On Higher Syzygies of Projective Toric Varieties

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Abstract

Let $A$ be an ample line bundle on a projective toric variety $X$ of dimension $n \geq 2$. It is known that the $d$-th tensor power $A^\otimes d$ embeds $X$ as a projectively normal variety in $Pr := P(H^0(X, L^\otimes d))$ if $d \geq n$. In this paper first we show that when $\dim X = 2$ the line bundle $A^\otimes d$ satisfies the property $N_p$ for $p \leq 3d - 3$. Second we show that when $\dim X = n \geq 3$ the bundle $A^\otimes d$ satisfies the property $N_p$ for $p \leq d + 2$ and $d \geq n$.

KEYWORDS: toric variety, syzygy
ON HIGHER SYZYGIES OF PROJECTIVE TORIC VARIETIES

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Abstract. Let $A$ be an ample line bundle on a projective toric variety $X$ of dimension $n$ ($\geq 2$). It is known that the $d$-th tensor power $A^\otimes d$ embeds $X$ as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X, L^\otimes d))$ if $d \geq n - 1$. In this paper first we show that when $\dim X = 2$ the line bundle $A^\otimes d$ satisfies the property $N_p$ for $p \leq 3d - 3$. Second we show that when $\dim X = n \geq 3$ the bundle $A^\otimes d$ satisfies the property $N_p$ for $p \leq d - n + 2$ and $d \geq n - 1$.

Introduction

The purpose of this article is to study the minimal free resolution of homogeneous coordinate rings of toric varieties.

Let $X$ be a projective toric variety of dimension $n$ and $L$ a very ample line bundle on $X$. Since projective toric variety of dimension one is isomorphic to the projective line, we may assume that $n \geq 2$.

Koelman showed that an ample line bundle on a projective toric surface $X$ is very ample and embeds $X$ as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X, L))$ [10], and obtained a criterion when the surface is defined by only quadrics [11]. When $n \geq 3$ an ample line bundle is not very ample in general. Ewald and Wessels [3] showed that for an ample line bundle $A$ on $X$ the $d$-th tensor power $L = A^\otimes d$ is very ample for $d \geq \dim X - 1$. Ogata and Nakagawa [13] showed that $L = A^\otimes d$ embeds $X$ as a projectively normal variety if $d \geq \dim X - 1$ and that the homogeneous ideal $I$ of $X$ in $\mathbb{P}^r := \mathbb{P}(H^0(X, L))$ is generated by quadrics if $d \geq \dim X$. In this paper, we study higher syzygies of the homogeneous ideal of $X$ in $\mathbb{P}^r$, especially the property $N_p$ introduced by Green and Lazarsfeld [7].

Definition 1. Let $X$ be a projective variety and $L$ a very ample line bundle on $X$ defining an embedding $X \hookrightarrow \mathbb{P}^r := \mathbb{P}(H^0(X, L))$. Denote by $S = \text{Sym} H^0(X, L)$ the homogeneous coordinate ring of the projective space $\mathbb{P}^r$. Consider the graded $S$-module $R = R(L) = \oplus_{i \geq 0} H^0(X, L^\otimes i)$, the homogeneous coordinate ring of $X$. Let $E_*$ be a minimal graded free resolution of $R$:

$$\cdots \to E_2 \to E_1 \to E_0 \to R \to 0,$$

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where $E_i = \bigoplus_j S(-a_{ij})$. Then the line bundle $L$ satisfies Property $(N_0)$ if $E_0 = S$. For an integer $p \geq 1$ the line bundle $L$ satisfies Property $(N_p)$ if $E_0 = S$ and if $a_{ij} = i + 1$ for $1 \leq i \leq p$.

Schenck and Smith [17] proved that for an ample line bundle $A$ on a projective toric variety of dimension $n$, the bundle $A^\otimes d$ satisfies Property $N_{d-n+1}$ for $d \geq n - 1$.

This paper improves their results by separately considering the case $n = 2$ and $n \geq 3$.

**Theorem 0.1.** Let $X$ be a projective toric surface and $A$ an ample line bundle on $X$. Then $A^\otimes d$ satisfies Property $N_p$ for $p \leq d - n + 2$ and $d \geq n - 1$.

This is given by Proposition 2.3.

**Theorem 0.2.** Let $X$ be a projective toric variety of dimension $n \geq 3$ and $A$ an ample line bundle on $X$. Then $A^\otimes d$ satisfies Property $N_p$ for $p \geq d - n + 2$ and $d \geq n - 1$.

This is given by Proposition 3.3.

1. Polarized Toric Varieties

First we mention the facts about toric varieties needed in this paper following Oda’s book [14], or Fulton’s book [5].

Let $N$ be a free $\mathbb{Z}$-module of rank $n$, $M$ its dual and $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_\mathbb{R} := N \otimes \mathbb{R}$ and $M_\mathbb{R} := M \otimes \mathbb{R}$. Let $T_N := N \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic $n$-torus over the field $\mathbb{C}$ of complex numbers, where $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$. Then $M = \text{Hom}_{\mathbb{gr}}(T_N, \mathbb{C}^*)$ is the character group of $T_N$. For $m \in M$ we denote $e(m)$ the corresponding character of $T_N$. Let $\Delta$ be a complete finite fan of $N$ consisting of strongly convex rational polyhedral cones $\sigma$, that is, there exist a finite number of elements $v_1, v_2, \ldots, v_s$ in $N$ such that

$$\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_s,$$

and $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N\text{emb}(\Delta) := \bigcup_{\sigma \in \Delta} U_\sigma$ of dimension $n$ (see Section 1.2 [14], or Section 1.4 [5]). Here $U_\sigma = \text{Spec } \mathbb{C}[\sigma^\vee \cap M]$ and $\sigma^\vee$ is the dual cone of $\sigma$ with respect to the pairing $\langle \cdot, \cdot \rangle$. For the origin $\{0\}$, the affine open set $U_{\{0\}} = \text{Spec } \mathbb{C}[M]$ is the unique dense $T_N$-orbit. We note that a toric variety is always normal.

Let $L$ be an ample $T_N$-equivariant invertible sheaf on $X$. Then the polarized variety $(X, L)$ corresponds to an integral convex polytope $P$ in $M_\mathbb{R}$ of dimension $n$. We call the convex hull $\text{Conv}\{u_0, u_1, \ldots, u_r\}$ in $M_\mathbb{R}$ of a
finite subset \( \{u_0, u_1, \ldots, u_r\} \subset M \) an integral convex polytope in \( M_\mathbb{R} \). The correspondence is given by the isomorphism
\[
H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C} e(m),
\]
where \( e(m) \) are considered as rational functions on \( X \) because they are functions on the open dense subset \( T_N \) of \( X \) (see Section 2.2 [14], or Section 3.5 [5]).

Let \( P_1 \) and \( P_2 \) be integral convex polytopes in \( M_\mathbb{R} \). Then we can consider the Minkowski sum \( P_1 + P_2 := \{x_1 + x_2 \in M_\mathbb{R}; x_i \in P_i \ (i = 1, 2)\} \) and the multiplication by scalars \( rP_1 := \{rx \in M_\mathbb{R}; x \in P_1\} \) for a positive real number \( r \). If \( l \) is a natural number, then \( lP_1 \) coincides with the \( l \)-times sum of \( P_1 \), i.e., \( lP_1 = \{x_1 + \cdots + x_l \in M_\mathbb{R}; x_1, \ldots, x_l \in P_1\} \). The \( l \)-th tensor power \( L \otimes l \) corresponds to the convex polytope \( lP := \{lx \in M_\mathbb{R}; x \in P\} \). Moreover the multiplication map
\[
H^0(X, L \otimes l) \otimes H^0(X, L) \rightarrow H^0(X, L \otimes (l+1))
\]
transforms \( e(u_1) \otimes e(u_2) \) for \( u_1 \in lP \cap M \) and \( u_2 \in P \cap M \) to \( e(u_1 + u_2) \) through the isomorphism (1.1). Therefore the equality \((lP \cap M) + (P \cap M) = (l+1)P \cap M\) is equivalent to the surjectivity of the multiplication map (1.2).

**Proposition 1.1** (Nakagawa-Ogata [13]). Let \( X \) be a projective toric variety of dimension \( n \) and \( L \) an ample line bundle on \( X \). Then the multiplication map
\[
H^0(X, L^{\otimes i}) \otimes H^0(X, L) \rightarrow H^0(X, L^{\otimes (i+1)})
\]
is surjective for all \( i \geq n - 1 \).

This implies that \( L^{\otimes d} \) satisfies Property \( N_0 \) for \( d \geq n - 1 \). By employing an analogous method of Mumford [12] we obtained that \( L^{\otimes d} \) satisfies Property \( N_1 \) for \( d \geq n \) (see Corollary 2.2 in [13]). Schenck and Smith [17] generalizes for \( L^{\otimes d} \) to satisfy Property \( N_p \) for \( d \geq n - 1 + p \).

### 2. Toric Surfaces

Ogata[16] generalize the result of [11] to higher dimension by using the method of Fujita’s regular ladder [4]. In this section we use the same method in the case of dimension two.

Let \( X \) be a projective toric surface and \( L \) an ample line bundle on \( X \). We consider a general hyperplane section \( C \) of the linear system \(|L|\). Since \( X \) is normal, we may assume that \( C \) is nonsingular. Set \( L_C = L|C \), the restriction of \( L \) to the curve \( C \). From easy calculation, we see that
\[
h^0(C, L_C) = h^0(X, L) - 1 = \frac{1}{2} P \cap M - 1,
\]
\begin{align}
(2.2) \quad h^1(C, \mathcal{O}_C) &= h^2(X, L^{-1}) = h^0(X, \omega_X \otimes L) = \# \text{Int } P \cap M, \\
(2.3) \quad h^1(C, L_C) &= 0.
\end{align}

Denote by \( g = h^1(C, \mathcal{O}_C) \) the genus of \( C \). Then from Riemann-Roch Theorem we have \( \deg L_C = 2g - 2 + \# \partial P \cap M \).

**Lemma 2.1** (Green [6], (4.a.1)). Let \( L \) be a line bundle of degree \( 2g + 1 + p \) \((p \geq 0)\) on a smooth irreducible projective curve of genus \( g \). Then \( L \) satisfies Property \( N_p \).

This is a generalization of Mumford [12] (the case \( p = 0 \)) and of Fujita [4] (the case \( p = 1 \)).

**Lemma 2.2** (Green [6], (3.b.7)). Let \( X \) be a compact complex manifold, \( L \) a line bundle on \( X \), \( Y \subset X \) a connected hypersurface in the linear system \( |L| \) and \( L_Y \) denote the restriction of \( L \) to \( Y \). Assume that

\[ H^1(X, L^{\otimes q}) = 0 \quad \text{for all } q \geq 0. \]

Then \( \mathcal{K}_{p,q}(X, L) \cong \mathcal{K}_{p,q}(Y, L_Y) \) for all \( p,q \).

Since \( H^1(X, L^{\otimes q}) = 0 \) for \( q \geq 0 \) and for any ample ample line bundle \( L \) on a toric variety \( X \), this lemma implies that if \( L_Y \) satisfies Property \( N_p \) for a general smooth hypersurface \( Y \) in \( |L| \), then \( L \) also satisfies Property \( N_p \).

**Proposition 2.3.** Let \( A \) be an ample line bundle on a projective toric surface \( X \) corresponding to an integral convex polygon \( Q \) in \( M_{\mathbb{R}} \) given by the isomorphism

\[ H^0(X, A) \cong \bigoplus_{m \in Q \cap M} \mathbb{C}e(m). \]

Then the \( d \)-th tensor power \( A^{\otimes d} \) satisfies Property \( N_p \) for \( p \leq d \# \text{Int } Q \cap M - 3 \). In particular, \( A^{\otimes d} \) satisfies Property \( N_p \) for \( p \leq 3d - 3 \).

**Proof.** Set \( L = A^{\otimes d} \). Let \( C \) be a general hyperplane section of \( |L| \). Set \( L_C = L|C \). Denote by \( g \) the genus of \( C \). Then we have

\[ \deg L_C = 2g - 2 + d \# \text{Int } Q \cap M \]

\[ = 2g + 1 + (d \# \text{Int } Q \cap M - 3). \]

From Lemma 2.1, the bundle \( L_C \) satisfies Property \( N_p \) for \( p \leq d \# \text{Int } Q \cap M - 3 \). From Lemma 2.2 we obtain a proof of Proposition. \( \square \)

This is a generalization of the case \((X, A) = (\mathbb{P}^2, \mathcal{O}(1))\) treated by Birkenhake in [1].
3. Higher Dimension

Lemma 3.1 (Ogata [15]). Let $A$ be an ample line bundle on a projective toric variety of dimension $n$ ($n \geq 3$). Then $A^{\otimes d}$ satisfies Property $N_1$ for $d \geq n - 1$.

A very ample invertible sheaf $L$ on a projective variety $X$ defines an embedding $\Phi_L : X \to \mathbb{P}(H^0(X, L)) = \mathbb{P}^r$. Set $M_L := \Phi_L^* \Omega^{1}_{\mathbb{P}^r}(1)$ so that there exists the following exact sequence of vector bundles

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0.$$  

Lemma 3.2 (Ein-Lazarsfeld [2]). Assume that $L$ is very ample and that $H^1(X, L^{\otimes k}) = 0$ for all $k \geq 1$. Then $L$ satisfies Property $N_p$ if and only if

$$H^1(X, \wedge^a M_L \otimes L^{\otimes b}) = 0 \quad \text{for } 1 \leq a \leq p + 1 \text{ and } b \geq 1.$$

Since in characteristic zero $\wedge^a M_L$ is a direct summand of $M_L^{\otimes a}$, we have only to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ as in [2] and [8].

Proposition 3.3. Let $p \geq 2$ be an integer. Let $A$ be an ample line bundle on a projective toric variety $X$ of dimension $n$ ($n \geq 3$). Then $A^{\otimes d}$ satisfies Property $N_p$ for $p \leq n - 2 - d$.

For a proof we have to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ for $1 \leq a \leq p + 1$ and $b \geq 1$ with $L = A^{\otimes d}$. We need a lemma.

Lemma 3.4 (Mumford [12]). Let $F$ be a coherent sheaf on a projective algebraic variety $X$. Let $A$ be a line bundle on $X$ generated by global sections. If $H^i(X, F \otimes A^{\otimes (i)}) = 0$ for all $i \geq 1$, then the multiplication map

$$H^0(X, F \otimes A^{\otimes j}) \otimes H^0(X, A) \to H^0(X, F \otimes A^{\otimes (j+1)})$$

is surjective for all $j \geq 0$.

For a proof see Theorem 2 in [12].

Proof of Proposition 3.3. Let $q \geq 2$ be an integer and $L = A^{\otimes d}$ with $d \geq n - q - 2$. We want to show that

$$H^1(X, M_L^{\otimes q} \otimes A^{\otimes j}) = 0 \quad \text{for } j \geq n + q - 3,$$

$$H^i(X, M_L^{\otimes q} \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 2 \text{ and } j \geq 0.$$

From Proposition 1.1 and the exact sequence (3.1) we see that

$$H^i(X, M_L \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 1, j \geq 0 \text{ and } d \geq n - 1.$$  

First we shall show the vanishing of (3.2) for $q = 2$. Taking tensor product of (3.1) with $M_L \otimes A^{\otimes j}$ we obtain an exact sequence

$$0 \to M_L^{\otimes 2} \otimes A^{\otimes j} \to H^0(X, L) \otimes \mathcal{O}_X \to M_L \otimes A^{\otimes j} \to M_L \otimes L \otimes A^{\otimes j} \to 0.$$
From (3.4) and (3.5) we have

\[(3.6) \quad H^i(X, M_L^{\otimes 2} \otimes A^{\otimes j}) = 0 \quad \text{for} \quad i \geq 2, \quad j \geq 0 \quad \text{and} \quad d \geq n - 1.\]

Second we shall show the vanishing of (3.2) for \( q = 2 \). From Lemmas 3.1 and 3.2, we see that \( H^1(X, \wedge^2 M_L \otimes L^{\otimes b}) = 0 \) for \( b \geq 1 \) and \( d \geq n - 1 \). Taking wedge product in (3.1) and twisting by \( L^{\otimes b} \), we obtain an exact sequence

\[(3.7) \quad 0 \rightarrow \wedge^2 M_L \otimes L^{\otimes b} \rightarrow \wedge^2 H^0(X, L) \otimes_C L^{\otimes b} \rightarrow M_L \otimes L^{\otimes (b+1)} \rightarrow 0.\]

The vanishing of the first cohomology group implies the surjectivity of the map

\[(3.8) \quad \wedge^2 H^0(X, L) \otimes H^0(X, L^{\otimes b}) \rightarrow H^0(X, M_L \otimes L^{\otimes (b+1)}).\]

Taking global sections of (3.7) with \( b = 0 \) we see that the surjective map (3.8) factors through \( \wedge^2 H^0(X, L) \rightarrow H^0(X, M_L \otimes L) \). Thus we have that

\[(3.9) \quad H^1(X, M_L^{\otimes 2} \otimes L^{\otimes b}) = 0 \quad \text{for} \quad b \geq 1 \quad \text{and} \quad d \geq n - 1\]

from the exact sequence (3.5) replacing \( A \) by \( L \). For line bundles \( L_1 \) and \( L_2 \), denote by \( R(L_1, L_2) \) the kernel of the multiplication map \( H^0(X, L_1) \otimes H^0(X, L_2) \rightarrow H^0(X, L_1 \otimes L_2) \). By taking a global section of the exact sequence

\[0 \rightarrow M_L \otimes A \rightarrow \Gamma(X, L) \otimes_C A \rightarrow L \otimes A \rightarrow 0,\]

we have \( H^0(X, M_L \otimes A) = R(L, A) \). Then we can rewrite the sequence (3.5) after taking its global sections as

\[
\begin{array}{ccc}
H^0(X, L) \otimes H^0(X, M_L \otimes A^{\otimes j}) & \longrightarrow & H^0(X, M_L \otimes L \otimes A^{\otimes j}) \\
\| & & \| \\
H^0(X, L) \otimes R(A^{\otimes j}, L) & \longrightarrow & R(L \otimes A^{\otimes j}, L).
\end{array}
\]

For simplicity we denote \( A^{\otimes i} \) as \( A^i \) and \( H^0(X, L) \) as \( \Gamma(L) \). From Corollary 2.2 in [13], we have that

\[\Gamma(A^i) \otimes R(A^d, A^j) \rightarrow R(A^{d+i}, A^j)\]

is surjective for \( d \geq n, \ i \geq 1 \) and \( j \geq 1 \). Hence we showed the vanishing of (3.2) for \( j \geq 1 \). We remain to show the vanishing of \( H^1(X, M_L^{\otimes 2} \otimes A^{\otimes (n-1)}) \).
for \( d \geq n \). Consider the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
R(A_{n-1}, A_{n-1}) \\
\otimes \\
\Gamma(A^d) \\
\downarrow \\
\Gamma(A_{n-1}) \\
\otimes \Gamma(A^{d+n-1}) \\
\downarrow \\
\Gamma(A^{d+n-1}) \\
\end{array}
\]

\[
\begin{array}{c}
0 \\
\downarrow \\
R(A_{n-1}, A^d) \\
\otimes \\
\Gamma(A^d) \\
\downarrow \beta \\
\Gamma(A^{2n-2}) \\
\otimes \\
\Gamma(A^{d+2n-2}) \\
\end{array}
\]

If \( \alpha \) is surjective, then \( \beta \) is also surjective. The vanishing (3.9) implies the surjectivity of

\[
R(A_{n-1}, A_{n-1}) \otimes \Gamma(A^{n-1}) \rightarrow R(A_{n-1}, A^{2n-2}).
\]

Since \( \Gamma(A^i) \otimes \Gamma(A^{n-1}) \rightarrow \Gamma(A^{n+i-1}) \) is surjective for \( i \geq 1 \), the map \( \alpha \) is surjective, hence,

\[
\Gamma(A^d) \otimes R(A^{n-1}, A^d) \rightarrow R(A^{d+n-1}, A^d)
\]

is surjective. This map is the same as

\[
\Gamma(L) \otimes \Gamma(M_L \otimes A^{n-1}) \rightarrow \Gamma(M_L \otimes L \otimes A^{n-1})
\]

with \( L = A^d \). Thus we obtain that \( H^1(X, M_L^{\otimes q} \otimes A^{n-1}) = 0 \).

For \( q \geq 2 \) and \( L = A^d \) with \( d \geq n + q - 2 \), if we have the equalities (3.2) and (3.3), then from the exact sequence

\[
(3.10) \quad 0 \rightarrow M_L^{\otimes (q+1)} \otimes A^j \rightarrow \Gamma(L) \otimes M_L^{\otimes q} \otimes A^j \rightarrow M_L^{\otimes q} \otimes L \otimes A^j \rightarrow 0
\]

we have

\[
H^i(X, M_L^{\otimes (q+1)} \otimes A^j) = 0 \quad \text{for } i \geq 2 \text{ and } j \geq 0,
\]

and from Lemma 3.4 we see the surjectivity of

\[
\Gamma(M_L^{\otimes q} \otimes A^j) \otimes \Gamma(A) \rightarrow \Gamma(M_L^{\otimes q} \otimes A^{j+1})
\]
for $j \geq n + q - 2$. From this and the exact sequence (3.10) we have

$$H^1(X, M_\ell^{\otimes (q+1)} \otimes A^j) = 0 \quad \text{for } j \geq n + q - 2.$$ 

In particular, we have

$$H^1(X, M_\ell^{\otimes (q+1)} \otimes L^\otimes b) = 0$$

for $b \geq 1$ and $L = A^{\otimes d}$ with $d \geq n + q - 2$. 

\[ \square \]

**References**


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