{Z}_2 \{ (\Sigma^4 i_{3,4}) (\Sigma^2 s)\eta_7^2 \}$. Since $[i_3, i_3] = 0$ and $(\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}]\eta_7 = 0$, we obtain

$[\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] (\Sigma^2 i_{1,4})\nu_5 \eta_8 = (\Sigma^2 i_{1,5})[i_3, i_3]\nu_5 \eta_8 = 0$

and

$[\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] (\Sigma^2 i_{3,4}) (\Sigma^2 s)\eta_7^2 = (\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}) (\Sigma^4 \gamma_4) \eta_8 = 0$.

Then $[\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ \pi_9 (\Sigma^4 P^4) = 0$. By (2.6), the element $[\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] \Sigma^2 s$ is changed as follows.

$[\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] \Sigma^2 s = (\Sigma^2 i_{3,5})[i_3 \Sigma^2 p_3, \Sigma^2 i_{1,3}] \Sigma^2 s$

$= (\Sigma^2 i_{3,5})[s, \Sigma^2 i_{1,3}] + (\Sigma^2 i_{2,5})[p_4, i_4] \Sigma^2 s$.

By the fact that $[p_4, i_4] \in [\Sigma^4 P^3, M^4] = (\Sigma^4 p_3, 2) \ast \pi_7 (M^4) \oplus (\Sigma^2 p_4) \ast [M^6, M^4]$, we obtain

$(\Sigma^2 i_{2,5})[p_4, i_4] \Sigma^2 s \in \Sigma^2 i_{2,5} \circ (\pi_7 (M^4) \circ \Sigma^4 p_3, 2 \ast [M^6, M^4] \circ \Sigma^2 p_4) \circ \Sigma^2 s$

$= \Sigma^2 i_{2,5} \circ \pi_7 (M^4)$.

We recall from [7, Lemma 2.2] that $\pi_7 (M^4) = \mathbf{Z}_2 \{ \lambda_2 \eta_6 \} \oplus \mathbf{Z}_2 \{ \overline{\eta}_3 \eta_5^2 \}$. Since $(\Sigma^2 i_{2,4}) \lambda_2 \eta_6 = 0$ [8, the proof of Lemma 2.2] and by (2.1), the group $\Sigma^2 i_{2,5} \circ \pi_7 (M^4)$ is 0. Then,

$[\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}] p_9 = [\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] (\Sigma^2 s) \eta_7 = (\Sigma^2 i_{3,5})[s, \Sigma^2 i_{1,3}] \eta_7$.

Here we consider an element $[\Sigma^2 \gamma_4, \Sigma^2 i_{2,4}] \in [M^9, \Sigma^2 P^4]$. Since $2\iota_M = i_4 \eta_3 p_4$ [11], $(\Sigma^2 i_{2,4}) \lambda_2 \eta_6 = 0$, $(\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}] \eta_7 = 0$ and $\eta_2 \wedge \iota_M = i_4 \eta_3 + \eta_3 p_5$, we obtain

$[\Sigma^2 \gamma_4, \Sigma^2 i_{2,4}] = (\Sigma^2 i_{3,4})[s \eta_5, \Sigma^2 i_{2,3}] + (\Sigma^2 i_{2,4})[2 \lambda_2, \iota_M]$

$= (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}] \circ \Sigma(\eta_4 \wedge \iota_M) + (\Sigma^2 i_{2,4})[\lambda_2, 2\iota_M]$.
SELF-HOMOTOPY OF $\Sigma^2\mathbb{P}^7$

$$= (\Sigma^2i_{3,4})[s, \Sigma^2i_{2,3}] \circ \Sigma(i_{7}\eta_6 + \tilde{\eta}_6p_8) + (\Sigma^2i_{2,4})[\lambda_2, i_4\eta_3p_4]$$

$$= (\Sigma^2i_{3,4})[s, \Sigma^2i_{1,3}]\overline{\eta}_7 + (\Sigma^2i_{2,4})[\lambda_2\eta_6, i_4]p_9$$

$$= (\Sigma^2i_{3,4})[s, \Sigma^2i_{1,3}]\overline{\eta}_7.$$ 

Thus, we get that

$$[\Sigma^2\gamma_5, \Sigma^2i_{1,5}]p_9 = (\Sigma^2i_{3,5})[s, \Sigma^2i_{1,3}]\overline{\eta}_7 = (\Sigma^2i_{4,5})[\Sigma^2\gamma_4, \Sigma^2i_{2,4}] = 0.$$ 

By use of the cofibre sequence $S^8 \xrightarrow{i_9} M^9 \xrightarrow{p_9} S^9 \xrightarrow{2\nu_9} S^9$, we have

$$[\Sigma^2\gamma_5, \Sigma^2i_{1,5}] \in 2\pi_9(\Sigma^2\mathbb{P}^5) = \mathbb{Z}_2\{2\eta_6^2\}.$$ 

Let $l_1 : \mathbb{P}^4/\mathbb{P}^3 = S^4 \to \mathbb{P}^5/\mathbb{P}^3 = S^4 \vee S^5$ be the canonical inclusion map. By Lemma 2.7 and by the relations $p_{5,3} \circ i_{4,5} = l_1 \circ p_{4,3}$ and $p_{5,3} \circ i_{1,5} = 0$, we obtain

$$\Sigma^2p_{5,3} \circ 2\eta_6^2 = \Sigma^2p_{5,3} \circ \Sigma^2i_{4,5} \circ \eta_6^3$$

$$= \Sigma^2l_1 \circ \Sigma^2p_{4,3} \circ \eta_6^3$$

$$= (\Sigma^2l_1)\eta_6^3 \neq 0 \in \pi_9(S^6 \vee S^7) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2$$

and

$$\Sigma^2p_{5,3} \circ [\Sigma^2\gamma_5, \Sigma^2i_{1,5}] = 0.$$ 

Therefore we have $[\Sigma^2\gamma_5, \Sigma^2i_{1,5}] = 0$ and the proof of Theorem 1.2 is complete.

By [8, the proof of Lemma 2.5], we have a relation $(\Sigma^2\gamma_5)\eta_7 = 0$ and we can define a coextension $\overline{\eta}_7' \in \pi_9(\Sigma^2\mathbb{P}^6)$ of $\eta_7$ as follows:

$$\overline{\eta}_7' \in \{\Sigma^2i_{5,6}, \Sigma^2\gamma_5, \eta_7\}.$$ 

Since $2\overline{\eta}_7' \in \{\Sigma^2i_{5,6}, \Sigma^2\gamma_5, \eta_7\} \circ 2\nu_9 = -(\Sigma^2i_{5,6} \circ \{\Sigma^2\gamma_5, \eta_7, 2\nu_8\})$ and

$$\Sigma^2p_{5,4} \circ \{\Sigma^2\gamma_5, \eta_7, 2\nu_8\} \subset \{2\nu_7, \eta_7, 2\nu_8\} = \eta_7^2,$$

we obtain $2\overline{\eta}_7' \equiv (\Sigma^2i_{5,6})\eta_6^3 \mod \Sigma^2i_{5,6} \circ 2\pi_9(\Sigma^2\mathbb{P}^5) + \Sigma^2i_{4,6} \circ \pi_9(\Sigma^2\mathbb{P}^4) = \Sigma^2i_{4,6} \circ \pi_9(\Sigma^2\mathbb{P}^4)$. From the exact sequence of a pair $(\Sigma^2\mathbb{P}^6, \Sigma^2\mathbb{P}^5)$ and by Theorem 1.2, we see that $(\Sigma^2i_{5,6})_+ : \pi_9(\Sigma^2\mathbb{P}^5) \to \pi_9(\Sigma^2\mathbb{P}^6)$ is a monomorphism. Thus, $\overline{\eta}_7'$ is of order 8 and the group structure of $\pi_9(\Sigma^2\mathbb{P}^6)$ is given as follows.

**Lemma 3.1.**

$$\pi_9(\Sigma^2\mathbb{P}^6) = \mathbb{Z}_8\{\overline{\eta}_7'\} \oplus \mathbb{Z}_2\{(\Sigma^2i_{2,6})\lambda_2\nu_6\} \oplus \mathbb{Z}_2\{(\Sigma^2i_{3,6})s\nu_5\eta_8\}$$

$$\oplus \mathbb{Z}_2\{(\Sigma^2i_{3,6})[\Sigma^2i_{1,3}, s], \Sigma^2i_{1,3}]\},$$

where $2\overline{\eta}_7' = (\Sigma^2i_{5,6})\eta_6^3$ for a suitable choice of $\eta_6^2$. 

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We denote by $s_1 : S^9 \to \Sigma^2 \mathbb{P}^7 = \Sigma^2 \mathbb{P}^6 \vee S^9$ the inclusion map to the second factor and by $q_1 : \Sigma^2 \mathbb{P}^7 = \Sigma^2 \mathbb{P}^6 \vee S^9 \to \Sigma^2 \mathbb{P}^6$ the map collapsing $S^9$ to one point. Finally we obtain the following.

**Theorem 3.2.**

$$[\Sigma^2 \mathbb{P}^7, \Sigma^2 \mathbb{P}^7] = \mathbb{Z}\{s_1 \Sigma^2 \mathbb{P}^7, 6\} \oplus \mathbb{Z}_8\{(\Sigma^2 i_{6,7})q_1\} \oplus \mathbb{Z}_8\{(\Sigma^2 i_{6,7})\eta_7 \Sigma^2 \mathbb{P}^7, 6\}$$

$$\oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})[s, \Sigma^2 i_{2,3}](\Sigma^2 \mathbb{P}^6, 4)q_1\}$$

$$\oplus \mathbb{Z}_2\{(\Sigma^2 i_{2,7})\lambda_2(\Sigma^2 \mathbb{P}^6, 4)q_1\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{2,7})[\lambda_2, i_4](\Sigma^2 \mathbb{P}^6, 5)q_1\}$$

$$\oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})s\nu_5(\Sigma^2 \mathbb{P}^6, 5)q_1\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{2,7})\lambda_2\nu_6 \Sigma^2 \mathbb{P}^7, 6\}$$

$$\oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})\nu_5\eta_8 \Sigma^2 \mathbb{P}^7, 6\}$$

$$\oplus \mathbb{Z}_2\{(\Sigma^2 i_{2,7})[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{2,3}][\Sigma^2 \mathbb{P}^7, 6\}.$$