On connectedness of strongly abelian extensions of rings

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ON CONNECTEDNESS OF STRONGLY ABELIAN EXTENSIONS OF RINGS

Dedicated to Prof. Noboru Itô on his 60th birthday

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A ring with an identity 1 is said to be a connected ring if 1 and 0 are only idempotents of its center. Recently in [1], we have studied on the connectedness of \( p \)-Galois extensions of connected rings of characteristic \( p > 0 \). In this paper, we shall continue the study on the connectedness of strongly abelian extensions of connected rings.

As is proved in [3], a strongly abelian extension of a ring is obtained by a homomorphic image of a skew polynomial ring of automorphism type, and so as a preparation, some general remarks about a skew polynomial ring of automorphism type are given in §1.

In §2, we study necessary and sufficient conditions for a strongly cyclic extension over a connected ring to be connected, and results of this section are applied in §3 for a study on the connectedness of some types of strongly abelian extensions.

1. Notations and general remarks. Throughout this paper, we assume that \( A \) is a ring with an identity 1 such that \( n(>1) \) is an invertible element for some integer \( n \) and the center \( C(A) \) contains a primitive \( n \)-th root \( \xi \) of 1 such that \( 1 - \xi^i \mid i=1, \ldots, n-1 \) are invertible elements.

Let \( \rho_i \mid i=1, \ldots, m \) be automorphisms of \( A \) and \( \mathcal{A} = \{ a_{ij} \in U(A) \mid i,j=1, \ldots, m \} \) where \( U(A) \) is the group of all invertible elements of \( A \). If \( \rho_i \mid i=1, \ldots, m \) and \( \mathcal{A} \) satisfy the conditions

1. \( a_{ij}a_{kl} = a_{il} = 1 \)

2. \( \rho_i \rho_j \rho_i^{-1} \rho_j^{-1} = a_{ij} \), the inner automorphism \( (a_{ij})(a_{ij})_r \)

3. \( a_{ij} \rho_{ij}(a_{ik})a_{jk} = \rho_{ik} \rho_{ik} \rho_{ik}(a_{ij}) \),

for all \( i,j,k=1, \ldots, m \), then the set of all polynomials \( \{ \sum X_1^{\nu_1}X_2^{\nu_2} \cdots X_m^{\nu_m}a_{\nu_1,\nu_2,\ldots,\nu_m} \mid a_{\nu_1,\nu_2,\ldots,\nu_m} \in \mathcal{A} \} \) becomes an associative ring by the rules

\[
aX_i = X_i \rho_i(a) \quad \text{for all } a \in A \text{ and}
\]

\[
X_iX_j = X_jX_i = a_{ij} \quad \text{for all } i,j.
\]

This ring is denoted by \( R_m = A[X_1, \ldots, X_m ; \rho_1, \ldots, \rho_m, \mathcal{A}] \) or \( R_m = \)
A[X_1, \ldots, X_m; \rho_1, \ldots, \rho_m; a_{ij}; i, j = 1, \ldots, m] and is called a skew polynomial ring of automorphism type. Moreover, by \( R_k (0 \leq k \leq m) \), we denote the skew polynomial ring \( A[X_1, \ldots, X_k; \rho_1, \ldots, \rho_k; a_{ij}; i, j = 1, \ldots, k] \) which is a subring of \( R_m \), where \( R_0 = A \). In particular, if \( m = 1 \), we denote it by
\[
R = A[X; \rho] = \{ \sum X^i a_i; a_i \in A \}
\]
and its multiplication is given by
\[
aX = X \rho(a) \quad \text{for} \quad a \in A.
\]

Remark 1.1. For a permutation \( \pi \) of \( k \) letters, \( 1, 2, \ldots, k \), we have an \( A \)-ring isomorphism \( R_k \cong A[X_{\pi(1)}, \ldots, X_{\pi(k)}; \rho_{\pi(1)}, \ldots, \rho_{\pi(k)}; a_{\pi(i) \pi(j)}; i, j = 1, \ldots, k] \) which maps \( X_i \) to \( X_{\pi(i)} \) (\( i = 1, \ldots, k \)).

Remark 1.2. \( \rho_{k+1} \) can be extended to an automorphism \( \rho_{k+1}^* \) of \( R_k \) by \( \rho_{k+1}^*(X_j) = X_j a_{jk+1} \) for \( j = 1, \ldots, k \) and \( \rho_{k+1}^* A = \rho_{k+1} A \). Moreover, there holds \( R_{k+1} \cong R_k [X_{k+1}; \rho_{k+1}^*] \).

Definition 1.3. Let \( g = X_i^s + \sum_{h=0}^{l-1} f_h(X_1, \ldots, X_{l+1}, \ldots, X_m) \in R_m \). \( g \) is said to be a generator in \( R_m \) if \( s \geq 1 \) and \( gR_m = R_m g \). A generator \( g \) in \( R = A[X; \rho] \) is said to be weakly irreducible (abbreviate \( w \)-irreducible) if \( g \) has no proper factors which are generators.

Let \( G = (\sigma_1) \times (\sigma_2) \times \cdots \times (\sigma_m) \) be an abelian group such that \( |\sigma_i| = n_i \) and \( n = \prod_{i=1}^{m} n_i \).

Definition 1.4. A \( G \)-Galois extension \( B \) of \( A \) is said to be a \( G \)-strongly abelian extension if \( B_\alpha \otimes A_i \) (i.e., \( A_i \) is an \( A \)-direct summand of \( B_\alpha \)) and there exist \( x_1, \ldots, x_m \in U(B) \) such that \( \sigma_i(x_j) = x_j(\xi_i)^{\varepsilon} \), where \( \xi_i = \xi^{n_i/n} \) and \( \varepsilon = \delta_{ii} \), the Kronecker's delta.

Remark 1.5. \( A \) has a \( G \)-strongly abelian extension if and only if there exist automorphisms \( \rho_i; i = 1, \ldots, m \) of \( A \) and a set of elements \( \mathcal{A} = \{ a_{ij}; i, j = 1, \ldots, m \} \) which satisfy conditions (1)-(3), \( \rho_i(\xi_i) = \xi_i \) and there exist elements \( \alpha_1, \ldots, \alpha_m \in U(A) \) such that \( X_k^{n_k} - \alpha_k \) is a generator in \( R_m = A[X_1, \ldots, X_m; \rho_1, \ldots, \rho_k, \mathcal{A}] \) for \( k = 1, \ldots, m \). Moreover, if this is the case, \( B \) is isomorphic to \( R_m/M \), \( M = (X_1^{n_1} - \alpha_1, \ldots, X_m^{n_m} - \alpha_m)R_m \) and \( \sigma_i(x_j) = x_j(\xi_i)^{\varepsilon} \) where \( x_j \) is the coset of \( X_j \) modulo \( M \). Hence, we may write

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2
ON CONNECTEDNESS OF STRONGLY ABELIAN EXTENSIONS

\[ B = A[x_1, \ldots, x_m; \rho_1, \ldots, \rho_m, \mathcal{A}] = \sum \oplus (x_1^{\nu_1}x_2^{\nu_2} \cdots x_m^{\nu_m})A \quad (0 \leqslant \nu_i \leqslant n_i - 1) \]

with \( x_i^{n_i} = a_i \in U(A) \), \( ax_i = x_i\rho_i(a) \) for \( a \in A \), \( \rho_1(\xi) = \xi \) and \( x_i x_j = x_j x_i a_{ij} \).

**Remark 1.6.** If \( \mathcal{A} = \{1\} \), we denote \( R_m \) by \( A[X_1, \ldots, X_m; \rho_1, \ldots, \rho_m] \). Then \( x_i^{n_i} - a_i \) is a generator in \( R_m \) if and only if \( a_i \in \bigcap_{j=1}^{n_j} A^{\rho_j} \) where \( A^{\rho_j} = \{ a \in A : \rho_j(a) = a \} \).

**Remark 1.7.** If \( G = (\sigma) \) is a cyclic group of order \( n \) then we say that a \( G \)-strongly abelian extension \( B \) of \( A \) is an \( n \)-strongly cyclic extension. In this case, \( B \) is obtained by \( A[x; \rho] = \sum_{i=0}^{n-1} x^i A \) with \( x^n = a \in U(A) \), \( ax = x\rho(a) \) for \( a \in A \), \( \rho(\xi) = \xi \) and \( \sigma(x) = x\xi \).

**Remark 1.8.** Let \( f(X) = X^8 + \sum_{i=0}^{s-1} X^i a_i \in R = A[X; \rho] \). If \( a_0 \in U(A) \) then \( f(X) \) is a generator in \( R \) if and only if \( a_i \in A^\rho \) and \( \rho^i(a) a_i = a_i \rho^s(a) \) for any \( a \in A \) and \( i = 0, 1, \ldots, s \). Hence \( f(X) \) is contained in \( C(A^n)[X] \).

Let \( H \) be a group. A normal subgroup \( N \) of \( H \) is said to be a small group (abbreviate an \( s \)-group) if only subgroup \( H' \) of \( H \) such that \( H = NH' \) is \( H \). The followings are proved in [1].

**Remark 1.9.** Let \( A \) be connected and let \( B/A \) be an \( H \)-Galois extension for a finite group \( H \).

1. If \( B \) is disconnected, then there exists a nontrivial idempotent \( e \in C(B) \) such that \( e \tau(e) = 0 \) or \( \tau(e) = e \) for every \( \tau \in H \).

2. If \( B^\circ \) is connected for an \( s \)-subgroup \( N \) of \( H \), then \( B \) is connected.

2. **Connected \( n \)-strongly cyclic extensions.** The purpose of this section is to study about the connectedness of an \( n \)-strongly cyclic extension of a connected ring. For this, we denote by \( B \) an \( n \)-strongly cyclic extension of \( A \). Thus we may assume that \( B = A[x; \rho] = \sum_{i=0}^{n-1} x^i A \) for some automorphism \( \rho \) of \( A \) and an element \( a \in U(A) \) such that \( \rho(\xi) = \xi \), \( x^n = a, \rho^n = \rho^{-1}, ax = x\rho(a) \) for \( a \in A \) and \( \sigma(x) = x\xi \).

**Theorem 2.1.** Let \( A \) be connected. Then \( B \) is connected if and only if \( f(X) = X^n - a \) is \( w \)-irreducible.
Proof. Let \( B \) be connected. If \( f(X) \) is not \( w \)-irreducible then \( f(X) = g(X)h(X) \) for some generators \( g(X) = \sum a_i X_i \) and \( h(X) = \sum b_i X_i \). Since both \( a_i \) and \( b_i \) are invertible elements, \( g(X) \) and \( h(X) \) are contained in \( C(A^o)[X] \) by Remark 1.8. Hence

\[
nX^{n-1} = f'(X) = g'(X)h(X) + g(X)h'(X)
\]

for an ordinal derivation \( \cdot \) of \( C(A^o)[X] \). Since \( nx^{n-1} = g'(X)h(x) + g(X)h'(x) \) is an invertible element in \( C(A^o)[x] \) which is a subring of \( B \), \( (g(x)) \) and \( (h(x)) \) are co-maximal ideals such that \( 0 = (g(x))(h(x)) = (g(x)) \cap (h(x)) \). Thus \( B = B/(g(x)h(x)) \cong B/((g(x)) \oplus B/((h(x))) \) and this is a contradiction.

Conversely, assume \( f(X) \) is \( w \)-irreducible and \( B \) is disconnected. Then there exists a nontrivial idempotent \( e \in C(B) \) such that either \( e \tau(e) = 0 \) or \( \tau(e) = e \) for \( \tau \in (\sigma) \) by Remark 1.9. Let \( H = \{ \tau \in (\sigma) \mid \tau(e) = e \} = (\sigma^n) \). Then \( T_\sigma(e; m) = e + \sigma(e) + \cdots + \sigma^{m-1}(e) \) is an idempotent of \( C(A) \) and so \( T_\sigma(e; m) \) is either 0 or 1. If \( T_\sigma(e; m) = 0 \) then we have a contradiction that \( 0 = eT_\sigma(e; m) = e \). Thus \( T_\sigma(e; m) = 1 \). Let \( T = B^{(m)} \). Then \( T = \sum_{i=0}^{n-1} \oplus (x^m)^i A \) where \( m' = |H| \) and \( m = n/m' \). Further we can easily see that \( T^{\sigma} = A \) and \( \{ x_i = y_i = \sigma_i^j(e) \mid i = 0, 1, \ldots, m-1 \} \) satisfies \( \sum_{i=0}^{n-1} x_i \sigma_i^j(e) = \delta_{i,j} \alpha_i \) for \( j = 1, \ldots, m \). Thus \( T/A \) is a \( \sigma \mid T \)-cyclic extension and, by ([6, Theorem 2.3]),

\[
T = Ae + A\sigma(e) + \cdots + A\sigma^{m-1}(e)
\]

and this sum is a direct sum since \( \sigma(e) \sigma'(e) = 0 \) for each \( i, j = 0, 1, \ldots, m-1 \) with \( i \neq j \).

(1) Let \( y = a_0 e + a_1 \sigma(e) + \cdots + a_{m-1} \sigma^{m-1}(e) \) where \( y = x^m \) and \( a_i \in A \).

Then

(2) \( y = x^{-1}yx = \rho(a_0)e + \rho(a_1)\sigma(e) + \cdots + \rho(a_{m-1})\sigma^{m-1}(e) \)

and

(3) \( ay = y\sigma^m(a) = a_0 e + a_1 \sigma(e) + \cdots + a_{m-1} \sigma^{m-1}(e) = a_0 \rho^m(a) e + a_1 \rho^m(a) \sigma(e) + \cdots + a_{m-1} \rho^m(a) \sigma^{m-1}(e) \)

for any \( a \in A \). Thus we obtain

(4) \( y\sigma^i(e) = a_i \sigma(e) = \rho(a_i) \sigma^i(e) \) for \( i = 0, 1, \ldots, m-1 \)

by (1) and (2), and

(5) \( a_i \sigma^i(e) = a_i \rho^m(a) \sigma^i(e) \) for \( a \in A \) by (3).

Noting that \( T_\sigma(\sigma^i(e); m) = 1 \), we have

(6) \( \rho(a_i) = T_\sigma(\rho(a_i) \sigma^i(e); m) = T_\sigma(a_i \sigma^i(e); m) = a_i \)

by (4) and
(7) \( a\alpha_i = T_\sigma(a_i \rho^i(e) ; m) = T_\sigma(a_i \rho \sigma^i(a) \sigma(e) ; m) = a_i \rho^m(a). \)

Further
\[ a \sigma^i(e) = y^m \sigma^i(e) = (y \sigma^i(e))^m = (a_i \sigma^i(e))^m = a_i^m \sigma^i(e) \] (by (4))
and hence, \( a = T_\sigma(a \sigma^i(e) ; m) = T_\sigma(a_i^m \sigma^i(e) ; m) = a_i^m. \)

Therefore
\[
X^n - a = (X^m)^m - (a_i)^m = (X^m - a_i)((X^m)^{m-1} + (X^m)^{m-2}a_i + \cdots + (a_i)^{m-1})
\]

by (6). Moreover, (6) and (7) show that \( X^m - a_i \) and \( X^m + (X^m)^{m-1}a_i + \cdots + (a_i)^{m-1} \) are generators. This is a contradiction.

We say that \( t \) is the index of \( \rho \) and denote it by \( \text{ind.} \rho \) if \( t \) is the index of the subgroup of inner automorphisms in \( \rho \). Since \( \rho^n = \bar{a}^{-1} \), \( \text{ind.} \rho \leq n \).

**Lemma 2.2.**

(i) If \( \text{ind.} \rho = n \) then \( X^n - a \) is \( w \)-irreducible.

(ii) If \( \text{ind.} \rho = 1 \) then we may assume \( R = A[X] ; \rho] = A[Y] \), a polynomial ring with a commutative indeterminate \( Y \in R \) and \( X^n - a = (Y^n - z) \alpha^n \) for some central polynomial \( Y^n - z \) and \( a \in U(A) \).

**Proof.**

(i) If a generator \( g(X) \) is a factor of \( X^n - a \), then the constant term \( a_0 \) of \( g(X) \) must be an invertible element and \( \rho^k = \bar{a}^{-1}_0 \) for \( k = \deg g(X) \). Thus \( k = n \) since \( \text{ind.} \rho = n \).

(ii) Let \( \rho = \bar{a}^{-1} \) for some \( a \in U(A) \). Then \( \bar{a}^{-1} = \rho^n = \bar{a}^{-n} \) implies \( a = a^n z \) for some \( z \in U(C(A))(= U(C(A))^{\rho}) \). Then \( Y = Xa^{-1} \) is central in \( A[X ; \rho] \) and \( A[X ; \rho] = A[Y] \). Further \( X^n - a = ((Xa^{-1})^n - z) \alpha^n = (Y^n - z) \alpha^n \) for a central polynomial \( Y^n - z \).

**Corollary 2.3.** Let \( A \) be connected and \( n \) a prime. Then \( X^n - a \) is either \( w \)-irreducible or a product of generators of degree 1.

**Proof.** \( \text{ind.} \rho \) is either \( n \) or 1. If \( \text{ind.} \rho = n \) then \( X^n - a \) is \( w \)-irreducible by Lemma 2.2.(i). While, if \( \text{ind.} \rho = 1 \), then \( R = A[Y] \) and \( X^n - a = (Y^n - z) \alpha^n \) for \( Y = Xa^{-1} \) with \( a \in U(A) \) by Lemma 2.2.(ii). Hence, if \( X^n - a \) is not \( w \)-irreducible, then \( Y^n - z \) is reducible in \( C(R) = C(A)[Y] \) and so a product of linear factor \( \prod \overline{\tau} \) \((Y - u \zeta^n) \) by ([5, Lemma 1.4]). Hence \( X^n - a = \prod \overline{\tau} \) \((Y - u \zeta^n) \alpha^n = \prod \overline{\tau} \) \((Ya - ua \zeta^n) \) and each \( X - ua \zeta^n \) is a generator.

Let \( U(A)^n = \{ a \in U(A)^n ; \rho^n = \bar{a} \} \). Then we have the following
Theorem 2.4. Let $A$ be connected and $n$ a prime. Then $A$ has a connected $n$-strongly cyclic extension if and only if one of the following conditions (a) and (b) is satisfied.

(a) $n \leq (U(C(A^r)) : U(C(A))^{n})$, the index of the subgroup $U(C(A))^{n} = \{ c^n : c \in U(C(A)) \}$.

(b) $A$ has an automorphism $\rho$ of index $n$ such that $\rho(\xi) = \xi$ and $\rho(\xi)^n \neq \phi$.

Proof. First, we assume that $A$ has a connected $n$-strongly cyclic extension $B$. Then there exist an automorphism $\rho$ of $A$ and an element $\alpha \in U(A)$ such that $X^n - \alpha$ is $\rho$-irreducible in $R = A[X ; \rho]$ and $B \cong R/(X^n - \alpha)$. If it is possible to choose $\rho$ as inner, then we may assume $R = A[Y]$ and $B \cong R/(Y^n - z)$ for some central polynomial $Y^n - z$ which is irreducible in $C(R) = C(A)[Y]$ by Corollary 2.3. Hence $z \in U(C(A)) \setminus U(C(A))^n$. Since $\xi U(C(A))^n (i = 0, 1, \ldots, n-1)$ are distinct cosets in $U(C(A))/U(C(A))^n$, there holds (a). On the other hand, if $\rho$ is non inner, then $\text{ind.}\rho = n$ and $\alpha \in U(A)^n$. Conversely, if (a) is hold then $C(A)$ has a commutative connected $n$-strongly cyclic extension $Z$ and $B = Z \otimes_{C(A)} A$ is a connected $n$-strongly cyclic extension of $A$. On the other hand, if (b) holds then $X^n - \alpha$ is $\rho$-irreducible for $\alpha \in U(A)^n$ by Lemma 2.2.(i) and $B = A[X ; \rho]/(X^n - \alpha)$ is connected by Theorem 2.1.

Let $n = p_1^{n_1}p_2^{n_2} \cdots p_s^{n_s}$ and $\tau = \sigma^{p_1^{\nu_1}p_2^{\nu_2} \cdots p_s^{\nu_s}}$ where $p_1, \ldots, p_s$ are distinct primes. Then $(\tau)$ is an $s$-subgroup of $(\sigma)$. Hence if $A$ is connected then $B$ is connected if $B^\tau$ is connected by Remark 1.9. Therefore, we may assume that $n = p_1p_2 \cdots p_s$ to study the connectedness of $B$ over a connected ring $A$.

Thus in the following we assume that $n = p_1p_2 \cdots p_s$, a product of distinct primes. Let $\tau_i = \sigma^{p_i}$ and $B_i = B^{\tau_i}$. Then $B_i = \sum_{l=0}^{p_i-1} (y_i)^l A \cong A[Y ; \rho^{q_i}]/(Y^{p_i} - \alpha)$ where $y_i = x^{q_i}$ and $q_i = n/p_i$.

Theorem 2.5. Let $A$ be connected. Then $B$ is connected if each $B_i$ is connected. Conversely, if $B$ is connected and $U(A)^{\sigma^{q_i}} = U(A)^{\sigma}$ for all $i$, then $B_i$ is connected for all $i$.

Proof. Let $B$ be connected and $U(A)^{\sigma^{q_i}} = U(A)^{\sigma}$ for all $i$. An element $c = \sum_{l=0}^{p_i-1} (y_i)^l a_i \in B_i$ is contained in $C(B_i)$ if and only if $\rho^{q_i}(a_i) = a_i$ and $(\rho^{q_i})^l(a_i a_j) = a_i a_j$ for any $a \in A$. But this means that $c$ is also contained in $C(B_i)$. Thus each $B_i$ is connected. Conversely, assume that each $B_i$ is connected and we put $S_i = B^{\sigma_i}$ where $\sigma_i = \sigma^{n_i}$. First, we show that $B$ is
connected if \( B_i \) and \( S_1 \) are connected. If this assertion is true then we can reduce the connectedness of \( S_1 \) from that of \( B_i, i = 2, \ldots, s \) applying the same methods on \( S_1 \). Suppose now \( B \) is disconnected. Then there exists a nontrivial idempotent \( e \in C(B) \) such that \( e \tau_i(e) = 0 \) and \( e \theta_i(e) = 0 \) by Remark 1.9. We now show that \( \theta_i^{j-i}(e) \theta_i^{j-i}(e) = 0 \) for \( i \neq j \). For, we may assume \( i < j \) and \( 1 \leq j - i \leq p_i - 1 \). Hence \( \theta_i^{j-i} \) is a generator of \( (\theta_i) \). Hence, if \( \theta_i^{j-i}(e) \theta_i^{j-i}(e) \neq 0 \), then \( \theta_i^{j-i}(e) \theta_i^{j-i}(e) \neq 0 \) and this implies a contradiction \( e = \theta_i^{j-i}(e) \in S_1 = B^{\mu_1} \). Therefore

\[
f = e + \theta_i(e) + \cdots + \theta_i^{p_i-1}(e) = 1
\]
since \( f \) is a central idempotent of \( S_1 \). Next we shall show that \( \tau_i(e) \theta_i^{j-i}(e) = 0 \) for \( i = 0, 1, \ldots, p_i - 1 \). For, \( \tau_i(e) \theta_i^{j-i}(e) = \sigma^{p_i^j}(e \sigma^{-p_i} \theta_i^{j-i}(e)) = \sigma^{p_i^j}(e \sigma^{q_i^j-p_i^j}(e)) \) and \( (q_i^j-p_i^j, p_i) = 1 \) for \( j = 1, \ldots, s \). Therefore we have a contradiction \( e = \sigma^{q_i^j-p_i^j}(e) \in B^{\sigma} = A \) if \( \tau_i(e) \theta_i^{j-i}(e) \neq 0 \). Thus we have a contradiction \( \tau_i(e) = \tau_i(e) f = 0 \) again.

**Corollary 2.6.** Let \( A \) be connected. Then \( A \) has a connected \( n \)-strongly cyclic extension \( B \) if there exist an automorphism \( \rho \) of \( A \) and an element \( a \in U(A)^{\mu_1} \) which satisfy

1. \( \rho(\zeta) = \zeta \)
2. ind. \( \rho^{q_i^j} = p_i \) for \( i = 1, \ldots, k \)
3. ind. \( \rho^{q_i^j} = 1 \) and \( a(\xi^{q_i^j})U(C(A))^{\mu_1} (j = 0, 1, \ldots, p_i - 1) \) are distinct cosets in \( (U(C(A)) : U(C(A))^{\mu_1}) \) for \( i = k + 1, \ldots, s \).

**Proof.** If there exist an automorphism \( \rho \) and an element \( a \in U(A)^{\mu_1} \) which satisfy conditions (1)–(3), then \( B = A[X : \rho]/(X^n - a) \) is an \( n \)-strongly cyclic extension. Further (2) and (3) show that each \( B_i \) is connected, and so \( B \) is connected.

If \( A \) is a connected commutative ring and \( B \) is a commutative \( n \)-cyclic extension, then \( B \) is an \( n \)-strongly cyclic extension, and so \( B \) is connected if and only if there exists an element \( a \in U(A) \) such that \( a \notin U(A)^{p_i} \) for each \( i \). Moreover, it is known that if \( A \) is a local ring (resp. a domain) then so is \( B \), and conversely (See [4, § 1]). Hence we have the following

**Corollary 2.7.** Let \( A \) be a connected commutative ring. Then \( A \) has a connected commutative \( n \)-cyclic extension \( B \) if and only if there exists \( a \in U(A) \) such that \( a \notin U(A)^{p_i} \) for \( i = 1, \ldots, s \). Further, if this is the case, \( B \) is a local ring (resp. a domain) if and only if so is \( A \).
Let $A$ be a two sided simple ring. Then each ideal of $A[X; \rho]$ is generated by a generator and the ideal is maximal if and only if the generator is $w$-irreducible. Since each generator $f(X)$ is decomposed into $\prod_{i=1}^{k} g_i(X)$, a product of $w$-irreducible polynomials, if $g_i(X) \neq g_j(X)$ for $i \neq j$ then $(f(X))$ is a product (= the intersection) of the maximal ideals $(g_i(X))$. Hence we have the following

**Theorem 2.8.** Let $A$ be a two sided simple ring (resp. a simple artinian ring).

(a) $B = A[X; \rho]/(X^n - a)$ is a finite direct sum of two sided simple rings (resp. simple artinian rings).

(b) $B = A[X; \rho]/(X^n - a)$ is a two sided simple ring (resp. a simple artinian ring) if and only if $B$ is connected.

**Proof.** (a) Let $X^n - a = \prod_{i=1}^{k} g_i(X)$ be a decomposition into $w$-irreducible polynomials in $A[X; \rho]$. Since $X^n - a$ is separable in $R$, $(g_i(X))$ and $(g_j(X))$ are co-maximal ideals if $i \neq j$ by ([7, Theorem 1.10]). Thus $B = \sum_{i=1}^{k} \oplus A[X; \rho]/(g_i(X))$ and each $A[X; \rho]/(g_i(X))$ is a two sided simple ring (resp. a simple artinian ring).

(b) is an immediate consequence of (a).

As is shown in ([1, Lemma 2.1]), if $A$ is of characteristic $p$ for a prime $p$ and $B$ is a connected $p^\infty$-cyclic extension of $A$ then $A$ is also connected. But the following example shows that there exists a connected $n$-strongly cyclic extension $B$ of $A$ even if $A$ is disconnected.

**Example 2.9.** Let $A = Q \oplus Q$ be the direct sum of 2-copies of the rational numbers field $Q$. Then $C(A) = A$ is disconnected. The map $\rho : A \to A$ such that $(q_1, q_2) \to (q_2, q_1)$ is an automorphism of $A$ of order 2, $A^\rho = Q = \{(q, q) : q \in Q\}$ and $X^2 - 2$ is a generator of $R = A[X; \rho]$ where $2 = (2, 2)$. Then $B = A[X; \rho]/(X^2 - 2)$ is a 2-strongly cyclic extension of $A$ with $C(B) = Q = \{(q, q) : q \in Q\}$.

3. **Connected strongly abelian extensions.** Let $G = (\sigma_1) \times (\sigma_2) \times \cdots (\sigma_m)$ be an abelian group of order $n$ such that $|\sigma_i| = n_i$. If $G$ is a $p$-group and $A$ is of characteristic $p$, then the connectedness of a $G$-abelian extension $B$ of $A$ implies that of $B_i = B^{\sigma_i}$ for $i = 1, \ldots, m$ and the connectedness of each $B_i$ implies that of $B$ (see [1]). But the following examples show that the above are not valid when $B$ is a $G$-strongly abelian extension.
Examples 3.1. (i) Let $A = Q[\sqrt{-1}]$ and $R = A[X_1, X_2; \{-1\}] = \left\{ \sum X_1^{i_1} X_2^{i_2} a_{i_1 i_2} : a_{i_1 i_2} \in A \right\}$ such that $aX_1 = X_2 a$ and $X_1 X_2 = -X_2 X_1$. Then $M = (X_1^2 - 1, X_1^2 - 1)R$ is a two sided ideal of $R$ and $B = A[x_1, x_2; \{-1\}] = R/M$ becomes a $G$-strongly abelian extension with respect to $G = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle$ by $\sigma_i(x_j) = (-1)^{e_i} x_j$ where $e = e_{ij}$. Then $C(B) = Q[\sqrt{-1}]$ is connected, $B^{\rho_1} = A \oplus x_1 A = C(B^{\rho_1})$ and $e = 1/2(1 + x_2)$ is a nontrivial idempotent of $C(B^{\rho_1})$.

(ii) Let $K = Q[\sqrt{-3}]$, $A = K[\sqrt{3}]$, $\rho(k_0 + k_1 \sqrt{3}) = k_0 - k_1 \sqrt{3}$ and $R = A[x_1, x_2; \rho_1 = \rho_2 = \rho]$. Then $M = (X_1^2 - 2, X_2^2 - 2)R$ is a two sided ideal of $R$ and $B = A[x_1, x_2; \rho_1 = \rho_2 = \rho] = R/M$ becomes a $G$-strongly abelian extension with respect to $G = \langle \sigma_i \rangle \times \langle \sigma_2 \rangle$ by $\sigma_i(x_j) = (-1)^{e_i} x_j$. Then $C(B) = K \oplus x_1 x_2 K$ and $e = 1/2(1 + 1/2(x_1 x_2))$ is a nontrivial idempotent of $C(B)$. On the other hand, $C(B^{\rho_1}) = C(B^{\rho_2}) = K$ is connected.

Hereafter we put $B$ is a $G$-strongly abelian extension of a connected ring $A$ such that $B = A[x_1, \ldots, x_m; \rho_1, \ldots, \rho_m]$ and $x_i^{n_i} = a_i$ (i.e., $\alpha x_i = x_i \rho_0(a)$ for $a \in A$ and $x_i x_j = x_j x_i$).

Let $r_i$ be the product of distinct prime factors of $n_i$. Then $B$ is connected if $T = B^{r_i}$ is connected for $H = \langle \sigma_1^{e_1} \rangle \times \cdots \times \langle \sigma_m^{e_m} \rangle$. Therefore we may assume that $n_i = \prod_{i=1}^m p_i^{e_{ij}}$, $e_{ij} = 1$ or 0 for all $i = 1, \ldots, m$ to study the connectedness of $B$. We now put

$$A_k = A[x_1, \ldots, x_k; \rho_1, \ldots, \rho_k]$$
$$Z_k = \cap_{j=1}^k U(A)^{\rho_j}.$$

Then we have the following

Lemma 3.2. Let $A_{k-1}$ be connected and $U(A)^{\rho_k} = U(A)^{\rho_k}$ for some $q = n_k/p$ where $p$ is a prime factor of $n_k$. Then the following conditions are equivalent.

1. $Y^p - a_k$ is $w$-irreducible in $A_{k-1}[Y; \rho_k^{*q}]$.
2. $a_k \in \Lambda_{p}$, $\{ a_{i_1}^{n_{i_1}} \cdots a_{i_k}^{n_{i_k}} : \mu_j$ is an integer such that $n_j \mu_j = p \nu_j$, for some integer $\nu_j$, $a \in Z_k$ and $\alpha \rho_k = \rho_{k-1}^{\nu_{i_1}} \cdots \rho_{k-1}^{\nu_{i_k}} \}$.

Proof. (1) $\rightarrow$ (2). If $a_k \in \Lambda_{p}$, then $a_k = a_{i_1}^{n_{i_1}} \cdots a_{i_k}^{n_{i_k}}$ where each $\mu_j$ is an integer such that $n_j \mu_j = p \nu_j$ for some integer $\nu_j$. Hence we have $a_k = (x_1^{n_1} \cdots x_k^{n_k})^{\rho_0}$ and $Y^p - a_k = (Y - \beta)(Y^{p-1} + Y^{p-2} \beta + \cdots + \beta^{p-1})$ where $\beta = x_1^{n_1} \cdots x_k^{n_k}$. Since $Y - \beta$ and $Y^{p-1} + Y^{p-2} \beta + \cdots + \beta^{p-1}$ are generators of $A_{k-1}[Y; \rho_k^{*q}]$ by the conditions $a \in Z_k$ and $\alpha \rho_k = \rho_{k-1}^{\nu_{i_1}} \cdots \rho_{k-1}^{\nu_{i_k}}$, $Y^p - a_k$ is not $w$-irreducible.
(2) → (1). Assume $Y^p - \alpha_k$ is not $w$-irreducible. Then there exists $f \in U(A_{k-1})$ such that $Y - f$ is a generator and a factor of $Y^p - \alpha_k$ by Corollary 2.3. Hence $f$ satisfies

(i) $\rho_j^*(f) = f$ for $1 \leq j \leq k-1$
(ii) $\rho_k^*(f) = f$
(iii) $gf = f \rho_k^*(g)$ for each $g \in A_{k-1}$.

Noting that $\rho_i^* \sigma_j = \sigma_j \rho_i^*$ for each $i, j$, we can see that $(f^{-1} \sigma_j(f)) = \sigma_j(f)^{-1}$ and $f^{-1} \sigma_j(f) \in C(A_{k-1})$ for $j = 1, \ldots, m$ by (i) − (iii). Further $(f^{-1} \sigma_j(f))^p = (f^{-p} \sigma_j(f^m)) = \alpha_k^{-1} \alpha_k = 1$ show that $\sigma_j(f) = f \eta_i$ for $\eta_i \in \left| \xi_i \right| ; i = 1, \ldots, n \}$ by ([2, Corollary 2.5]). Thus

$$f = x_i^{\nu_i} \cdots x_k^{\nu_k} a$$

for $a \in Z_k$ and $\bar{a} \rho_k^q = \rho_k^{\nu_k \cdots \rho_i \nu_i}$.

Consequently, we have

$$\alpha_k = f^p = (x_i^{\nu_i} \cdots x_k^{\nu_k})^{\nu_k \cdots \nu_i} a^p.$$ For $\left| x_i^{\nu_i} \cdots x_k^{\nu_k} ; 0 \leq \xi_i \leq n_i - 1 \right|$ is linearly independent over $A$, this means that $\nu_i \mu_i$ for some $\mu_i$ and $\alpha_k = \alpha_i^{\mu_i} a^{\nu_i \cdots \nu_k} a^p$.

**Corollary 3.3.** Let $A$ be connected. If there exist automorphisms $\rho_1, \ldots, \rho_m$ of $A$ and elements $a_1, \ldots, a_m \in U(A)$ such that

(i) $\rho_j^a = \alpha_i^{-1}$, $\rho_j(a_i) = a_i$, $\rho_j(\xi) = \xi$ for $i, j = 1, \ldots, m$,

(ii) $U(A)^{\alpha_i^q} = U(A)^{\alpha_i^q} (i = 1, \ldots, m)$ for each prime factor $p$ and $q = n_i/p$.

(iii) For each prime factor $p$ of $n_k$ ($k = 1, \ldots, m$), $\alpha_k \not\in \Lambda_p = \left| \alpha_i^{\mu_i} \cdots \alpha_k^{\nu_k} \cdots \alpha_i^{\nu_i} a \right|$, each $\mu_i$ is an integer such that $n_i \mu_i = p \nu_i$ for some integer $\nu_i$,

then $A$ has a connected $G$-strongly abelian extension $B$.

**Proof.** By (i), $B = A[X_1, \ldots, X_m ; \rho_1, \ldots, \rho_m] / (X_1^{a_1} - a_1, \ldots, X_m^{a_m} - a_m)$ is a $G$-strongly abelian extension of $A$. Then (ii) and (iii) show that each $A_k$ is connected by Lemma 3.2 and Theorem 2.5.

Let $A$ be a commutative ring. Then it is known that a commutative $G$-abelian extension of $A$ is a $G$-strongly abelian extension. Hence, if $B$ is a commutative $G$-abelian extension of $A$, then $B$ is obtained by $A[X_1, \ldots, X_m] / (X_i^{a_i} - a_i, \ldots, X_m^{a_m} - a_m)$ for $a_i \in U(A), i = 1, \ldots, m$. Assume now $A_{k-1}$ is connected and $Y^p - \alpha_k$ is not $w$-irreducible in $A_{k-1}[Y]$ for some prime factor $p$ of $n_k$. Then, as is shown in the proof of Lemma 3.2.(2), there exists
ON CONNECTEDNESS OF STRONGLY ABELIAN EXTENSIONS

69

Kishimoto: On connectedness of strongly abelian extensions of rings

\[ f = x_1^{\nu_1} \cdots x_k^{\nu_k} a, \quad a \in U(A) \] such that \( a_k = f^p = x_1^{\nu_1/p} \cdots x_k^{\nu_k/p} a^p \), and hence

\[ p \nu_j = n_j \mu_j \] for some \( \mu_j \). Let \( \mu_j = ph_j + s_j, (0 \leq s_j < p) \). Then \( x_j^{\nu_j} = x_j^{n_j \mu_j} = a_j^{s_j} \beta_j \), where \( \beta_j = (a_j^{n_j})^p \). Hence we may put \( \Lambda_p = \{ a_1^{\mu_1} \cdots a_k^{\mu_k} U(A)^p \}; \] \( 0 \leq \mu_i < p \) in Lemma 3.2. Combining this with Lemma 3.2, we have the following

**Theorem 3.4.** Let \( A \) be a connected commutative ring.

(a) \( A \) has a connected commutative \( G \)-abelian extension \( B \) if and only if there exist \( a_1, \ldots, a_m \in U(A) \) such that \( a_i U(A)^p \) is a distinct cosets from \( \{ a_1^{\mu_1} \cdots a_k^{\mu_k} U(A)^p \}; \) \( 0 \leq \mu_i \leq n_i - 1 \} \) in \( U(A)/U(A)^p \) for each prime factor \( p \) of \( n_i \) and \( i = 1, 2, \ldots, m \).

(b) If each \( n_i = p_i p_1 \cdots p_s \), then \( A \) has a connected commutative \( G \)-abelian extension \( B \) if and only if there exist elements \( a_1, \ldots, a_m \in U(A) \) such that \( a_1^{\mu_1} \cdots a_m^{\mu_m} U(A)^p \) \( 0 \leq \mu_i < p_i \) are distinct cosets in \( U(A)/U(A)^p \) for \( i = 1, \ldots, s \).

**Proof.** (a) is a direct consequence of Lemma 3.2 and Corollary 3.3.

(b) Let \( B \cong A[X_1, \ldots, X_m]/(X_1^{n_1} - a_1, \ldots, X_m^{n_m} - a_m) \) be a connected commutative \( G \)-abelian extension of \( A \). If \( a_1^{\mu_1} \cdots a_m^{\mu_m} U(A)^{p_i} = a_1^{\nu_1} \cdots a_m^{\nu_m} U(A)^{p_i} \) for some \( j \) with \( \mu_i \neq \nu_j \), then we may assume \( a_i = a_i^{\nu_i} \cdots a_j^{\nu_j} \cdots a_m^{\nu_m} \) for some \( c \in U(A) \) and \( 0 \leq \xi_i < p_i \) since \( a_j^{\mu_j - \nu_j} U(A)^{p_i} \) is a generator of a cyclic group \( (a_j U(A)^{p_i}) \) in \( U(A)/U(A)^{p_i} \). But this means that \( a_i = (x_i^{n_i/p_i})^{\xi_i} \cdots (x_{j-1}^{n_{j-1}/p_i})^{\xi_{j-1}} (x_j^{n_j/p_i})^{\xi_j} \cdots (x_m^{n_m/p_i})^{\xi_m} \) and this contradicts to the irreducibility of \( X_j^{\nu_j} - a_j \) in \( A[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m]/[X_j] \). Conversely, if there exist \( a_1, \ldots, a_m \in U(A) \) which satisfy the condition, then \( B = A[X_1, \ldots, X_m]/(X_1^{n_1} - a_1, \ldots, X_m^{n_m} - a_m) \) is a connected commutative \( G \)-abelian extension of \( A \).

When \( A \) is a two sided simple ring, we can characterize a connected \( G \)-strongly abelian extension \( S = A[x_1, \ldots, x_m : \rho_1, \ldots, \rho_m, \mathcal{A}] \) as follow.

**Theorem 3.5.** Let \( A \) be a two sided simple ring (resp. a simple artinian ring). Then a \( G \)-strongly abelian extension \( S \) of \( A \) is a two sided simple ring (resp. a simple artinian ring) if and only if \( S \) is connected.

**Proof.** Since \( A[X_i : \rho_i]/(X_i^{n_i} - a_i) \) is a finite direct sum of two sided simple rings (resp. simple artinian rings) by Theorem 2.8, we can see that \( S \) is also a finite direct sum of two sided simple rings (resp. simple artinian rings) by inductive argument. Thus \( S \) is a two sided simple ring (resp. a simple artinian ring) if and only if \( S \) is connected.
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