On subgroups of convergence or divergence type of $U(1,n;\mathbb{C})$

Shigeyasu Kamiya*
ON SUBGROUPS OF CONVERGENCE OR DIVERGENCE TYPE OF $U(1, n; \mathbf{C})$

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0. Introduction. Let $V = V^{1,n}(\mathbf{C})$ ($n \geq 1$) denote the vector space $\mathbf{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(z, w) = -z_0 \bar{w}_0 + z_1 \bar{w}_1 + z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n$$

for $z = (z_0, z_1, \ldots, z_n)$ and $w = (w_0, w_1, \ldots, w_n)$. An automorphism $g$ of $V$, that is a linear bijection of $V$ onto $V$ such that $\Phi(g(z), g(w)) = \Phi(z, w)$ for $z, w \in V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, n; \mathbf{C})$. A unitary transformation operates in $P(V)$, leaving $H^n(\mathbf{C})$ invariant. Since $H^n(\mathbf{C})$ is identified with the unit ball $B^n(\mathbf{C})$, discrete subgroups of $U(1, n; \mathbf{C})$ are considered as generalized Fuchsian groups.

In this paper, we shall classify discrete subgroups of $U(1, n; \mathbf{C})$ into convergence type and divergence type as in Fuchsian groups and generalize some results in [3] to them.

1. Preliminaries. Let $V_0 = \{ z \in V \mid \Phi(z, z) = 0 \}$ and $V_- = \{ z \in V \mid \Phi(z, z) < 0 \}$. $V_0$ and $V_-$ are invariant under $U(1, n; \mathbf{C})$. Let $P(V)$ be the projective space obtained from $V$. This is defined, as usual, by using the equivalent relation in $V-\{ 0 \} : u \sim v$ if there exists $\lambda \in \mathbf{C}-\{ 0 \}$ such that $u = v\lambda$. $P(V)$ is the set of equivalence classes, with the quotient topology. Let $P : V-\{ 0 \} \rightarrow P(V)$ denote the projection map. We define $H^n(\mathbf{C}) = P(V_-)$. Let $H^n(\mathbf{C})$ denote the closure of $H^n(\mathbf{C})$ in the projective space $P(V)$. An element $g$ in $U(1, n; \mathbf{C})$ operates in $P(V)$, leaving $H^n(\mathbf{C})$ invariant. If $z = (z_0, z_1, \ldots, z_n) \in V_-$, then the condition $-|z_0|^2 + \sum_{k=1}^{n} |z_k|^2 < 0$ implies that $z_0 \neq 0$. Therefore we may define a set of coordinates $\zeta = (\xi_1, \xi_2, \ldots, \xi_n)$ in $H^n(\mathbf{C})$ by $\xi_i(P(z)) = z_i z_0^{-1}$. In this way $H^n(\mathbf{C})$ becomes identified with the unit ball $B = B^n(\mathbf{C}) = \{ \zeta = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{C}^n \mid \sum_{k=1}^{n} |\xi_k|^2 < 1 \}$. Next we shall consider the metric in $H^n(\mathbf{C})$. Let $V_- = \{ z \in V \mid \Phi(z, z) = -1 \}$. Let $T_z(V_-)$ be the tangent space. This contains the $\mathbf{C}$-subspace $T_z'(V_-) = \{ v \in V \mid \Phi(z, v) = 0 \}$. Thus the restriction $P_z' = P_z^* : T_z'(V_-)$ is a $\mathbf{C}$-linear isomorphism of $T_z'(V_-)$ onto
$T_{rz}(B)$, where $P_2^* : T_2(V; -) \to T_2(B)$. We define the form $\Psi$ in $T_{rz}(B)$ by $\Psi(P_2(v), P_2(w)) = z_0 \Phi(v, w) z_0^{-1}$ which is Hermitian. We can compute this form explicitly, with respect to the standard basis $|f_1, f_2, \ldots, f_n|$ in $C^n$. We have

$$\Psi(f_i, f_j) = \delta_{ij}(1 - \sum_{k=1}^n |\xi_k|^2)^{-1} + \xi_i \bar{\xi}_j(1 - \sum_{k=1}^n |\xi_k|^2)^{-2}$$

(c.f. [1], Proposition 2.3.1).

2. The metric $\delta$. We introduce another metric $\delta(a, b)$ for two points $a, b$ in $H^n(U)$ as follows:

$$\delta(a, b) = \left[1 - |\Phi(a^*, a^*)| |\Phi(b^*, b^*)|^{-2}\right]^{1/2},$$

where $a^* \in P^{-1}(a)$ and $b^* \in P^{-1}(b)$. We see that $\delta(a, b) = \delta(b, a) \geq 0$ and $\delta(a, c) \leq \delta(a, b) + \delta(b, c)$, where $a, b, c \in H^n(U)$. Also, if $f$ is an element of $U(1, n; U)$, then $\delta(f(a), f(b)) = \delta(a, b)$. We define $\|a\| = \left[\sum_{k=1}^n |a_k|^2\right]^{1/2}$, where $a = (a_1, a_2, \ldots, a_n) \in H^n(U)$. Let $\rho$ be a real number satisfying $0 \leq \rho < 1$. We define

$$C(a, \rho) = \{z \in H^n(U) \mid \delta(a, z) < \rho\},$$

and then we have the following proposition.

**Proposition 2.1.**

(i) $C(0, \rho) = \{z \mid \|z\| < \rho\}$.

(ii) $f(C(a, \rho)) = C(f(a), \rho)$ for any unitary transformation $f$.

(iii) If $\|a\| < r < 1$, then $C(a, \rho)$ is contained in $\{z \mid \|z\| < (r + \rho)(1 + r\rho)^{-1}\}$.

(iv) $C(a, \rho) \subset \{z \mid \|z - a\| < [\rho(1 - \|a\|^2)(1 - \rho^2)^{-1}]^{1/2}\}$.

**Proof.** The first is immediate.

(ii) First we note that $f(C(0, \rho)) = C(f(0), \rho)$. Using Proposition 2.1.2 in [1], we can find $g \in U(1, n; U)$ such that $g(a) = 0$. From this, we obtain

$$g^{-1}(C(0, \rho)) = C(g^{-1}(0), \rho) = C(a, \rho),$$

and therefore

$$f(C(a, \rho)) = fg^{-1}(C(0, \rho)) = C(fg^{-1}(0), \rho) = C(f(a), \rho).$$

(iii) Without loss of generality, we may assume $a = (t, 0, \ldots, 0)$ ($t > 0$).
ON SUBGROUPS OF CONVERGENCE OR DIVERGENCE TYPE

Simple computation yields:

\[ |z| \delta(a, z) < \rho \iff |z = (z_1, \ldots, z_n)| \left| \left(1 - t^2 \rho^2 \right)^{1/2} \right| < \rho \sqrt{(1 - t^2)^2} \]

\[-\left(1 - \rho^2 \right) t \left(1 - t^2 \rho^2 \right)^{-1/2} \leq \sum_{j=1}^{n} |z_j|^2 < \rho^3 (1 - t^2)^2 \]

\[ (1 - t^2 \rho^2)^{-1} \]

It follows that

\[ C(a, \rho) \subset |z| \left\| z \right\| < (\rho + t)(1 + t\rho)^{-1} \]

\[ \subset |z| \left\| z \right\| < (\rho + r)(1 + r\rho)^{-1}. \]

(iv) By computation, we obtain the result.

**Proposition 2.2.** Let \(|a_n|\) and \(|b_n|\) be sequences in \(H^n(C)\). Suppose that \(\delta(a_n, b_n) = \rho(\text{constant}) < 1\) and \(\lim_{n \to \infty} \|a_n\| = 1\). Then \(\lim_{n \to \infty} \|a_n - b_n\| = 0\).

Proof. By (iv) in Proposition 2.1, we see that \(\|a_n - b_n\| < \rho^3 (1 - \|a_n\|^2)(1 - \rho^2)^{-1/2}\). As \(\|a_n\| \to 1\), the value on the right side goes to 0. Therefore we have \(\lim_{n \to \infty} \|a_n - b_n\| = 0\).

**3. The fundamental polyhedron.** Let \(G\) be a discrete subgroup of \(U(1, n; C)\). Let \(a \in H^n(C)\). Suppose the isotropy group \(G_a = \{f \in G \mid f(a) = a \} = \{\text{identity}\}\). We define the fundamental polyhedron \(D_a\) by \(|z| \delta(z, a) < \delta(z, f(a))\) for all \(f \in G \setminus \{\text{identity}\}\). Obviously, we see

\[ D_a = |z| \mid d(z, a) < d(z, f(a))\] for all \(f \in G \setminus \{\text{identity}\}\),

where \(d\) is the metric derived from \(\Psi\). Let

\[ f_k = \begin{pmatrix} a_{1,1}^{(k)} & \cdots & a_{1,n+1}^{(k)} \\ \cdots & \cdots & \cdots \\ a_{n+1,1}^{(k)} & \cdots & a_{n+1,n+1}^{(k)} \end{pmatrix} \]

We find that

\[ D_0 = |z| \left\| f_k(z) \right\| > \|z\| \text{ for all } f_k \in G \setminus \{\text{identity}\}.\]

\[ = |z = (z_1, z_2, \ldots, z_n)| \left| a_{1,1}^{(k)} + \sum_{j=2}^{n+1} a_{1,j}^{(k)} z_{j-1} | > 1 \right. \]

for all \(f_k \in G \setminus \{\text{identity}\}\).

Following the methods of Tsuji [3], we can prove that
(1) No two points in $D_\alpha$ are equivalent under $G$.
(2) Every point in $H^n(\mathcal{C})$ has its equivalent point in $\overline{D_\alpha}$.

4. The counting function $n(r, a)$. Unless otherwise stated, we shall always take $G$ to be a discrete subgroup of $U(1, n; \mathcal{C})$ with $G_0 = \{\text{identity}\}$. Let $a$ be a point in $H^n(\mathcal{C})$. Let $n(r, a)$ be the number of the elements $f$ in $G$ such that $\|f(a)\| < r$. First we prove

**Proposition 4.1.** For $0 \leq r < 1$, the following inequality is satisfied.

$$n(r, a) \leq B(1-r)^{-n},$$

where $B$ is a constant independent of $a$.

For the proof of this proposition, we need two lemmas.

**Lemma 4.2.** $n(r, a) = \# \{f \in G \mid f(0) \in C(a, r)\}$.

**Proof.** Let us write $G = \{f_0, f_1, \ldots\}$. Suppose that $\|f_\alpha(a)\| < r$. This means that $\delta(f_\alpha(a), 0) < r$ and so $\delta(a, f_\alpha(0)) < r$. Then $f_\alpha(0)$ lies in $C(a, r)$. It is similarly seen that $f_\alpha(0) \in C(a, r)$ implies $\|f_\alpha(0)\| < r$.

**Lemma 4.3.**

(i) The volume element $dV$ at $z$ in $H^n(\mathcal{C})$ is

$$K \cdot (1 - \|z\|^2)^{-(n+1)}dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

where $K$ is a constant.

(ii) $\int_{\|z\| < r_1} dV \leq K_1 \cdot (1 - r_1)^{-n}$, where $K_1$ is a constant.

**Proof.** (i) Consider $\det |\mathcal{L}(f_\alpha, f_i)|$ to obtain the result. (ii) We see that

$$\int_{\|z\| < r_1} dV \leq \text{constant} \int_0^{r_1} (1 - r^2)^{-(n+1)}rdr \leq K_1(1 - r_1)^{-n}.$$

**Proof of Proposition 4.1.** We first note that $G$ is discontinuous in $H^n(\mathcal{C})$. From this fact, we can choose $s > 0$ so small that $f_\alpha(C(0, s)) \cap f_\beta(C(0, s)) = \emptyset$ for $f_\alpha \neq f_\beta$. Suppose $\delta(f(0), a) < r$. By Proposition 2.1, we have $f(C(0, s)) = C(f(0), s) \subset C(a, r_1)$, where $r_1 = (r+s)(1+rs)^{-1}$. By Lemma 4.2, we see that the number of images $f_\alpha(C(0, s))$ in $C(a, r_1)$ is $n(r, a)$. Therefore it follows that

$$n(r, a) \text{ vol}(C(0, s)) \leq \text{ vol}(C(a, r_1)),$$

where $\text{ vol}(\cdot)$ denotes the volume. By Proposition 2.1.2 in [1], there exists
g \in U(1,n;\mathbb{C}) such that g(a) = 0, so we see that g(C(a, r_1)) = C(g(a), r_1) = C(0, r_1). Hence we obtain the inequality

$$n(r, a) \leq vol(C(0, r_1))(vol(C(0, s))^{-1}.$$

It follows from $r_1 = (r+s)(1+rs)^{-1}$ that $(1-r_1)^{-n} \leq 2(1-s)^{-n}(1-r)^{-n}$. Using this inequality and Lemma 4.3, we have

$$n(r, a) \leq constant \cdot (1-s)^{-n}(1-r)^{-n}(vol(C(0, s))^{-1}.$$

The quantity $(1-s)^{-n}(vol(C(0, s))^{-1}$ depends on $G$. Thus we have the desired inequality.

In the same manner as in Theorem XI. 10 of [3], we obtain

**Proposition 4.4.** Suppose $vol(D_0) < \infty$. Let $a \in D_0$ and $\|a\| < \rho < 1$. There exists $r_0$ such that the following inequality is satisfied for $r_0 \leq r < 1$.

$$A(1-r)^{-n} \leq n(r, a) \leq B(1-r)^{-n},$$

where $A$ is a constant, which depends on $\rho$ and $B$ is a numerical constant.

5. **Convergence type or divergence type.** In this section we shall classify discrete subgroups of $U(1,n;\mathbb{C})$ into convergence type and divergence type and discuss their properties.

**Theorem 5.1.** Let us write $G = |f_0,f_1,\cdots|$. Then either

(i) $\sum_{g \in G} (1-\|f_k(a)\|)^n < \infty$ for each $a \in H^\infty(\mathbb{C})$, or

(ii) $\sum_{g \in G} (1-\|f_k(a)\|)^n = \infty$ for each $a \in H^\infty(\mathbb{C})$.

**Proof.** Let $0^* = (\lambda,0,\cdots,0)$ and $a^* = (a_1,a_2,\cdots,a_{n+1})$ in $V_-$ such that $P(0^*) = 0$ and $P(a^*) = a$, respectively.

Let

$$f_k = \begin{pmatrix} a_{1,1}^{(k)} & \cdots & a_{1,n+1}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{n+1,1}^{(k)} & \cdots & a_{n+1,n+1}^{(k)} \end{pmatrix}.$$ 

We have

$$1 - \|a\|^2 = |\Phi(f_k(a^*),f_k(a^*))| \Phi(f_k(0^*),f_k(0^*))| \Phi(f_k(a^*),f_k(0^*))| -^2$$
\[
\begin{align*}
&(1 - \|f(a)\|)(1 - \|f(0)\|) \left| 1 - \sum_{m=1}^{n} \left( \frac{1}{\sum_{j=1}^{n} a_{m,j}^{[k]} a_{j,m}^{[k]} \left( \sum_{j=1}^{n} a_{m,j}^{[k]} a_{j,m}^{[k]} \right)^{-1}} \right) \right|^{-2} \\
&\leq (1 + \|f(a)\|)(1 - \|f(0)\|)(1 + \|f(a)\|)(1 - \|f(0)\|)
\end{align*}
\]

Noting that \(1 + \|f(a)\| < 2\) and \(1 + \|f(0)\| < 2\), we obtain the next inequality:

\[
\begin{align*}
1 - \|a\|^2 &\leq 4(1 - \|f(a)\|)(1 - \|f(a)\|)(1 - \|f(0)\|)^{-2} \\
&\leq \begin{cases} 
4(1 - \|f(a)\|)(1 - \|f(0)\|)^{-1} \\
4(1 - \|f(0)\|)(1 - \|f(a)\|)^{-1}.
\end{cases}
\end{align*}
\]

Therefore, we see that

\[
\frac{1}{4}(1 - \|a\|^2)^n(1 - \|f(0)\|)^n \leq (1 - \|f(a)\|)^n \leq 4^n(1 - \|f(0)\|)^n(1 - \|a\|^2)^{-n}.
\]

Hence it follows that if \(\sum_{f(a)\in G} (1 - \|f(0)\|)^n < \infty\), then \(\sum_{f(a)\in G} (1 - \|f(a)\|)^n < \infty\) and that if \(\sum_{f(a)\in G} (1 - \|f(0)\|)^n = \infty\), then \(\sum_{f(a)\in G} (1 - \|f(a)\|)^n = \infty\).

Thus our proof is completed.

**Definition.** \(G\) is called of **convergence type**, or **divergence type** according to the case (i) or (ii).

Next we shall show that the power \(n\) is the best number for the classification of discrete subgroups of \(U(1, n; \mathbb{C})\).

**Theorem 5.2.** If \(\epsilon > 0\), then \(\sum_{f(a)\in G} (1 - \|f(a)\|)^{n+\epsilon} < \infty\).

**Proof.** Using Proposition 4.1, we have

\[
\sum_{\|f(a)\| < r} (1 - \|f(a)\|)^{n+\epsilon} = \int_{0}^{r} (1 - t)^{n+\epsilon} dn(t, a)
\]

\[
= (1 - r)^{n+\epsilon} n(r, a) - n(0, a) + (n + \epsilon) \int_{0}^{r} (1 - t)^{-n-\epsilon-1} n(t, a) dt
\]

\[
\leq B(1 - r)^\epsilon + (n + \epsilon) B \int_{0}^{r} (1 - t)^{-\epsilon-1} dt
\]
Therefore, if the condition is satisfied, then $\sum_{f_k \in G} (1 - \|f_k(a)\|)^{n-\varepsilon}$ is convergent as $r \to 1$.

**Theorem 5.3.** The following three are equivalent.

(i) $G$ is of divergence type.

(ii) $\sum_{f_k \in G} (1 - \|f_k(a)\|)^2(1 - \|a\|)^{-n} = \infty$ for all $a \in H^n(C)$.

(iii) $\int_0^1 (1-t)^{n-1}n(t,a)dt = \infty$.

**Proof.** First we shall prove that (i) and (ii) are equivalent. Noting that $\|f_k(a)\| < 1$, we obtain the following inequalities:

\[
(1 - \|f_k(a)\|)^2(1 - \|a\|)^{-n} = (1 - \|f_k(a)\|)^n(1 + \|f_k(a)\|)^n
\]

\[
\leq 2^n(1 - \|f_k(a)\|)^n(1 - \|a\|)^{-n} \quad (1)
\]

\[
\geq (1/2)^n(1 - \|f_k(a)\|)^n(1 - \|a\|)^{-n}. \quad (2)
\]

Considering (1), we see that (i) implies (ii). The inequality (2) shows that (ii) yields (i).

Next we shall show that (i) and (iii) are equivalent. We see that

\[
\sum_{\|f_k(a)\| < r} (1 - \|f_k(a)\|)^n = \int_0^r (1-t)^n dt = \frac{r^n}{n+1}
\]

\[
= (1-r)^n n(r,a) - n(0,a) + \int_0^r n(1-t)^{n-1}n(t,a)dt
\]

It follows from Proposition 4.1 that $(1-r)^n n(r,a) - n(0,a)$ is convergent as $r \to 1$. Therefore we obtain the stated conclusion.

**Theorem 5.4.** If $vol(D_0) < \infty$, then $G$ is of divergence type.

**Proof.** Let $a$ be a point in $D_0$ and $\|a\| < \rho < 1$. Using Proposition 4.4, we see that

\[
\int_0^1 (1-t)^{n-1}n(t,a)dt \geq \int_0^1 A(1-t)^{-1}dt = \infty.
\]

It follows from Theorem 5.3 that $G$ is of divergence type.

Next we consider the case where $G$ is of convergence type.

**Theorem 5.5.** The following (i) and (ii) are equivalent.

(i) $G$ is of convergence type.
(ii) \( \sum_{g \in G} |\log \delta(f(\phi(z), z))^{-1}|^n < \infty \) for some \( z \) in \( H^n(\mathbb{C}) \).

Furthermore if the above are satisfied, the series in (ii) is uniformly convergent in any compact subset of \( H^n(\mathbb{C}) \).

For the proof, we need two lemmas.

**Lemma 5.6.** Let \( z^*(z_0, z_1, \ldots, z_n) \) and \( a^*(a_0, a_1, \ldots, a_n) \) be in \( V_- \). Then

\[
(1 - \| P(a^*) \| \| P(z^*) \|)^2 \leq |a_0|^{-2} |z_0|^{-2} |\Phi(a^*, z^*)|^2 \\
\leq (1 + \| P(a^*) \| \| P(z^*) \|)^2 \\
\leq 4.
\]

**Proof.** Since \( |\Phi(a^*, z^*)|^2 = |a_0|^2 |z_0|^2 - 1 + \sum_{j=1}^n a_j z_j z_0^{-1} a_0^{-1} \), we have

\[
(1 - \sum_{j=1}^n a_j \| z_j \| z_0^{-1} |a_0|^{-1})^2 \leq |a_0|^{-2} |z_0|^{-2} |\Phi(a^*, z^*)|^2 \\
\leq (1 + \sum_{j=1}^n a_j \| z_j \| z_0^{-1} |a_0|^{-1})^2.
\]

Using Schwartz’s inequality, we obtain

\[
(1 - \| P(z^*) \| \| P(a^*) \|)^2 = \left[ 1 - \left( \frac{\sum_{j=1}^n |a_j a_0^{-1}|^2}{\sum_{j=1}^n |z_j z_0^{-1}|^2} \right)^{1/2} \right]^2 \\
\leq |a_0|^{-1} |z_0|^{-1} |\Phi(a^*, z^*)|^2 \\
\leq \left[ 1 + \left( \frac{\sum_{j=1}^n |a_j a_0^{-1}|^2}{1} \right)^{1/2} \left( \frac{\sum_{j=1}^n |z_j z_0^{-1}|^2}{1} \right)^{1/2} \right]^2 \\
= (1 + \| P(z^*) \| \| P(a^*) \|)^2.
\]

Since \( \| P(z^*) \| < 1 \) and \( \| P(a^*) \| < 1 \), we obtain the result.

**Lemma 5.7.** If \( a^* \) and \( z^* \) are in \( V_- \), then

\[
(1/2) \Phi(a^*, a^*) \phi(z^*, z^*) \phi(a^*, z^*)^{-1} \leq \log [\delta(P(a^*), P(z^*))]^{-1} \\
\leq (1/2) \phi(a^*, a^*) \phi(z^*, z^*) \phi(a^*, z^*)^{-1} \phi(z^*, z^*)^{-1}.
\]

**Proof.** Since \( \log (1+x) \leq x(x \geq 0) \) and \( \log (1-x)^{-1} \geq x(0 \leq x \leq 1) \), we have

\[
\log |\delta(P(a^*), P(z^*))|^{-2} = \log [1 + |\delta(P(a^*), P(z^*))|^{-2} - 1] \\
\leq |\delta(P(a^*), P(z^*))|^{-2} - 1
\]
ON SUBGROUPS OF CONVERGENCE OR DIVERGENCE TYPE

\[ = \Phi(a^*, a^*) \Phi(z^*, z^*) + \Phi(a^*, z^*) |^2 - \Phi(a^*, a^*) \Phi(z^*, z^*) |^{-1}, \text{ and} \]
\[ \log | \delta(P(a^*), P(z^*)) |^{-1} = \log [1 - (\delta(P(a^*), P(z^*))]^2]^{-1} \]
\[ \geq 1 - | \delta(P(a^*), P(z^*)) |^2 \]
\[ = \Phi(a^*, a^*) \Phi(z^*, z^*) | \Phi(a^*, z^*) |^{-2}. \]

Proof of Theorem 5.5. By Lemma 5.6, we have
\[(1/8)(1 - \|a\|)(1 - \|z\|^2) \leq (1/2) \Phi(a^*, a^*) \Phi(z^*, z^*) | \Phi(a^*, z^*) |^{-2}.\]

Using Lemma 5.7, we obtain
\[\sum_{k \in G} k^{-n}(1 - \|f_k(a)\|)(1 - \|z\|^2) \leq \sum_{k \in G} [\log (| \delta(f_k(a), z)^{-1} |)^n].\]

If (ii) is satisfied, then the series on the left side is convergent. Thus G is of convergence type. Next we shall show that (i) implies (ii). By Lemmas 5.6 and 5.7, we see that
\[(1/2) \Phi(a^*, a^*) \Phi(z^*, z^*) | \Phi(a^*, z^*) |^{-2} \leq (1/2)(1 - \|a\|^2)(1 - \|z\|^2)(\|a\| - \|z\|)^{-2}.\]

Now we assume that \(\|z\| < r_1 < 1\). Since G is discontinuous in \(H^n(\mathcal{C})\), there exists an integer N such that \(\|f_k(a)\| > r\) for \(n > N\). So we obtain
\[\sum_{k \in G} 2^{-n}(1 - \|z\|^2)^n(1 - \|f_k(a)\|^2)^n(\|f_k(a)\| - \|z\|)^{-2n} \leq \sum_{k \in G} (1 - \|f_k(a)\|^n)^n(r_1 - r)^{-2n}.\]

If G is of convergence type, then the series on the right side is convergent. Thus it is seen that \(\sum_{k \in G} [\log (| \delta(f_k(a), z)^{-1} |)^n]\) is uniformly convergent. So the proof of Theorem 5.5 is complete.

If G is of convergence type, we can denote \(\sum_{k \in G} [\log (| \delta(f_k(a), z)^{-1} |)^n]\) by \(g_a(z)\). Since \(\delta(a, b)\) is \(U(1, n; \mathcal{C})\)-invariant, we have
\[g_a(h(z)) = \sum_{k \in G} [\log (| \delta(f_k(a), h(z))^{-1} |)^n] = \sum_{k \in G} [\log (| \delta(h^{-1}f_k(a), z)^{-1} |)^n] \]
for any \(h\) in G. Set \(h^{-1}f_k = h_k\). We see that
\[g_a(h(z)) = \sum_{h_k \in G} [\log (| \delta(h_k(a), z)^{-1} |)^n] = g_a(z)\]
for any \(h\) in G. So we have proved
Theorem 5.8. If $G$ is of convergence type, then $g_\alpha(z)$ is $G$-invariant.

Theorem 5.9. Let $G$ be a discrete subgroup of $U(1, n; \mathcal{C})$. Then $G$ and the conjugate group $fGf^{-1}$ are of the same type for $f \in U(1, n; \mathcal{C})$.

Proof. Note that $\delta(a, b)$ is $U(1, n; \mathcal{C})$-invariant. Set $b = f(a)$ and $w = f(z)$. Then we obtain
\[
\sum_{\alpha \in G} [\log |\delta(f^*\alpha, f(z))|^{-1}]^n = \sum_{\alpha \in G} [\log |\delta(f^*\alpha, w)|^{-1}]^n.
\]
Thus our proof is complete.

Let $\sigma$ denote the rotation-invariant positive Borel measure on $\partial H^n(\mathcal{C})$ for which $\sigma(\partial H^n(\mathcal{C})) = 1$. We shall show a sufficient condition for $G$ to be of convergence type.

Theorem 5.10. Let $E$ be the subset with positive measure in $\partial H^n(\mathcal{C})$. If $g(E) \cap h(E) = \emptyset$ for any different elements $g$ and $h$ in $G$, then $G$ is of convergence type.

Before proving Theorem 5.10, we give the definition of Poisson kernel and discuss its properties. Let $z$ and $\xi$ be in $H^n(\mathcal{C})$ and $\partial H^n(\mathcal{C})$, respectively. We define Poisson kernel as follows:
\[ P(z, \xi) = \int \phi(z^*, z^*) |\phi(z^*, \xi^*)|^{-2^n} d\sigma, \]
where $z^* = (z_0^*, z_1^*, \ldots, z_n^*) \in P^{-1}(z)$ and $\xi^* = (\xi_0^*, \xi_1^*, \ldots, \xi_n^*) \in P^{-1}(\xi)$.
It is easy to show that the above definition is well-defined. First we show

Proposition 5.11. Let $z$ be a point in $H^n(\mathcal{C})$. Let $\xi$ and $\eta$ be in $\partial H^n(\mathcal{C})$. Let $g$ be an element in $U(1, n; \mathcal{C})$. We have the following properties.

1. $P(g(z), g(\xi)) = |(g(\xi^*))_0|^{2n} |\xi_0^*|^{-2n} P(z, \xi)$.
2. $P(g(z), \xi) = P(z, g^{-1}(\xi))P(g(0), \xi)$.
3. $P(k\eta, \xi) = P(k\xi, \eta)$ for $0 \leq k < 1$.
4. $\int_{\partial H^n(\mathcal{C})} P(k\eta, \xi) d\sigma(\eta) = \int_{\partial H^n(\mathcal{C})} P(k\xi, \eta) d\sigma(\eta) = 1$ for $0 \leq k < 1$.
5. $\int_{\partial H^n(\mathcal{C})} P(g^{-1}(0), \xi) d\sigma(\xi) = \int_{\partial H^n(\mathcal{C})} |\xi_0^*|^{2n} |(g(\xi^*))_0|^{-2n} d\sigma(\xi)
   = \int_{\partial H^n(\mathcal{C})} d\sigma$. 

ON SUBGROUPS OF CONVERGENCE OR DIVERGENCE TYPE

Proof. Noting that
\[ |\Phi(g(z))^*, (g(\zeta))^*)| = |(g(z))^*|| (g(z^*)^*)^{-1}|(g(\zeta))^*|| (g(\zeta^*)^*)^{-1}| \Phi(g(z^*), g(\zeta^*))|, \]
we easily obtain (1), (2) and (3).

(4) The first equality follows from (2). Set \( w = k\eta \). By the Cauchy Formula, we obtain
\[
\int_{\mathbb{H}} P(w, \zeta) d\sigma(\zeta) = \int_{\mathbb{H}} (|\zeta^*|^2 |\Phi(w^*, w^*)| |\Phi(w^*, \zeta^*)|^{-2}) d\sigma(\zeta)
\]
\[
= \int_{\mathbb{H}} (-w_0^* \zeta^* \Phi(\zeta^*, w^*)^{-1}) |\zeta^*| |\Phi(w^*, w^*)| (-w_0^* \Phi(w^*, \zeta^*))^{-1} d\sigma(\zeta)
\]
\[
= 1.
\]

(5) It is easy to show that \( P(g^{-1}(0), \zeta) = 1 |\zeta^*| ||g(\zeta^*)^*||^{-2}. \) Using (2) and (4), we have
\[
\int_{\mathbb{H}} P(g^{-1}(k\eta), \zeta) d\sigma(\zeta)
\]
\[
= \int_{\mathbb{H}} P(k\eta, g(\zeta)) P(g^{-1}(0), \zeta) d\sigma(\zeta)
\]
\[
= P(g^{-1}(0), \zeta) \int_{\mathbb{H}} P(k\eta, g(\zeta)) d\sigma(\zeta)
\]
\[
= P(g^{-1}(0), \zeta).
\]

It follows from (4) that
\[
\int_{\mathbb{H}} P(g^{-1}(0), \zeta) d\sigma(\zeta)
\]
\[
= \int_{\mathbb{H}} \left[ \int_{\mathbb{H}} P(g^{-1}(k\eta), \zeta) d\sigma(\zeta) \right] d\sigma(\eta)
\]
\[
= \int_{\mathbb{H}} d\sigma.
\]

Lemma 5.12. \( P(z, \zeta) \leq |(1 + \|z\|)(1 - \|z\|)^{-1}|^n \leq 2^n (1 - \|z\|)^{-n}. \)

Proof. First we note that \( |\Phi(z^*, \zeta^*)|^2 \geq |z^*|^2 |\zeta^*|(1 - \|z\|^2) \) for \( z^* = (z_0^*, z_1^*, \cdots, z_n^*) \) and \( \zeta^* = (\zeta_0^*, \zeta_1^*, \cdots, \zeta_n^*) \). Using the above fact, we easily see that
$$P(z, \zeta) \leq \left\{ (1 - \| z \| ) (1 - \| z \| )^{-1} \right\}^n$$
$$\leq \left\{ (1 + \| z \| ) (1 - \| z \| )^{-1} \right\}^n$$
$$\leq 2^n (1 - \| z \| )^{-n}.$$ 

Now we are ready to prove our theorem.

**Proof of Theorem 5.10.** Put $u(z) = \int_E P(z, \zeta) d\sigma(\zeta)$. We have $u(0)$

$$= \int_E P(0, \zeta) d\sigma(\zeta) = \sigma(E).$$

Using (1) and (5) in Proposition 5.11 and Lemma 5.12, we have

$$u(0) = \int_E P(0, \zeta) d\sigma(\zeta)$$

$$= \int_E \left\{ \left\| \frac{\zeta^*}{\left\| (g(\zeta^*))_0 \right\|} \right\|^{1/2} P(g(0), g(\zeta)) d\sigma(\zeta).$$

$$\leq 2^n (1 - \| g(0) \| )^{-n} \int_E \left\{ \left\| \frac{\zeta^*}{\left\| (g(\zeta^*))_0 \right\|} \right\|^{1/2} d\sigma(\zeta).$$

$$= 2^n (1 - \| g(0) \| )^{-n} \int_{g(E)} \left\{ \left\| \frac{\zeta^*}{\left\| (g(\zeta^*))_0 \right\|} \right\|^{1/2} d\sigma(\zeta).$$

$$= 2^n \sigma(g(E))(1 - \| g(0) \| )^{-n}.$$ 

It follows from the above fact that

$$\sigma(E) \leq 2^n \sigma(g(E))(1 - \| g(0) \| )^{-n}.$$ 

Since $g(E) \cap h(E) = \emptyset$, then we have

$$\sum_{g \in G} (1 - \| g(0) \| )^n \leq 2^n (\sigma(E))^{-1} \sum_{g \in G} \sigma(g(E))$$

$$= 2^n (\sigma(E))^{-1} \sigma\left( \bigcup_{g \in G} g(E) \right)$$

$$\leq 2^n (\sigma(E))^{-1} \sigma(\partial H^n(\mathcal{C})) < \infty.$$ 

Thus our theorem is completely proved.

**References**


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ON SUBGROUPS OF CONVERGENCE OR DIVERGENCE TYPE


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