A Theorem on Radicals in Functor Categories with Application to Torsion Theories

Yao Musheng*
A THEOREM ON RADICALS IN FUNCTOR CATEGORIES WITH APPLICATION TO TORSION THEORIES

YAO MUSHENG

In this paper, we obtain a result concerning radicals in functor categories and give an application to torsion theories.

1. Let $C$ be a small preadditive category, and $\text{Ab}$ the category of abelian groups. $(C^e, \text{Ab})$ denotes the category of all additive contravariant functors from $C$ to $\text{Ab}$. Kernels, cokernels and direct sums in $(C^e, \text{Ab})$ can be defined componentwise. Then $(C^e, \text{Ab})$ is a Grothendieck category. For details, we refer to [5].

The Jacobson radical $J(T)$ of an object $T$ in a Grothendieck category is defined to be the subobject which is the intersection of all its maximal subobjects. If $T$ has no maximal subobjects, then define $J(T) = T$. A subobject $S \leq T$ is said to be superfluous if for any subobject $N \leq T$, $S + N = T$ always implies $N = T$. We use the notation $S \ll T$ to denote that $S$ is a superfluous subobject of $T$.

**Lemma 1.** Let $S$, $T$ be objects in a Grothendieck category. Then for any $f \in \text{Mor}(S, T)$, $f(J(S)) \leq J(T)$, where $\text{Mor}(S, T)$ is the set of morphisms from $S$ to $T$.

This is well known. The following two results are also known.

**Lemma 2.** $J$ commutes with finite direct sums.

**Lemma 3.** Let $T$ be an object in a Grothendieck category. Then $\sum T_\alpha \ll J(T)$, where $T_\alpha \ll T$.

Let $F = (C^e, \text{Ab})$ be the functor category, and $T$ an object of $F$. If $B$ is an object of $C$, then $T(B)$ is an abelian group. Let $a$ be an element of $T(B)$. We define a contravariant functor $S$ from $C$ to $\text{Ab}$ as follows: $S(C) = \{|T(f)(a)| : f \in \text{Mor}(C, B)\}$ for any $C \in C$. If $a$ is a morphism from $C$ to $C'$, then define $S(a) : S(C) \to S(C')$ to be the restriction of $T(a)$. It is easy to see that $S$ is an object of $F$ and actually a subobject of $T$. We call $S$ the subobject of $T$ generated by the element $a$. It is easy to verify that every object $T$ is the sum of its subobject which can be generated by one
element.

Lemma 4. Let $F = (C^0, Ab)$, and $T$ a nonzero object of $F$. If $B$ is an object of $C$ such that $T(B) \neq 0$, then for any nonzero element $a \in T(B)$, there is a maximal subobject $W$ of $T$ with respect to $a \in W(B)$.

Proof. Since $a \neq 0$, the set of subobjects $|W_a|$ of $T$ with the property $a \in W_a(B)$ is not empty (at least contains the zero subobject). Consider any chain of the set $|W_a|$: 

$$W_1 \leq W_2 \leq W_3 \leq \cdots \quad (\ast)$$

It is clear that $\bigcup_{n=1}^\infty W_n$ is a subobject of $T$ which does not contain $a$ (i.e. $a \notin (\bigcup_{n=1}^\infty W_n)(B)$). So it is the upper bound of the chain (\ast). By Zorn's lemma, there is a maximal element in $|W_a|$.

Theorem 5. Let $F = (C^0, Ab)$. Then $J(T) = \sum |T_\alpha| T_\alpha \ll T$ for any object $T$ in $F$.

Proof. By lemma 3, we need only to verify that $J(T) \leq \sum T_\alpha$. Let $a \in J(T)(B)$ for some $B \in C$, and $S$ the subobject of $J(T)$ generated by $a$. We want to show that $S$ is superfluous. Let $N$ be a subobject of $T$ such that $S+N = T$. We may assume $S \ll N$ which means $a \in N(B)$. Then, there is a nonempty set $|W_a|$ of subobjects of $T$ such that $N \leq W_a$ and $a \in W_a(B)$.

By lemma 4, there is a maximal element $W$ in the set $|W_a|$. Obviously $S+W = T$. But now $W$ is a maximal subobject of $T$. In fact, let $K$ be a subobject of $T$ such that $W \leq K$, then $S \leq K$ which implies $K = T$. According to the definition of radical, $S \leq W$, which is a contradiction. Therefore $S$ is superfluous. Consequently, we obtain $J(T) \leq \sum T_\alpha$.

Proposition 6. Let $P$ be a projective object in $F$ such that $J(P) \ll P$. Then

$$\text{End } P / J(\text{End } P) = \text{End } P / J(P)$$

where $\text{End } P$ denotes the endomorphism ring of $P$.

Proposition 7. Let $P$ be a projective object in $F$, and $E = \text{End } P$. If
A THEOREM ON RADICALS IN FUNCTOR CATEGORIES

Musheng: A Theorem on Radicals in Functor Categories with Application to

\( f \) is an element of \( E \), then \( f \in J(E) \) if and only if \( \text{Im } f \ll P \), where \( J(E) \) is the Jacobson radical of \( E \).

The next result is a corollary of Theorem 5:

**Proposition 8.** Let \( P \neq 0 \) be a projective object in \( F \). Then \( J(P) \neq P \), or equivalently, \( P \) has at least one maximal subobject.

**Proof.** Suppose \( J(P) = P \). By Theorem 5, \( P = \sum T_a \| T_a \ll P \| \). There is a canonical epimorphism \( f: \oplus T_a \to P \). Since \( P \) is projective, there is \( g: P \to \oplus T_a \) such that \( fg = f \). The canonical projection \( p_a: \oplus T_a \to T_a \) induces a morphism \( g_a = p_a g: P \to T_a \). Let \( \tau_a \) be the injection of \( T_a \to P \), then \( \tau_a g_a \in \text{End } P \). Since \( T_a \ll P \), \( \text{Im } \tau_a g_a \ll P \). This implies that \( \text{Im } \sum \tau_a g_a \ll P \) for any finite subset \( I \subseteq I \). By Proposition 7, \( \sum \tau_a g_a \in J(\text{End } P) \). Hence \( 1 - \sum \tau_a g_a \) is an automorphism of \( P \). Since \( P \neq 0 \), there is an object \( B \in C \) such that \( P(B) \neq 0 \). Let \( \alpha \) be a nonzero element of \( P(B) \). \( P(B) = \sum T_a(B) \). It is easy to see \( 1 - \sum (\tau_a g_a)_a \) and \( g_a \alpha(a) \) \( \neq 0 \) for only finite numbers of \( a \). Hence, there is a finite subset \( I' \) of \( I \) such that \( (1 - \sum (\tau_a g_a)_a)(a) = 0 \). This means \( 1 - \sum \tau_a g_a \) is not an automorphism which contradicts the fact mentioned above.

2. Let \( R \) be a ring with identity, and \( \text{Mod-} R \) the category of all unital right \( R \)-modules. Let \( \sigma \) be a hereditary torsion theory on \( \text{Mod-} R \). The quotient category of \( \text{Mod-} R \) with respect to \( \sigma \) is denoted by \( \text{Mod-} R / \sigma \). For a right \( R \)-module \( M \), the \( \sigma \)-Jacobson radical (or simply \( \sigma \)-radical) of \( M \) is defined by

\[
J_\sigma(M) = \cap |N| M/N \text{ is a } \sigma\text{-cocritical } R\text{-module} |.
\]

If there is no such \( N \), we define \( J_\sigma(M) = M \). \( J_\sigma(M) \) is a \( \sigma \)-pure submodule of \( M \); i.e., \( M/J_\sigma(M) \) is \( \sigma \)-torsionfree. It is easy to see that in the quotient category \( \text{Mod-} R / \sigma \), \( J_\sigma(M) \) coincides with \( J(M) \), where \( J(M) \) is the Jacobson radical of the object \( M \). We say a submodule \( N \) of \( M \) is \( \sigma \)-superfluous if and only if there is no proper \( \sigma \)-pure submodule \( K \) of \( M \) such that \( N + K \) is \( \sigma \)-dense in \( M \), where \( N^\sigma \) denotes the \( \sigma \)-closure of \( N \) in \( M \). We use the notation \( N \ll \sigma M \) to denote that \( N \) is a \( \sigma \)-superfluous submodule of \( M \). In the theory of modules, there is a well known result:

\[
J(M) = \sum |M_\sigma| M_\sigma \ll M |.
\]

One may ask if there is a relative version of this result. In general the
sum of \( \sigma \)-superfluous submodules of \( M \) need not be a \( \sigma \)-pure submodule. However, we may ask that, with what condition, the \( \sigma \)-radical of \( M \) coincides with the closure of the sum of its \( \sigma \)-superfluous submodules, i.e. \( J_{\sigma}(M) = (\sum M_\alpha)^c \), where \( M_\alpha \triangleleft_{\sigma} M \).

**Lemma 9.** Let \( N \) be a submodule of \( M \). Then \( N \) is a \( \sigma \)-superfluous submodule of \( M \) if and only if \( N \) is a superfluous subobject of \( M \) in the quotient category \( \text{Mod-R}/\sigma \).

**Proof.** Let \( N \triangleleft_{\sigma} M \) in \( \text{Mod-R} \). Then \( N_\sigma = (N^c)_\sigma \), where \( N_\sigma \) and \( (N^c)_\sigma \) denote the \( \sigma \)-localization of \( N \) and \( N^c \) respectively. From the definition of \( \sigma \)-superfluous submodule, \( N^c \triangleleft_{\sigma} M \). Since the lattice of pure submodules of \( M \) is isomorphic to the lattice of subobjects of \( M_\sigma \), \( N_\sigma \) is superfluous in \( M_\sigma \).

Conversely, let \( N_\sigma \triangleleft M_\sigma \) in the category \( \text{Mod-R}/\sigma \), and \( K \) a \( \sigma \)-pure submodule of \( M \) such that \( N^c + K \) is dense in \( M \). Then \( (N^c + K)_\sigma = M_\sigma \), which implies \( N_\sigma + K_\sigma = M_\sigma \). So that \( K_\sigma = M_\sigma \) which means that \( K \) is dense in \( M \). But \( K \) is \( \sigma \)-pure, so \( K = M \).

The following lemma is a result due to Freyd (see [2]).

**Lemma 10.** Let \( F \) be a Grothendieck category. If \( F \) has a family of finitely generated projective generators, then \( F \) is equivalent to a functor category \( (C^0, \text{Ab}) \), where \( C \) is a suitable small preadditive category.

Now we prove the following theorem:

**Theorem 11.** Let \( \sigma \) be a hereditary torsion theory on \( \text{Mod-R} \). If the quotient category \( \text{Mod-R}/\sigma \) has a family of finitely generated projective generators, then for any right \( R \)-module \( M \), \( J_{\sigma}(M) = (\sum M_\alpha)^c \), where \( M_\alpha \) runs over all the \( \sigma \)-superfluous submodules of \( M \).

**Proof.** Let \( F \) be an equivalence functor from a Grothendieck category \( C \) to another Grothendieck category \( D \). For any object \( C \) in \( C \), the lattice of subobjects of \( C \) is isomorphic to the lattice of \( F(C) \). Therefore, a subobject \( B \) of \( C \) is maximal iff \( F(B) \) is a maximal subobject of \( F(C) \). This means \( J(F(C)) = F(J(C)) \). It is also clear that a subobject \( B \) of \( C \) is superfluous iff \( F(B) \) is a superfluous subobject of \( F(C) \). By Lemma 10, the category \( \text{Mod-R}/\sigma \) is equivalent to a functor category. Therefore \( J_{\sigma}(M) = (\sum M_\alpha)^c \) by using Theorem 5 and Lemma 9.

**Corollary 12.** Let \( \sigma \) be a right perfect hereditary torsion theory on
A THEOREM ON RADICALS IN FUNCTOR CATEGORIES

$\text{Mod-R. Then, for any right } R\text{-module } M, J_\sigma(M) = \sum |M_\alpha| \leq M_s M|$.

Proof. Since $\sigma$ is right perfect, the quotient category $\text{Mod-R}/\sigma$ has a finitely generated projective generator. By Theorem 11, $J_\sigma(M) = (\sum M_\alpha)^c$. We need only to show that $\sum M_\alpha$ is $\sigma$-pure in $M$. Let $x \in M$ such that $xI \subseteq \sum M_\alpha$, where $I$ is a dense right ideal of $R$. Since $\sigma$ is perfect, we may assume that $I$ is finitely generated. So, there is a finite subset $A$ such that $xI \subseteq \sum A M_\alpha$. However, every finite sum of $\sigma$-superfluous submodules is still superfluous. Hence $xI$ is $\sigma$-superfluous in $M$. The closure of $xI$ is also $\sigma$-superfluous. It means that $x \in \sum M_\alpha$. Therefore $\sum M_\alpha$ is $\sigma$-pure in $M$.

Acknowledgements. The author is grateful to the Japan Society for Promotion of Science. This paper was finished while he was a JSPS fellow and visited Okayama University. The author also would like to express his hearty thanks to Professor H. Tominaga for his kind help and hospitality.

References


Department of Mathematics
Fudan University
Shanghai, China

(Received September 8, 1989)