Nearly Triply Regular Hadamard Designs and Tournaments

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1. Introduction. Let $A$ be a Hadamard $2 \cdot (4 \lambda + 3, 2 \lambda + 1, \lambda)$ design. Namely $A$ is a $(0, 1)$-matrix of degree $4 \lambda + 3$, where $\lambda$ is a positive integer, such that

\[ AA^t = (\lambda + 1)I + \lambda J, \]

where $t$ denotes the transposition, and $I$ and $J$ are the identity and all one matrices of degree $4 \lambda + 3$ respectively. A Hadamard design $A$ is called a Hadamard tournament if $A$ satisfies the following equation

\[ A + A^t + I = J. \]

We choose the mode that each row vector of $A$ is a block, or more precisely the incidence vector of a block. If $A$ is a Hadamard tournament, then a block is an out-neighborhood of a vertex.

A Hadamard design is called nearly triply regular, if $|a \cap \beta \cap \gamma|$ takes only two distinct values $\mu$ and $\nu$ for any three distinct blocks, $a$, $\beta$ and $\gamma$, where $|X|$ denotes the number of elements of a finite set $X$. We assume that $\mu > \nu$. It is known that $|a \cap \beta \cap \gamma|$ takes at least two distinct values. In fact, if $|a \cap \beta \cap \gamma|$ equals a constant $\mu$, then fixing $a$ and $\beta$ and varying $\gamma$, we get the equation $(4\lambda + 1)\mu = (2\lambda - 1)\lambda$, which is absurd.

The concept of nearly triple regularity is first introduced by M. Herzog and K. B. Reid [2] for Hadamard tournaments. They have shown that Hadamard tournaments of quadratic residue type of orders 7 and 11 are only nearly triply regular Hadamard tournaments with $\nu = 0$. However, its dual concept which is named quasi 3 has been introduced by P. Cameron several years earlier [1]. Furthermore, he showed, in particular, that only quasi 3-Hadamard designs are (i) Hadamard designs of projective geometry over GF(2) type and (ii) Hadamard designs of quadratic residue type of order 11. Since the dual (converse) of a Hadamard tournament is a Hadamard tournament, the result of M. Herzog and K. B. Reid may be regarded as a special case of the result of P. Cameron.

In the present paper we give a more direct and more elementary proof to the mentioned results of P. Cameron based on the nearly triple regularity, even though we have to separate the case $\lambda = 8$ (For this see [3]).

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Now we formulate the theorem.

**Theorem.** (i) Hadamard designs of projective geometry over GF(2) type are the only nearly triply regular Hadamard designs with \( \mu = \lambda \). These designs are also called of affine geometry over GF(2) type, or of group type.

(ii) Nearly triply regular Hadamard designs with \( \mu < \lambda \) may exist only when \( \lambda = 2 \) and 8.

(iii) Among the designs of (i) only the one with \( \lambda = 1 \) is equivalent to a Hadamard tournament.

2. Hadamard designs of projective geometry over GF(2) type. Let \( V \) be an \((n+2)\)-dimensional vector space over GF(2) and let PG be the \((n+1)\)-dimensional projective space over GF(2) where a point of PG is a 1-dimensional subspace of \( V \). Let \( P \) and \( B \) be the sets of points and hyperspaces of PG respectively. Then \( D_n = (P, B) \) is a Hadamard design of projective geometry over GF(2) type with \( \lambda = 2^n-1 \). It is obvious that \( D_n \) is nearly triply regular with \( \mu = 1 \) and \( v = (\lambda - 1)/2 \), and that \( D_n \) is self-dual.

**Proposition 1.** Let \( D = (P, B) \) be a nearly triply regular Hadamard 2-(4\(\lambda +3\), 2\(\lambda +1\), \(\lambda\)) design with \( \mu = \lambda \), where \( P \) and \( B \) denote the sets of points and blocks of \( D \) respectively. Then \( D \) is equivalent to \( D_n \) where \( \lambda = 2^n-1 \).

**Proof.** The nearly triple regularity with \( \mu = \lambda \) implies that for any two distinct blocks \( \alpha \) and \( \beta \) there exists a unique block \( \gamma \) such that \( \Delta = \alpha \cap \beta = \beta \cap \gamma = \gamma \cap \alpha \). So we define an addition in \( B \cup \{ P \} \) as follows: \( P + P = P \), \( \alpha + \alpha = P \), \( \alpha + P = \alpha \) and \( \alpha + \beta = \gamma \). We notice that this addition is a natural one, namely \( \alpha + \beta = (\alpha \cap \beta) \cup (\alpha^c \cap \beta^c) \), where \( c \) denotes the complementation. So the associative law holds and \( (B \cup \{ P \}, +) \) forms an elementary Abelian group of order \( 4\lambda + 4 \). We put \( 4\lambda + 4 = 2^{n-1} \). Then \( \lambda = 2^n-1 \). This shows that \( D \) is equivalent to the dual of \( D_n \) and, since \( D_n \) is self-dual, \( D \) is equivalent to \( D_n \).

**Remark.** Proposition 1 is generalized to a general symmetric design by Blessilda Raposa in Ateneo de Manila University, the Philippines, unaware of the results of P. Cameron [4].

**Proposition 2.** Let \( D_n \) be equivalent to a Hadamard tournament. Then \( n = 1 \).
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Proof. Assume that \( n > 1 \). Then for any three distinct blocks \( a, \beta \) and \( \gamma \), \( a \cap \beta \cap \gamma \) is not empty. Let \( a, \beta \) and \( \gamma \) be the out-neighborhoods of vertices \( a, \beta \), and \( c \). Let \( d \) be a vertex of \( a \cap \beta \cap \gamma \), and consider the out-neighborhood \( \delta \) of \( d \). Then, since \( \delta \) is a hyperspace, \( \delta \) contains either \( a \) or \( b \) or \( c \). This is a contradiction.

From now on we assume that \( \mu < \lambda \).

3. The case where \( \mu < \lambda \). Let \( D \) be a nearly triply regular Hadamard 2-(4\( \lambda + 3 \), 2\( \lambda + 1 \), \( \lambda \)) design with \( \mu < \lambda \). Apparently we may assume that

\[
\lambda \geq 3.
\]

Let \( a \) and \( \beta \) be two distinct blocks of \( D \) and put \( \Delta = a \cap \beta \). Let \( c \) and \( d \) be the numbers of blocks \( \gamma \) and \( \delta \) such that \( |D \cap \gamma| = \mu \) and \( |D \cap \delta| = \nu \) respectively. Then we have the following equations:

\[
\begin{align*}
(4) & \quad c + d = 4\lambda + 1, \\
(5) & \quad c\mu + d\nu = \lambda(2\lambda - 1).
\end{align*}
\]

and

\[
(6) \quad c(\mu - 1)\mu + d(\nu - 1)\nu = (\lambda - 2)(\lambda - 1)\lambda.
\]

Eliminating \( c \) and \( d \) from (4), (5) and (6) we obtain that

\[
(7) \quad (4\lambda + 1)\mu\nu = \lambda[(\mu + \nu - 1)(2\lambda - 1) - (\lambda - 1)(\lambda - 2)].
\]

If \( \nu = 0 \), then we obtain that \((\mu - 1)(2\lambda - 1) = (\lambda - 1)(\lambda - 2)\), which implies that \( \lambda = 2 \) against (3). So we may put

\[
(8) \quad \mu\nu = A\lambda,
\]

where \( A \) is a positive integer. Then from (7) and (8) we obtain that

\[
(9) \quad A(4\lambda + 1) = (\mu + \nu - 1)(2\lambda - 1) - (\lambda - 1)(\lambda - 2),
\]

which implies that

\[
(10) \quad 12A + 3 = (2\lambda - 1)(4\mu + 4\nu + 1 - 8A - 2\lambda).
\]

Since \( \lambda > \mu > \nu > A \), \( 12A + 3 \leq 12(\lambda - 3) + 3 = 12\lambda - 33 \). Hence from (10) we obtain that

\[
(11) \quad 4\mu + 4\nu + 1 - 8A - 2\lambda = 1, 3, \text{ or } 5.
\]

If \( 4\mu + 4\nu + 1 - 8A - 2\lambda = 5 \), from (8), (10) and (11) we obtain that
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(12) \[ 6A = 5\lambda - 4, \]
(13) \[ 6\mu\nu = (5\lambda - 4)\lambda \]
and
(14) \[ 6\mu + 6\nu = 13\lambda - 2. \]

So \(6\mu\) and \(6\nu\) are roots of the polynomial \(F(X) = X^2 - (13\lambda - 2)X + 6\lambda(5\lambda - 4)\). Since \(F(6\lambda) = -12\lambda^2 - 12\lambda < 0\), the larger root of \(F(X)\) is greater than \(6\lambda\). Since \(6\nu < 6\mu < 6\lambda\), this is a contradiction.

If \(4\mu + 4\nu + 1 - 8A - 2\lambda = 3\), from (8), (10) and (11) we obtain that
(15) \[ 2A = \lambda - 1, \]
(16) \[ 2\mu\nu = (\lambda - 1) \]
and
(17) \[ \mu + \nu = \lambda. \]

So \(2\mu\) and \(2\nu\) are roots of the polynomial \(F(X) = X^2 - (3\lambda - 1)X + 2(\lambda - 1)\lambda = (X - 2\lambda)(X - \lambda + 1)\), which implies that \(\mu = \lambda\) against our assumption.

Hence we have that \(4\mu + 4\nu + 1 - 8A - 2\lambda = 1\). From (8), (10) and (11) we obtain that
(18) \[ 6A = \lambda - 2, \]
(19) \[ 6\mu\nu = (\lambda - 2)\lambda \]
and
(20) \[ 6\mu + 6\nu = 5\lambda - 4. \]

So \(6\mu\) and \(6\nu\) are roots of the polynomial \(F(X) = X^2 - (5\lambda - 4)X + 6\lambda(\lambda - 2) = (X - 3\lambda)(X - 2\lambda + 4)\). Hence we obtain that
(21) \[ \mu = \lambda/2 \text{ and } \nu = (\lambda - 2)/3. \]

Then going back to (4) and (5) we obtain that
(22) \[ d = 9 - (36/(\lambda + 4)). \]

Hence by (3) we obtain that \(\lambda = 8, 14\) or 32. Now we count the number \(t\) of trios \(\{\alpha, \beta, \gamma\}\) of blocks such that \(|\alpha \cap \beta \cap \gamma| = \nu\). For each choice of pairs \(\{\alpha, \beta\}\) of blocks there exist \(d\) blocks \(\gamma\) such that \(|\alpha \cap \beta \cap \gamma| = \nu\). Hence we obtain that
(23) \[ t = (4\lambda + 3)(2\lambda + 1)d/3. \]
However, for $\lambda = 14$ and $32$ we have that $d = 7$ and $8$ respectively. So the right hand side of (23) is not an integer for $\lambda = 14$ and $32$. For $\lambda = 8$ we have that $d = 6$ and $t = 1190$. This completes the proof of Theorem.

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