Some commutativity results for rings with certain polynomial identities

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SOME COMMUTATIVITY RESULTS FOR RINGS WITH CERTAIN POLYNOMIAL IDENTITIES

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Throughout the present paper, $R$ will represent an associative ring (with or without 1), $N$ the set of all nilpotent elements in $R$, and $Z$ the center of $R$. Generalized commutators $[x, x_1, x_2, \ldots, x_k]$, for integers $k \geq 1$, are defined as follows: $[x, x_1] = xx_1 - x_1x$, if $k = 1$, and $[[x, x_1, x_2, \ldots, x_{k-1}], x_k]$, if $k \geq 2$. For $x_1 = x_2 = \cdots = x_k = y$, $[x, y, y, \ldots, y]$ is abbreviated as $[x, y]_k$. As is well known, if $[x, y] = 0$ then $[x, y]^m = m^{m-1}[x, y]$ for any positive integer $m$. We denote by $Z(k)$ the set of all $x \in R$ such that $[x, x_1, x_2, \ldots, x_k] = 0$ for all $x_1, x_2, \ldots, x_k \in R$. Following [5], a ring $R$ is said to be $s$-unital if for each $x$ in $R$, $x \in xR \cap Rx$. As stated in [5], if $R$ is an $s$-unital ring, then for any finite subset $F$ of $R$ there exists an element $e$ in $R$ such that $ex = xe = x$ for all $x \in F$. Such an element $e$ will be called a pseudo-identity of $F$.

Now, let $n$ be a fixed positive integer. Awtar [2] showed that if $R$ is an $n!$-torsion free ring with 1 satisfying the polynomial identity $[x^n, y^n] = 0$ then it must be commutative. On the other hand, Bell [3] showed that an $n$-torsion free ring with 1 satisfying the same polynomial identity need not be commutative. More recently, Abu-Khuzam and Yaqub [1] proved that an $n$-torsion free ring with 1 satisfying the same polynomial identity must be commutative under some additional condition such as $x^ky^k - y^kx^k \in Z$ or $(xy)^k - (yx)^k \in Z$ with $(n,k) = 1$.

Let $n$ be a fixed positive integer, and consider the following properties:

\( (I)_n \quad \text{if } x, y \in R \text{ and } n[x, y] = 0, \text{ then } [x, y] = 0. \)

\( (II)_n \quad [x^n, y^n] = 0 \text{ for all } x, y \in R. \)

The major purpose of this paper is to prove the following theorem which generalizes [1, Theorems 2 and 3] and [5, Theorem 1].

**Theorem.** Let $n$, $k$ be fixed positive integers with $(n,k) = 1$. Let $R$ be an $s$-unital ring satisfying $(I)_n$ and $(II)_n$. Then the following are equivalent:

(i) $R$ is commutative.

(ii) For every $x, y \in R$ there exists a positive integer $m = m(x,y)$ such that $[x^k, y^k]_m = 0$.

(iii) For every $x, y \in R$ there exists a positive integer $m = m(x,y)$
such that \([(xy)^k-(yx)^k, y^k]\)_{m} = 0.

(iv) If \(x, y \in R \) and \(x-y \in N \) then either \(x^k-y^k \in Z(m) \) with some positive integer \(m = m(x, y) \) or both \(x \) and \(y \) commute with all elements in \(N\).

(v) For every \(x \in R \) and \(a \in N \) there exists a positive integer \(m = m(x, a) \) such that \([x(1+a)]^n-x^n(1+a)^n, x]_{m} = 0 \) (formally written).

In preparation for proving our main theorem, we quote the following lemmas which are stated in [5].

Lemma 1. Let \(R \) be an \(s\)-unital ring, \(e \) a pseudo-identity of \(|a, b| \subseteq R\). If \(a^m b = (a+e)^m b \) for some positive integer \(m \), then \(b = 0\).

Lemma 2. Let \(R \) be an \(s\)-unital ring satisfying (\(I\)) and (\(II\)). Then \([a, a^n] = 0 \) for all \(a \in N \) and \(x \in R \). \(N \) is a commutative ideal containing the commutator ideal of \(R \), and \(N^2 \subseteq Z \); in particular, \((x+ax)^n-(x+xa)^n = [a, x^m] \) for all \(a \in N \), \(x \in R \), and positive integers \(m \).

Proof of Theorem. We start the proof by showing that either of (iii) and (iv) implies (ii). Suppose (iii). Let \(x, y \in R \), and \(a = [x^k, y^k] \in N \) (Lemma 2). Then \(0 = [(y+ay)^k-(y+ya)^k, y^k]_{m} = [[a, y^k], y^k]_{m} = [x^k, y^k]_{m+1}\) for some positive integer \(m \) (Lemma 2), proving (ii). Next, suppose (iv). Let \(x, y \in R \), and \(a = [x^k, y^k] \), as above. Since \((a+y)-y \in N \), we see that \([a, y] = 0 \) or \((a+y)^k-y^k \in Z(m) \) for some \(m \geq 1 \). If \((a+y)^k-y^k \in Z(m) \) then

\[((a+y)^k-y^k, a+y) = -[y^k, a+y] = [a, y^k],
\]

and therefore \([x^k, y^k]_{m+1} = [a, y^k]_{m} = 0 \). Needless to say, if \([a, y] = 0 \) then \([x^k, y^k]_{m+1} = [a, y^k]_{m} = 0 \), proving (ii).

Now we prove that (v) implies (ii). Let \(a \in N \) and \(y \in R \). Then there exists a positive integer \(m \) such that \([y(1+a)]^n-y^n(1+a)^n, y]_{m} = 0 \), that is, \([y^n(1+a)^n, y]_{m} = [y(1+a)]^n, y]_{m} \). Since

\([y(1+a)]^n, y] = y[(1+a)y^n-y(1+a)]^n] = y[a, y^n] = 0\)

by Lemma 2, we get \(y^n[(e+a)^n, y]_{m} = [y^n(1+a)^n, y]_{m} = 0 \), where \(e \) is a pseudo-identity of \(|a, y| \). Similarly, \((y+e)^n[(e+a)^n, y]_{m} = 0 \) for some \(m \geq 1 \), where \(e' \) is a pseudo-identity of \(|e, a, y| \). Without loss of generality, we may assume that \(m = m' : y^n[(e+a)^n, y]_{m} = 0 \). But \(N^2 \subseteq Z \) by Lemma 2. Hence \(n[a, y]_{m} = 0 \), and therefore \([a, y]_{m} = 0 \). Now, let \(x \in R \). Then \([x, y] \in N \) (Lemma 2), and we conclude that \([x, y]_{m+1} = 0 \) for some
$m \geq 1$, proving (ii).

Finally, we prove that (ii) implies (i). In view of [4, Proposition 1], we may assume that $R$ has 1. Suppose that $[r^k, s^k] \neq 0$ for some $r, s \in R$. Then $[r^k, s^k]_m = 0$ and $[r^k, s^k]_{m-1} \neq 0$ for some $m > 2$. According to Lemma 2, $t = [r^k, s^k]_{m-2} \in N$ and $[t, x^m] = 0$ for all $x \in R$. Hence $[t, s^k] = 0 = [t, (s^k + 1)^x]$. Notice that $[t, s^k]_2 = [t, s^k]_2 = 0$. Then

$$ns^{k(n-1)}[t, s^k] = 0 = n(s^k + 1)^{n-1}[t, s^k + 1] = n(s^k + 1)^{n-1}[t, s^k].$$

Thus by Lemma 1 and (I)$_n$, we obtain $[r^k, s^k]_{m-1} = [t, s^k] = 0$. This contradiction proves that $[x^k, y^k]_2 = 0$ for all $x, y \in R$. Now, let $u, v \in |x^k|, x \in R$. Since $[u^n, v]_2 = 0 = [u^n, v+1]_2$ and $[u^n, v^n] = 0 = [u^n, (v+1)^n]$, we obtain $n[u^n]_1[u^n, v] = 0 = n(v+1)^{n-1}[u^n, v]$, and therefore $[u^n, v] = 0$ by Lemma 1 and (I)$_n$. Noting that $[(u+1)^n, v]_2 = \sum_{i=0}^{n-1}[u^{n-i}, v]_2 = 0$ by what proved just above, we can repeat the above argument for $u+1$ instead of $u$ to see that $[(u+1)^n, v] = 0$. Combining these with $[v, u+1]_2 = 0 = [v, u]_2$, by repeated use of the above argument, we can see that $nu^n[u^n, v] = 0 = n(u+1)^{n-1}[u, v]$. Hence again by Lemma 1 and (I)$_n$, we get $[u, v] = 0$. This proves that $R$ satisfies the polynomial identity $[x^k, y^k] = 0$. Hence, by [1, Theorem 2], $R$ is commutative, proving (i).

Needless to say, every commutative ring $R$ satisfies (ii) $-$ (v). This completes the proof of the theorem.

**Remark.** The example of Johnsen, Outcalt and Yaqub cited in [1] shows that (I)$_n$ cannot be omitted in Theorem. Also, the existence of a finite non-commutative nil ring shows that the hypothesis that $R$ is $s$-unital cannot be deleted. Finally, the following example shows that we cannot drop (II)$_n$:

Let $R_m$ be the ring consisting of $m \times m$ matrices over $Z$ of the form

$$\begin{pmatrix}
a & \cdots & \ast \\
0 & \cdots & a
\end{pmatrix}$$

Here, $Z \neq Z(m-1) = R_m$ if $m > 2$.

Now, let $R$ be the ring $R_1 \oplus R_2 \oplus R_3 \oplus \cdots$. Then $R$ is an $s$-unital ring (without 1) and $Z(m) \subseteq Z(m+1)$ for all positive integers $m$. As is easily seen, $R$ satisfies the condition (ii) in Theorem (for $k = 1$).

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