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NOTE ON RIGHT S-IDEMPOTENT IDEALS

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Let $R$ be a ring, and $M(\neq 0)$ a right $R$-module. If $u \in uR$ for every $u \in M$, $M$ is said to be $s$-unital. In particular, if $R_n$ is $s$-unital, $R$ is called a right $s$-unital ring. Given a finite subset $U$ of an $s$-unital module $M_n$, there exists an element $e$ in $R$ such that $ue = u$ for all $u \in U$ (see [7, Theorem 1]). Following Lanski [6], a right ideal $I$ of $R$ is called right $s$-idempotent if $TI = T$ for every right ideal $T$ of $R$ contained in $I$, or equivalently, if $a \in I(a)I$ for each $a \in I$, where $I(a)$ is the principal right ideal generated by $a$. Finally, following [4], $R$ is called almost right Noetherian if for each infinite ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of right ideals of $R$ there exists a positive integer $k$ such that $I_nR^k \subseteq I_k$ for all $n$.

The purpose of this note is to give the following theorem which includes [6, Theorems 2 and 3] and leads also to [6, Theorem 4].

**Theorem 1.** (1) Let $I$ be a non-zero right ideal of a ring $R$. Then the following are equivalent:

1) $I$ is right $s$-idempotent.
2) $I^2 = I$ and $RI$ is a right $s$-unital ring.

(2) Let $R$ be either i) a right Goldie ring, ii) an almost right Noetherian ring, iii) a ring satisfying the minimum condition on right annihilators (equivalently, the maximum condition on left annihilators), or iv) a ring satisfying the maximum condition on principal left ideals. Let $I$ be a non-zero right ideal of $R$. Then the following are equivalent:

1) $I$ is right $s$-idempotent.
2) $I^2 = I$ and $RI$ has a right identity.

If, furthermore, $R$ is semiprime (resp. prime), then 1) is equivalent to 2'). $RI$ (resp. $RI = R$) has an identity.

In preparation for proving Theorem 1, we state the next lemma.

**Lemma 1.** (1) Let $R$ be as in Theorem 1 (2). Then every right $s$-unital subring $A$ of $R$ has a right identity.

(2) Let $I$ be an ideal of a semiprime ring $R$. If $I$ has a right identity $e$, then $e$ is the identity of $I$.

**Proof.** (1) In view of [5, Theorems 3, 4 and Corollary 6], it suffices
to prove the case iv). Choose \( a \in A \) such that the principal left ideal \((a|\) generated by \(a\) is maximal in \(|(x| | x \in A|\), and take \(e \in A\) with \(ae = a\). Then we can easily see that \((e| = (a|\) and \(e^2 = e\). Suppose \(Ae \neq A\), and choose a non-zero \(b \in A(1-e)\). Take \(c \in A\) such that \(ec = e\) and \(bc = b\). Then \(Rc \supseteq Re \oplus (b| \supseteq Re = (a|\). This contradiction proves that \(e\) is a right identity of \(A\).

(2) Since \(|(1-e)I|^2 \subseteq I(1-e)I = 0\), we have \((1-e)I = 0\), which proves that \(e\) is a left identity of \(I\).

The next improves [1, Theorems 1 and 2].

**Corollary 1.** Let \(R\) be a ring in which every element is a product of idempotents. If \(R\) satisfies the minimum condition on right annihilators or the maximum condition on principal left ideals, then \(R\) is a Boolean ring.

**Proof.** By Lemma 1 (1), \(R\) has a right identity \(e\). Now, for any \(x \in R\) we have \(R(ex-x) = R(e-1)x = 0\), whence it follows that \(ex = x\). This proves that \(e\) is the identity 1 of \(R\). Hence, \(R\) is a Boolean ring by [3, Lemma (2)].

**Proof of Theorem 1.** (1) \(1 \Rightarrow 2\). It is easy to see that \(I^2 = I\) and \(I_{RI}\) is \(s\)-unital. Now, let \(x = \sum_{i=1}^{n} x_i a_i \ (x_i \in R, a_i \in I)\) be an arbitrary element of \(RI\), and choose \(e \in RI\) such that \(a_i e = a_i \ (i = 1, \cdots, n)\). Then \(x = xe\).

2) \(\Rightarrow 1\). Obviously, \(I = I^2 \subseteq RI\). Hence, for any \(a \in I\) we get \(a \in aRI \subseteq |a|I\).

(2) In view of (1) and Lemma 1 (1), it is immediate that 1) and 2) are equivalent. We assume henceforth that \(R\) is semiprime. Then, by Lemma 1 (2), we see that 1) implies 2)'. In order to see the converse, let \(e\) be an identity of \(RI\). Then \(RI(1-e) = 0\) shows that \(I(1-e) = 0\), and therefore \(I = Ie \subseteq I^2\). Hence, \(I\) is right \(s\)-idempotent by (1).

**Corollary 2.** (1) Let \(I\) be a non-zero ideal of a ring \(R\). Then the following are equivalent:
1) \(I\) is right \(s\)-idempotent,
2) \(I\) is a right \(s\)-unital ring.

(2) Let \(R\) be as in Theorem 1 (2), and \(I\) a non-zero ideal of \(R\). Then the following are equivalent:
1) \(I\) is right \(s\)-idempotent.
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2) \( I \) has a right identity.

The next includes [6, Theorem 4] (see also [2, Corollaries 4 and 5]).

**Corollary 3.** The following are equivalent:
1) Every right ideal of \( R \) is right s-idempotent.
2) Every ideal of \( R \) is right s-idempotent.
3) \( R \) is fully right idempotent, namely every right ideal of \( R \) is idempotent.
4) \( R \) is as in Theorem 1 (2) then 1) is equivalent to

If, furthermore, \( R \) is as in Theorem 1 (2) then 1) is equivalent to

**Proof.** Obviously, 4) implies 3). In view of Theorem 1 (1), it is easy
to see that 1) and 2) are equivalent and imply 3). Now, suppose 3). Then \( R \)
is semiprime and every non-zero ideal of \( R \) is a right s-unital ring. Hence,
by Corollary 2 (1), \( R \) satisfies 2). If, furthermore, \( R \) is as in Theorem
1 (2) then Corollary 2 (2) and Lemma 1 (2) show that every non-zero
ideal of \( R \) has an identity and is a direct summand of \( R \), and therefore \( R \)
satisfies 4).

**Remark.** A ring \( R \) is said to have the finite intersection property on
right annihilators provided that whenever \( r(I) = 0 \) for a right ideal \( I \) of \( R \)
there exists a finite subset \( F \) of \( I \) such that \( r(F) = 0 \). On the other hand,
\( R \) is called a right strongly semiprime ring provided if \( I \) is an ideal of \( R \)
and is essential as a right ideal then there exists a finite subset \( F \) of \( I \) such that
\( r(F) = 0 \). In [2, Theorem 2], it has been proved that the following are
equivalent:
1) \( R \) is a right strongly semiprime, fully right idempotent ring.
2) \( R \) is a fully right idempotent ring and possesses the finite inter-
section property on right annihilators.
3) \( R \) is a finite direct sum of simple rings with identity.

As a matter of fact, [2, Corollary 4] was obtained as a corollary to the
theorem.

**References**


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