On some series associated with discrete subgroups of U(1,n;C)

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ON SOME SERIES ASSOCIATED WITH
DISCRETE SUBGROUPS OF U(1, n ; ℂ)

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0. Let $F$ be a fuchsian group acting on the unit disk. An element $g_k$ of $F$ is of the form

$$g_k(z) = \frac{a_k z + c_k}{c_k z + a_k}, \quad |a_k|^2 - |c_k|^2 = 1.$$ 

It is well-known about the convergence or divergence of the series $\sum_{g_k \in F} |c_k|^{-t}$ (see [4]). In this paper we show some generalized results on the series associated with discrete subgroups of $U(1, n ; ℂ)$.

1. Let us recall some definitions and notation. Let $V = V^{1,n}(ℂ)$ ($n \geq 1$) denote the vector space of $ℂ^{n+1}$, together with the unitary structure defined by the Hermitian form

$$Φ(z, w) = -\overline{z}_0w_0 + \overline{z}_1w_1 + \cdots + \overline{z}_nw_n$$

for $z = (z_0, z_1, \ldots, z_n)$ and $w = (w_0, w_1, \ldots, w_n)$. An automorphism $g$ of $V$, that is a linear bijection of $V$ onto $V$ such that $Φ(g(z), g(w)) = Φ(z, w)$ for $z, w \in V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, n ; ℂ)$. Let $V_- = \{z \in V \mid Φ(z, z) < 0\}$. Obviously $V_-$ is invariant under $U(1, n ; ℂ)$. Let $P(V)$ be the projective space obtained by $V$.

We define $H^α(ℂ) = P(V_-)$. Let $\overline{H^α(ℂ)}$ denote the closure of $H^α(ℂ)$ in projective space $P(V)$. An element $g$ in $U(1, n ; ℂ)$ operates in $P(V)$, leaving $\overline{H^α(ℂ)}$ invariant. Since $H^α(ℂ)$ is identified with the unit ball $B^α(ℂ)$ $= \{ξ \mid \|ξ\|^2 = \sum_{k=1}^{n} |ξ_k|^2 < 1\}$, we can regard discrete subgroups of $U(1, n ; ℂ)$ as generalized fuchsian groups (see [2]).

2. Let $g_k = (d^{(k)}_{i,j})_{1 \leq i,j \leq n+1}$ be an element in $U(1, n ; ℂ)$. We denote a point of $P^{-1}(0)$ by $0^*$. Let $d$ be the derived metric from $Ψ$ (see [2, Proposition 2.4.4]). We easily obtain

**Proposition 2.1.** $|a_{i,i}^{(k)}| = |Φ(g_k(0^*), 0^*)| |Φ(0^*, 0^*)|^{-1}$ $= \cosh d(0, g_k(0))$. 

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For the sake of simplicity and brevity, we denote $2 \mid a^{(k)}_{i,j} \mid$ by $\nu(g_k)$.

**Proposition 2.2.** If $g$ and $h$ are elements of $U(1,n; \mathcal{C})$, then

1. $\nu(g^{-1}) = \nu(g)$,
2. $\nu(gh) \leq \nu(g)\nu(h)$,
3. $\nu(hgh^{-1}) \leq [\nu(h)]^2 \nu(g) \leq [\nu(h)]^4 \nu(hgh^{-1})$.

**Proof.** The first is immediate.

2. Using Proposition 2.1, we have

\[
\nu(g)\nu(h) = 2 \cosh d(0,g(0))2 \cosh d(0,h(0)) \\
= 2 \cosh d(0,g(0))2 \cosh d(g(0),h(0)) \\
\geq \exp d(0,gh(0)) + \exp -d(0,gh(0)) \\
= 2 \cosh d(0,gh(0)) \\
= \nu(gh).
\]

3. It follows from (1) and (2) that

\[
\nu(hgh^{-1}) \leq \nu(h)\nu(g)\nu(h^{-1}) \\
= [\nu(h)]^2 \nu(g) \\
= [\nu(h)]^2 \nu(h^{-1}hgh^{-1}h) \\
\leq [\nu(h)]^4 \nu(hgh^{-1}).
\]

3. Unless otherwise stated, we shall always take $G$ to be a discrete subgroup of $U(1,n; \mathcal{C})$. First we give

**Definition 3.1 (cf. [3, Theorem 5.1]).** For any point $a \in H^n(\mathcal{C})$, $G$ is called of convergence type or divergence type according as $\sum_{g_k \in G} (1 - \|g(a)\|)^n$ converges or diverges.

**Theorem 3.2.** $G$ is of convergence type or divergence type according as $\sum_{g_k \in G} |a^{(k)}_{i,j}|^{-2n}$ converges or diverges.

**Proof.** Noting that $1 - \|g_k(0)\|^2 = 1 - \sum_{j=2}^{n+1} |a^{(k)}_{i,j}|^2 |a^{(k)}_{i,1}|^{-2} = |a^{(k)}_{i,1}|^{-2}$, we see

\[
(1/2)(1 - \|g_k(0)\|)^{-1} \leq |a^{(k)}_{i,1}|^2 \leq (1 - \|g_k(0)\|)^{-1}.
\]

Therefore we have

\[
\sum_{g_k \in G} (1 - \|g_k(0)\|)^n \leq \sum_{g_k \in G} |a^{(k)}_{i,1}|^{-2n} \leq 2^n \sum_{g_k \in G} (1 - \|g_k(0)\|)^n.
\]
Thus our proof is complete.

By using (3) in Proposition 2.2, we obtain

**Corollary 3.3** (3. Theorem 5.9). For any element $h$ in $U(1,n; \mathbb{C})$, the conjugate group $hGh^{-1}$ is of the same type as $G$.

Next we shall make the estimate of \[ \sum_{g \in G, \|g\| < r} [\nu(g)]^{-t} \] as $r \to \infty$. From now on we assume that $G_0 = \{ \text{identity} \}$.

We now state our results.

**Theorem 3.4.** Let $r > 2$ and $t$ any real number. Then

\[
\sum_{g \in G, \|g\| < r} [\nu(g)]^{-t} = \begin{cases} 
O(1) & \text{as } r \to \infty \text{ if } t > 2n; \\
O(\log r) & \text{as } r \to \infty \text{ if } t = 2n; \\
O(r^{2n-t}) & \text{as } r \to \infty \text{ if } t < 2n.
\end{cases}
\]

**Theorem 3.5.** Let $D_0$ be a fundamental polyhedron with respect to 0 for $G$. If $\text{vol}(D_0)$ is finite, then there exist positive numbers $m_1$ and $m_2$ such that

\[ m_1 \log r \leq \sum_{g \in G, \|g\| < r} [\nu(g)]^{-2n} \leq m_2 \log r, \]

and if $t < 2n$, then

\[ m_1 r^{2n-t} \leq \sum_{g \in G, \|g\| < r} [\nu(g)]^{-t} \leq m_2 r^{2n-t}. \]

**Remark 3.6.** When $n = 1$, $G$ is a fuchsian group acting on the unit disk. Noting that the radii of isometric circles are bounded, we see that Theorems 3.4 and 3.5 yield some familiar classical results (see [4]).

For proving the above theorems, we need two lemmas.

**Lemma 3.7** (3. Proposition 4.1). For $0 \leq r < 1$, the following inequality is satisfied.

\[ n(r,a) \leq B(1-r)^{-n}, \]

where $B$ is a constant independent of $a \in H^n(\mathbb{C})$.

**Lemma 3.8** (3. Proposition 4.4). Let $D_0$ be a fundamental polyhedron with respect to 0 for $G$. Suppose $\text{vol}(D_0) < \infty$. Let $a \in D_0$ and $\|a\| < \rho$.
< 1. Then there exists \( r_0 \) such that the following inequality is satisfied for \( r_0 \leq r < 1 \).

\[
A(1-r)^{-n} \leq n(r, a) \leq B(1-r)^{-n},
\]

where \( A \) is a constant which depends on \( \rho \) and \( B \) is a numerical constant.

We shall prove Theorems 3.4 and 3.5 in the same manner as in the proof of [1, Theorems 2 and 3].

**Proof of Theorems 3.4 and 3.5.** Let \( \chi_0(r) = \# \{ g \in G \mid \nu(g) < r \} \).

By Lemma 3.7, we have

\[
\chi_0(r) = \# \{ g \in G \mid \| g(0) \| < 1 - (4/r^2)^{1/2} \}
\]

\[
= n(1 - (4/r^2)^{1/2}, 0)
\]

\[
\leq B(1 - 1 - (4/r^2)^{1/2})^{-n} \leq 2^{-n} B r^{-n}. \quad (1)
\]

For each real number \( t \), we define

\[
\chi_t(r) = \sum_{\theta \in G, 0 < r \leq t} [\nu(g)]^{-t}.
\]

If \( r > 2 \), then

\[
\chi_t(r) = \int_2^r \frac{d\chi_0(s)}{s^t} = \frac{\chi_0(r)}{r^t} + t \int_2^r \frac{\chi_0(s)}{s^{t+1}} ds. \quad (2)
\]

Using this equation, together with the inequality (1), we obtain Theorem 3.4.

Lemma 3.8 establishes

\[
\chi_0(r) \geq A(1 - 1 - (4/r^2)^{1/2})^{-n} \geq 2^{-n} A r^{-n}. \quad (3)
\]

By (2) and (3), we complete our proof of Theorem 3.5.

Theorems 3.2 and 3.5 lead to

**Corollary 3.9 (3, Theorem 5.4).** If \( \text{vol} (D_0) < \infty \), then \( G \) is of divergence type.

**References**


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(Received August 8, 1984)