Multipliers and Cyclic Vectors on the Weighted Bloch Space

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MULTIPLIERS AND CYCLIC VECTORS ON THE WEIGHTED BLOCH SPACE

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Abstract. In this paper we study the pointwise multipliers and cyclic vectors on the weighted Bloch space \( \beta_L = \{ f \in H(D) : \sup_D (1 - |z|^2) \ln \left( \frac{2}{1 - |z|} \right) |f'(z)| < +\infty \} \). We obtain a characterization of multipliers on \( \beta_L \) and little \( \beta^0_L \). Also, a sufficient condition and a necessary condition are given for which \( f \) is a cyclic vector in \( \beta^0_L \).

1. Introduction

Let \( D = \{ z : |z| < 1 \} \) be the open unit disk in the complex plane \( \mathbb{C} \), and \( H(D) \) denote the set of all analytic functions on \( D \). For \( f \in H(D) \), Let

\[
\| f \|_{\beta_\alpha} = \sup \{ (1 - |z|^2)^\alpha |f'(z)| : z \in D \}, \quad 0 < \alpha < +\infty,
\]

\[
\| f \|_{\beta_L} = \sup \{ (1 - |z|^2) \ln \left( \frac{2}{1 - |z|} \right) |f'(z)| : z \in D \}.
\]

As in [7], [9], the \( \alpha \)-Bloch space \( \beta_\alpha \) consists of all \( f \in H(D) \) satisfying \( \| f \|_{\beta_\alpha} < +\infty \) and the little \( \alpha \)-Bloch space \( \beta^0_\alpha \) consists of all \( f \in H(D) \) satisfying \( \lim_{|z| \to 1^-} (1 - |z|^2)^\alpha |f'(z)| = 0 \); the logarithmic weighted Bloch space \( \beta_L \) consists of all \( f \in H(D) \) satisfying \( \| f \|_{\beta_L} < +\infty \) and the little logarithmic weighted Bloch space \( \beta^0_L \) consists of all \( f \in H(D) \) satisfying \( \lim_{|z| \to 1^-} (1 - |z|^2) \ln \left( \frac{2}{1 - |z|} \right) |f'(z)| = 0 \). It can easily proved that \( \beta_L \) is a Banach space under the norm \( \| f \|_L = |f(0)| + \| f \|_{\beta_L} \) and that \( \beta^0_L \) is a closed subspace of \( \beta_L \). It is well known that with the norm \( \| f \|_\alpha = |f(0)| + \| f \|_{\beta_\alpha} \) \( \beta_\alpha \) is a Banach space and \( \beta^0_\alpha \) is a closed subspace of \( \beta_\alpha \). It is easily proved that for \( 0 < \alpha < 1 \), \( \beta_\alpha \subsetneq \beta_L \subsetneq \beta_1 \). For more information about \( \beta_\alpha \), see, for example, [9].

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The space of analytic functions on $D$ of bounded mean oscillation, denoted by $BMOA$, consists of $f$ in $H^2$ for which

$$
\|f\|_{BMOA} = \sup_I \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z) < +\infty,
$$

where $dA(z)$ denotes the Lebesgue measure on $D$, $I$ denotes a subarc of $\partial D$, $|I|$ denotes the arclength measure of $I$ and $S(I) = \{re^{i\theta} : 1-r \leq |I|, e^{i\theta} \in I\}$. The subset of $BMOA$, denoted by $VMOA$, consists of $f$ for which

$$
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z) = 0.
$$

For more details, see [5].

Let $X$ be an analytic function space. We say a function $\phi$ is a pointwise multiplier on $X$, if $\phi f \in X$ for all $f \in X$. Let $M(X)$ denote the space of all pointwise multipliers on $X$. By $M_\phi$ we denote the operator of multiplication by $\phi$: $M_\phi f = \phi f, f \in X$. An application of the closed graph theorem shows that if $\phi \in M(X)$, then $M_\phi$ is a bounded linear transformation. Hence it has a finite norm $\|M_\phi\|$.

In [2], K. R. M. Attle showed that for $f \in L^2_a(D)$, the Hankel operator $H_f : L^1_a \to L^1$ is bounded if and only if $f \in \beta_L$, and in [3], L. Brown and A. L. Shields proved that $M_\phi$ is bounded on the classical Bloch space $\beta_1(\beta^0_L)$ if and only if $\phi \in \beta_L \cap H^\infty$. R. Yoneda [7] studied the composition operator in $\beta_L$ space. In Section 2 we will characterize multiplier spaces $M(\beta_L)$ and $M(\beta^0_L)$.

Let $Y$ be an analytic Banach function space and the polynomials are dense in it. For $f \in Y$ and let $[f]$ be the closure in $Y$ of the polynomial multiples of $f$. Thus $f$ is called a cyclic vector in $Y$ if and only if $[f]=Y$. In [1], [3], L. Brown and A. L. Shields studied cyclic vectors in the classical Bloch space $\beta_1(\beta^0_1)$. In the $BMOA(VMOA)$ space, the author in [6] characterized the cyclic vectors. There are just the following theorem.

**Theorem A.**

1. For $f \in BMOA(VMOA)$, then $f$ is a cyclic vector on $BMOA(VMOA)$ if and only if $f$ is an outer function.
2. If $f$ is an outer function in $\beta_1(\beta^0_1)$, then $f$ is cyclic in $\beta_1(\beta^0_1)$.
3. There exists a singular inner function that is cyclic in $\beta_1$.

In Section 3 we study cyclic vectors in $\beta^0_L$.

2. Multipliers in the weighted Bloch space

In this section we shall characterize the pointwise multipliers space $M(\beta_L)$ and $M(\beta^0_L)$. For this purpose, we need the following lemmas.
Lemma 2.1. If \( f \in \beta_L \), then

(i) \( |f(z)| \leq (2 + \ln(\ln \frac{2}{1 - |z|})) \|f\|_L \);

(ii) \( |f(z) - f(tz)| \leq \ln \left( \frac{\ln(\frac{2}{1 - |z|})}{\ln(\frac{2}{1 - |tz|})} \right) \|f\|_{\beta_L} \), for every \( t \) with \( 0 \leq t < 1 \).

Proof. Suppose \( f \in \beta_L \) and \( z \in D \), then

\[
|f(z) - f(tz)| = |z \int_t^1 f'(zt)dt| \leq \|f\|_{\beta_L} \int_t^1 \frac{|z|}{(1 - |zt|^2) \ln \frac{2}{1 - |zt|}} dt \\
\leq \|f\|_{\beta_L} \int_{|zt|}^{\infty} \frac{dx}{(1 - x) \ln \frac{2}{1-x}} \\
= \|f\|_{\beta_L} (\ln \ln \frac{2}{1-*} - \ln \ln \frac{2}{1-|tz|}) \\
\leq \ln\left( \frac{\ln(\frac{2}{1-|z|})}{\ln(\frac{2}{1-|tz|})} \right) \|f\|_{\beta_L}.
\]

Especially, \( |f(z) - f(0)| \leq \|f\|_{\beta_L} (\ln \ln \frac{2}{1-|z|} - \ln \ln 2) \), hence

\[
|f(z)| \leq (2 + \ln \ln \frac{2}{1-|z|}) \|f\|_L.
\]

\[ \square \]

Lemma 2.2. If \( f \in \beta_L^0 \), then \( \lim_{|z| \to 1-} \frac{|f(z)|}{\ln (\frac{2}{1-|z|})} = 0 \).

The proof is similar to Lemma 2.1. The details are omitted.

Lemma 2.3. Let \( f(z) = \frac{(1 - |z|) \ln \frac{2}{1-|z|}}{|1-z| \ln \frac{4}{|1-z|}} \), \( z \in D \). Then \( |f(z)| < 2 \).

Proof. Since \( r(x) = x \ln \frac{2}{x} \) is increasing on \((0, \frac{2}{e}]\), decreasing on \([\frac{2}{e}, 1]\) and \( r(\frac{2}{e}) = \frac{2}{e} < 1 \), then \( |f(z)| < 1 \) where \( z \in D_1 = \{z \in D : |1-z| < \frac{2}{e}\} \).

On the other hand, for \( z \in D \setminus D_1 \),

\[
|f(z)| \leq \frac{(1 - |z|) \ln \frac{2}{|1-z|}}{\frac{2}{e} \ln 2} \leq \frac{\frac{2}{e}}{\frac{2}{e} \ln 2} < 2,
\]

hence \( |f(z)| < 2 \). \[ \square \]

Theorem 2.4. The following are equivalent:

(a) \( \phi \in M(\beta_L) \);
(b) \( \phi \in M(\beta_L^0) \);
(c) $\phi \in H^\infty$ and
\[
(2.1) \quad \sup_D (1 - |z|^2) \ln \frac{2}{1 - |z|} \ln\left(\ln\frac{2}{1 - |z|}\right) |\phi'(z)| < +\infty.
\]

**Proof.** (c)⇒(a). Assume $\phi \in H^\infty$ and (2.1) holds. For every $f \in \beta_L$, by Lemma 2.1, we have
\[
(1 - |z|^2) \ln \frac{2}{1 - |z|} |(M\phi f)'(z)|
\]
\[
\leq (1 - |z|^2) \ln \frac{2}{1 - |z|} |\phi(z)||f'(z)| + (1 - |z|^2) \ln \frac{2}{1 - |z|} |\phi'(z)||f(z)|
\]
\[
\leq \|\phi\|_\infty \|f\|_{\beta_L} + (1 - |z|^2) \ln \frac{2}{1 - |z|} (2 + \ln(\ln\frac{2}{1 - |z|})) |\phi'(z)||f||_L < +\infty.
\]
Thus $f\phi \in \beta_L$.

(a)⇒(c). Suppose that $\phi$ is a multiplier of $\beta_L$. Then by [4, Proposition 3] $\phi \in H^\infty$ and $|\phi(z)| \leq \|M\phi\|$. Let $z_0 = re^{i\theta}$. We take the test function
\[
f(z) = \ln(\ln\frac{4}{1 - e^{-i\theta}z}).
\]
By Lemma 2.3 we know that $f \in \beta_L$ and $\|f\|_L \leq 5$. We have
\[
\|f\phi\|_L \leq \|M\phi\| \|f\|_L \leq 5 \|M\phi\|.
\]
It follows that
\[
(1 - |z|^2) \ln \frac{2}{1 - |z|} \ln(\ln\frac{4}{1 - e^{-i\theta}z}) |\phi'(z)|
\]
\[
\leq (1 - |z|^2) \ln \frac{2}{1 - |z|} |\phi(z)||f'(z)| + 5 \|M\phi\|
\]
\[
\leq 5(\|\phi\|_\infty + \|M\phi\|) < +\infty.
\]
Let $z = z_0$. Hence
\[
(1 - |z_0|^2) \ln \frac{2}{1 - |z_0|} \ln(\ln\frac{4}{1 - |z_0|}) |\phi'(z_0)| \leq 5(\|\phi\|_\infty + \|M\phi\|) < +\infty.
\]
Thus
\[
\sup_D (1 - |z|^2) \ln \frac{2}{1 - |z|} \ln(\ln\frac{2}{1 - |z|}) |\phi'(z)| < +\infty.
\]
(b)⇒(c). Given $z_0 = re^{i\theta}$ and $\alpha \in (0, 1)$. Let
\[
f_\alpha(z) = (\ln(\ln\frac{4}{1 - e^{-i\theta}z}))^\alpha.
\]
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A calculation shows that \( f_\alpha \in \beta^0_L \) and \( \sup_\alpha \|f\|_L = k < +\infty \). In a manner similar to the proof \((a) \Rightarrow (c)\), one obtains that if \( \phi \) is a multiplier of \( \beta^0_L \), then for each \( \alpha \),

\[
(1 - |z_0|^2) \ln \frac{2}{1 - |z_0|} (\ln(\ln \frac{4}{1 - |z_0|}))^{\alpha} \|\phi'(z)\| \leq k(\|\phi\|_\infty + \|M_\phi\|) < +\infty.
\]

Hence

\[
(1 - |z_0|^2) \ln \frac{2}{1 - |z_0|} \ln(\ln \frac{4}{1 - |z_0|}) \|\phi'(z_0)\| < +\infty,
\]

which shows that (1) holds.

\((c) \Rightarrow (b)\). Assume \( \phi \in H^\infty \) and \( \sup_D (1 - |z|^2) \ln \frac{2}{1 - |z|} \ln(\ln \frac{2}{1 - |z|}) \|\phi'(z)\| = M < +\infty \). For every \( f \in \beta^0_L \), by Lemma 2.2, we have

\[
(1 - |z|^2) \ln \frac{2}{1 - |z|} (M_\phi f)'(z) \leq (1 - |z|^2) \ln \frac{2}{1 - |z|} \phi(z) \|f'(z)\| + (1 - |z|^2) \ln \frac{2}{1 - |z|} \phi'(z) \|f(z)\| \\
\leq \|\phi\|_\infty (1 - |z|^2) \ln \frac{2}{1 - |z|} f'(z) + \frac{M}{\ln(\ln \frac{2}{1 - |z|})} f(z) \to 0 (|z| \to 1).
\]

Thus \( f\phi \in \beta^0_L \). \( \square \)

3. CYCLIC VECTORS IN THE LITTLE WEIGHTED BLOCH SPACE

**Lemma 3.1.** Let \( g(x) = (1 - x) \ln \frac{2}{1 - x}, x \in [0, 1) \). Then \( \frac{g(x)}{g(tx)} \leq 2 \) for each \( t \in [0, 1] \).

**Proof.** Let \( x_0 = 1 - \frac{2}{e} \). A calculation shows that \( \frac{4}{3} x_0 < 1 \). We know that \( g(x) \) is increasing on \( [0, x_0] \), and decreasing on \( [x_0, 1) \).

First, suppose \( t > \frac{4}{3} \) and \( x > \frac{4}{3} x_0 \). Then \( x \geq tx > x_0 \), hence \( g(x) \leq g(tx) \).

Next, suppose \( t > \frac{4}{3} \) and \( x \leq \frac{4}{3} x_0 \). Then

\[
\frac{g(x)}{g(tx)} \leq \frac{g(x_0)}{\min(g(0), g(\frac{4}{3} x_0))} = \frac{2/e}{\ln 2} < 2.
\]

Finally, suppose \( t \leq \frac{3}{4} \). A calculation shows that

\[
\frac{3}{4} \ln 2 \leq g(tx) \leq \frac{2}{e}.
\]
Then
\[ \frac{g(x)}{g(tx)} \leq \frac{2/e}{3} \ln 2 < 2. \]

Lemma 3.2. Let \( h(x) = (1 - x) \ln^2 \frac{2}{1-x}, x \in [0, 1] \). Then there exists a constant \( M > 0 \) such that \( \frac{h(x)}{h(tx)} \leq M \) for each \( t \in [0, 1] \).

The proof is similar to Lemma 3.1. We omit the details.

Lemma 3.3. Suppose \( f \in \beta L \), then \( f \in \beta^0 L \) if and only if \( \| f_t - f \|_L \to 0 \) (\( t \to 1^- \)), where \( f_t(z) = f(tz) \).

Proof. Suppose \( f \in \beta^0 L \), then given any \( \epsilon > 0 \), there exists \( \delta \in (0, 1) \) such that
\[ (1 - |z|) \ln \frac{2}{1 - |z|} |f'(z)| \leq (1 - |z|^2) \ln \frac{2}{1 - |z|} |f'(z)| < \epsilon \] for all \( \delta^2 < |z| < 1 \). Consider
\[ \| f_t - f \|_L = \sup_D (1 - |z|^2) \ln \frac{2}{1 - |z|} |tf'(tz) - f'(z)| \]
\[ \leq \sup_{|z| > \delta} (1 - |z|^2) \ln \frac{2}{1 - |z|} |tf'(tz) - f'(z)| \]
\[ + \sup_{|z| \leq \delta} (1 - |z|^2) \ln \frac{2}{1 - |z|} |tf'(tz) - f'(z)| \]
\[ = I_1 + I_2. \]

If \( |z| > \delta \) and \( t > \delta \), then \( |tz| > \delta^2 \). By Lemma 3.1 we have
\[ I_1 \leq \sup_{|z| > \delta} (1 - |z|^2) \ln \frac{2}{1 - |z|} |f'(z)| + \sup_{|z| > \delta} (1 - |z|^2) \ln \frac{2}{1 - |z|} |tf'(tz)| \]
\[ \leq 2 \sup_{|z| > \delta} (1 - |z|) \ln \frac{2}{1 - |z|} |f'(z)| + 2 \sup_{|z| > \delta} (1 - |z|) \ln \frac{2}{1 - |z|} |f'(tz)| \]
\[ \leq 2\epsilon + 4 \sup_{|z| > \delta} (1 - |zt|) \ln \frac{2}{1 - |zt|} |f'(tz)| \]
\[ \leq 2\epsilon + 4\epsilon = 6\epsilon. \]

On the other hand, \( I_2 \to 0 \) as \( t \to 1^- \) since \( tf'(tz) \to f'(z) \) uniformly for \( |z| \leq \delta \). Thus \( \lim_{t \to 1^-} \| f_t - f \|_L = 0 \).
Conversely, suppose \( f \in \beta^0_L \) and \( \lim_{t \to 1} \|f_t - f\|_L = 0 \). Then for \( \epsilon > 0 \) there exists \( t \in (0, 1) \) such that \( \|f_t - f\|_L < \epsilon \). It follows that
\[
(1 - |z|^2) \ln \frac{2}{1 - |z|} |f'(z)| \leq \|f_t - f\|_L + (1 - |z|^2) \ln \frac{2}{1 - |z|} |(f_t)'(z)|< \epsilon + (1 - |z|^2) \ln \frac{2}{1 - |z|} |(f_t)'(z)|.
\]
Now let \( |z| \to 1 \) then \( (1 - |z|^2) \ln \frac{2}{1 - |z|} |(f_t)'(z)| \to 0 \) because \( f_t \in \beta^0_L \).
Hence \( f \in \beta^0_L \).

**Proposition 3.4.** The polynomials are dense in \( \beta^0_L \).

**Proof.** Let \( f \in \beta^0_L \) and \( t_n = 1 - \frac{1}{n} \), then \( f(t_n z) \) is analytic in \( |z| \leq 1 \). Hence there exists a polynomial \( p_n(z) \) such that
\[
|f(t_n z) - p_n(z)| < \frac{1}{n}, \quad |f'(t_n z) - p'(n)(z)| < \frac{1}{n}
\]
for all \( |z| \leq 1 \). Then by Lemma 3.3 we get
\[
\|f(z) - p_n(z)\|_L \leq \|f(z) - f(t_n z)\|_L + \|f(t_n z) - p_n(z)\|_L
\leq \|f(z) - f(t_n z)\|_L + \frac{(1 + \frac{4}{n})}{n} \to 0 \quad (n \to \infty).
\]
Thus the polynomials are dense in \( \beta^0_L \). \( \square \)

**Proposition 3.5.** \( \beta_L \subset VMOA \).

**Proof.** Let \( I \) is an arc in \( \partial D \) and \( S(I) \) is the Carleson box based on \( I \), i.e, \( S(I) = \{ re^{i\theta} : 1 - r \leq |I|, e^{i\theta} \in I \} \). For \( f \in \beta_L \), it follows that
\[
\int_{S(I)} |f'(z)|^2(1 - |z|^2)dA(z)
\leq \int_{S(I)} \frac{\|f\|^2_{\beta_L}}{(1 - |z|^2) \ln^2 \frac{2}{1 - |z|}}dA(z)
\leq \|f\|^2_{\beta_L} |I| \int_{1 - |I|}^1 \frac{1}{(1 - r) \ln^2 \frac{2}{1 - r}}dr = \|f\|^2_{\beta_L} \frac{|I|}{\ln \frac{2}{|I|}}.
\]
Then
\[
\frac{1}{|I|} \int_{S(I)} |f'(z)|^2(1 - |z|^2)dA(z) \leq \frac{\|f\|^2_{\beta_L}}{\ln \frac{2}{|I|}} \to 0 \quad (|I| \to 0).
\]
Hence \( f \in VMOA \). \( \square \)

Since \( \beta_\alpha \subset \beta_L \) for \( 0 < \alpha < 1 \), we have the following corollary.
Corollary 3.6. For $0 < \alpha < 1$, $\beta_{\alpha} \subset VMOA$.

This fact was proved in [8, Theorem 3]. However this proof is much easier than the one in [8].

Theorem 3.7.

(1) Let $f \in \beta_{L}^{0}$, if $|f(z)| \geq \sigma > 0 (|z| < 1)$, then $f$ is a cyclic vector in $\beta_{L}^{0}$.

(2) If $f$ is a cyclic vector in $\beta_{L}^{0}$, then $f$ is an outer function.

Proof. (1) For $0 < t < 1$, $f_{t}(z) = f(tz)$. Since $\frac{f}{f_{t}}$ is analytic in $|z| \leq 1$, we can easily prove that there exists polynomials $p_{n}$ such that $\|p_{n}f - \frac{f}{f_{t}}\|_{L} \rightarrow 0$ as $n \rightarrow \infty$. Thus we have $\frac{f}{f_{t}} \in [f]$. If $\|\frac{f}{f_{t}} - 1\|_{L} \rightarrow 0$ as $t \rightarrow 1^{-}$, then $1 \in [f]$, hence by Proposition 3.4, $f$ is cyclic in $\beta_{L}^{0}$. Now we are going to show that $\|\frac{f}{f_{t}} - 1\|_{L} \rightarrow 0 (t \rightarrow 1^{-})$.

We have

$$\|\frac{f}{f_{t}} - 1\|_{L} \leq \frac{1}{\sigma} \|f - f_{t}\|_{L} + \frac{1}{\sigma^{2}} \sup_{D}(1 - |z|^{2}) \ln \frac{2}{1 - |z|} |f(z) - f(tz)||tf'(tz)|$$

$$\leq \frac{1}{\sigma} I_{3} + \frac{1}{\sigma^{2}} I_{4}. $$

By Lemma 3.3 we know $I_{3} \rightarrow 0 (t \rightarrow 1^{-})$, then we only prove $I_{4} \rightarrow 0 (t \rightarrow 1^{-})$.

Since $f \in \beta_{L}^{0}$, for a given any $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$(1 - |z|^{2}) \ln \frac{2}{1 - |z|} |f'(z)| < \epsilon$$

for all $\delta^{2} < |z| < 1$. If $|z| > \delta$ and $t > \delta$, then $|tz| > \delta^{2}$. By Lemmas 2.1 and 3.2 it follows that

$$\sup_{|z| > \delta}(1 - |z|^{2}) \ln \frac{2}{1 - |z|} |f(z) - f(tz)||tf'(tz)|$$

$$\leq \epsilon(1 - |z|^{2}) \ln \frac{2}{1 - |z|} |f(z) - f(tz)|$$

$$\leq \epsilon\|f\|_{\beta_{L}} \frac{(1 - |z|^{2}) \ln \frac{2}{1 - |z|}}{(1 - |tz|^{2}) \ln \frac{2}{1 - |tz|}} \ln \left( \frac{\ln \frac{2}{1 - |z|}}{\ln \frac{2}{1 - |tz|}} \right)$$

$$\leq \epsilon\|f\|_{\beta_{L}} \frac{2(1 - |z|) \ln^{2} \frac{2}{1 - |z|}}{(1 - |tz|) \ln^{2} \frac{2}{1 - |tz|}}$$

$$\leq 2M\|f\|_{\beta_{L}} \epsilon.$$
On the other hand,
\[
\sup_{|z| \leq \delta} (1 - |z|^2) \ln \frac{2}{1 - |z|} |f(z) - f(tz)| |tf'(tz)|
\]
\[
\leq \|f\|_{\beta_L^0}^2 \sup_{|z| \leq \delta} (1 - |tz|^2) \ln \frac{2}{1 - |tz|} \ln \left(\frac{ln \frac{2}{1 - |z|}}{ln \frac{2}{1 - |tz|}}\right) \to 0 \quad (t \to 1^-).
\]

Hence \( I_4 \to 0 \quad (t \to 1^-) \). Thus \( f \) is a cyclic vector in \( \beta_L^0 \).

(2) If \( f \) is a cyclic vector in \( \beta_L^0 \), then, according to Proposition 3.5 and [4, Proposition 6], \( f \) is a cyclic vector in \( VMOA \). Hence \( f \) is an outer function by Theorem A. This completes the proof of Theorem 3.1. \( \Box \)

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