On Harada Rings III

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ON HARADA RINGS III

Dedicated to Professor Teruo Kanzaki on his 60th birthday

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In [2], we showed that left Harada (abbreviated $H$-) rings can be represented as suitable generalized matrix rings. In the present paper, using this result, we shall further show that all left $H$-rings can be constructed by suitable extension rings of QF-rings and their factors. As a result, we see that left $H$-rings are (left and) right artinian.

Preliminaries. Throughout this paper, rings $R$ considered are associative rings with identity and all $R$-modules are unitary. The notation $M_R$ (resp. $sM$) is used to stress that $M$ is a right (resp. left) $R$-module. For an $R$-module $M$, $J(M)$ and $S(M)$ denote its Jacobson radical and socle, respectively, and $|J_i(M)|$ and $|S_i(M)|$ denote its descending Loewy chain and ascending Loewy chain, respectively.

For $R$-modules $M$ and $N$, for the sake of convenience, we put $(M, N) = \text{Hom}_R(M, N)$ and in particular, put $(e, f) = (eR, fR) = \text{Hom}_R(eR, fR)$.

Now, in what follows, we assume that $R$ is a basic left $H$-ring and $E$ a complete set of orthogonal primitive idempotents of $R$. Namely, $R$ is a basic left artinian ring and $E$ is arranged as

$$E = \{ e_{11}, \ldots, e_{1n_1}, \ldots, e_{m_1}, \ldots, e_{m_{n_2}} \}$$

for which

1) each $e_{ii}R_R$ is injective,

2) there exists an isomorphism from $e_{ik}R_R$ to $e_{i,k-1}J(R)_R \cong J(e_{i,k-1}R_R)$ for $1 \leq i \leq m$ and $2 \leq k \leq n(i)$.

we represent $R$ as

$$R = \begin{pmatrix}
(e_{11}, e_{11}) & \cdots & (e_{mnm_1}, e_{11}) \\
\vdots & \ddots & \vdots \\
(e_{11}, e_{mm}) & \cdots & (e_{mm}, e_{mm})
\end{pmatrix}
$$

$$= \begin{pmatrix}
e_{11}Re_{11} & \cdots & e_{11}Re_{mm} \\
\vdots & \ddots & \vdots \\
e_{mn}Re_{11} & \cdots & e_{mn}Re_{mn}
\end{pmatrix}.$$
a) Each $S(e_{ij}R_R)$ is simple,

$$S(e_{i1}R_R) = \cdots = S(e_{im}R_R),$$

$$S(e_{ij}R_R) \neq S(e_{ki}R_R) \text{ if } i \neq k.$$ 

b) For each $e_{ii}$, there exists a unique $g_i \in E$ such that $(e_{ii}R_R; Rg_i)$ is an injective pair, i.e., $g_iR_R/J(g_iR_R) = S(e_{ii}R_R)$ and $Rg_i/J(Rg_i) = S_R(Rg_i)$. Then $Rg_i$ is injective, and

$$S_k(Rg_i) = S(e_{i1}R_R) + \cdots + S(e_{ik}R_R)$$

for $1 \leq i \leq m$, $1 \leq k \leq n(i)$. So, $S_k(Rg_i)$ is a two-sided ideal. In particular, $S(Rg_i) = S(e_{ii}R_R)$ is a simple ideal. In the matrix representation,

$$S_k(Rg_i) = \begin{pmatrix}
0 \\
\vdots \\
0 \\
X_i \\
0 \\
\vdots \\
0 \\
X_k \\
0
\end{pmatrix}$$

where $X_j = S(e_{ij}Rg_i, e_{ij}Rg_i) = S(e_{ij}Rg_i, e_{ij}Rg_i)$ for $1 \leq j \leq k$.

Here we define two mappings:

$$\sigma : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\}$$

$$\rho : \{1, \ldots, m\} \rightarrow \{1, \ldots, n(1) \cup \cdots \cup \{1, \ldots, n(m)\}\}$$

by the rule $\sigma(i) = k$ and $\rho(i) = l$ if $g_i = e_{kl}$; namely $(e_{ii}R_R; Rg_{\sigma(i)} \rho(i))$ is an injective pair. We note that $\sigma(1), \ldots, \sigma(m) \subseteq \{1, \ldots, m\}$ and $\rho(i) \leq n(\sigma(i))$.

**Definition.** We say that $R$ is a left $H$-ring of type $(\ast)$ if $\sigma(1), \ldots, \sigma(n)$ is a permutation of $\{1, \ldots, n\}$ and $\rho(i) = n(\sigma(i))$ for $i = 1, \ldots, n$.

We define $R_{ij}$ by putting

$$R_{ij} = \begin{pmatrix}
(e_{j1}, e_{i1}) & \cdots & (e_{jm1}, e_{im1}) \\
\vdots & \ddots & \vdots \\
(e_{j1}, e_{im1}) & \cdots & (e_{jm1}, e_{in1})
\end{pmatrix}$$
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\[
\begin{pmatrix}
e_{ii} R_{jj} & \cdots & e_{ii} R_{jn(j)} \\
& \ddots & \\
e_{in(i)} R_{jj} & \cdots & e_{in(i)} R_{jn(j)}
\end{pmatrix}.
\]

Then

\[
R = \begin{pmatrix}
R_{11} & \cdots & R_{1m} \\
& \ddots & \\
R_{m1} & \cdots & R_{mm}
\end{pmatrix}.
\]

Notation. For the sake of convenience, we put \(e_i = e_{ii}\), \(A_{ij} = e_i R_{jj}\) \((i \neq j)\) and \(Q_i = e_i R_{jj}\) for \(1 \leq i \leq m\) and \(1 \leq j \leq m\).

Now, we define \(P_{ik,jl}\) corresponding to \(e_{ik} R_{jl}\) as follows:

\[
P_{ik,jl} = \begin{cases} 
A_{ij} & (i \neq j) \\
Q_i & (i = j, \ k \leq t) \\
J(Q_i) & (i = j, \ k > t)
\end{cases}
\]

and put

\[
P_{ij} = \begin{pmatrix}
P_{i1,j1} & \cdots & P_{i1,jn(j)} \\
& \ddots & \\
P_{in(i),j1} & \cdots & P_{in(i),jn(j)}
\end{pmatrix}.
\]

Namely, when \(i \neq j\),

\[
P_{ij} = \begin{pmatrix}
A_{ij} & \cdots & A_{ij} \\
& \ddots & \\
A_{ij} & \cdots & A_{ij}
\end{pmatrix}
\]

and when \(i = j\),

\[
P_{ij} = P_{ii} = \begin{pmatrix}
Q_i & \cdots & Q_i \\
& \ddots & \\
J(Q_i) & \cdots & Q_i
\end{pmatrix}.
\]

We put

\[
P(R) = \begin{pmatrix}
P_{11} & \cdots & P_{1m} \\
& \ddots & \\
P_{m1} & \cdots & P_{mm}
\end{pmatrix}.
\]

Then \(P = P(R)\) becomes a ring by usual matrix operations. Let \(p_{ij}\) be the
element of $P$ such that its $(ij, ij)$ position is the unity of $P_{ij, ij}$ and all other positions are zero. Then $|p_{11}, ..., p_{1n}, ..., p_{m1}, ..., p_{mn}|$ is a complete set of orthogonal primitive idempotents of $P$; $P = p_{11}P \oplus \ldots \oplus p_{1n}P \oplus \ldots \oplus p_{m1}P \oplus \ldots \oplus p_{mn}P$.

We put

$$K_{ij} = \begin{pmatrix} K_{i1,j1} & \cdots & K_{i1,jn} \\ \vdots & \ddots & \vdots \\ K_{in,j1} & \cdots & K_{in,jn} \end{pmatrix}$$

where

$$K_{ik,ji} = \begin{cases} 0 & j \neq \sigma(i) \\ 0 & j = \sigma(i), t \leq \rho(i) \\ S(P_{ik,ji}) & j = \sigma(i), t > \rho(i) \end{cases}$$

Namely

$$K_{ij} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} (j \neq \sigma(i))$$

$$K_{ij} = \begin{pmatrix} 0 & \cdots & S & \cdots & \cdots & S \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & S & \cdots & \cdots & S \end{pmatrix} (j = \sigma(i), S = S(P_{i\sigma(i),i\sigma(i)}))$$

The following holds ([2]).

**Theorem 1.** There is a ring epimorphism $\tau$ of $P$ to $R$ such that

$$Ker \tau = \begin{pmatrix} K_{11} & \cdots & K_{1m} \\ \vdots & \ddots & \vdots \\ K_{m1} & \cdots & K_{mm} \end{pmatrix}.$$ So, $R \cong P/\text{Ker} \tau$.

**Remark.** In

$$P_{i\sigma(i)} = \begin{pmatrix} P_{i1,1} & \cdots & P_{i1,m} \\ \vdots & \ddots & \vdots \\ P_{in,1} & \cdots & P_{in,m} \end{pmatrix}$$

we replace $P_{i\sigma(i)k}$ by $P_{i\sigma(i)k}^* = P_{i\sigma(i)k}/S(P_{i\sigma(i)k})$ for $1 \leq j \leq n(i), \rho(i) + 1 \leq k \leq n(\sigma(i))$, and denote it by $P_{i\sigma(i)}^*$. 

$$\ln P(R) = \begin{pmatrix} P_{11} & \cdots & P_{1m} \\ \vdots & \ddots & \vdots \\ P_{m1} & \cdots & P_{mm} \end{pmatrix}$$
we replace $P_{i\sigma;0}$ by $P_{i\sigma;e}$ ($i = 1, ..., m$) and denote it by $R^*$. Then $R^*$ canonically becomes a ring and isomorphic to $R$; so $R^*$ is a representative matrix ring of $R$. We identify $R$ with $R^*$ or $R/\text{Ker } \tau$.

We can easily show the following by using injective pairs.

**Proposition 1.** If $R$ is a left $H$-ring of type (*), then the ring

$$
\begin{pmatrix}
Q_{11} & A_{12} & \cdots & A_{1m} \\
A_{21} & Q_{22} & A_{23} & \cdots & A_{2m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{m1} & \cdots & A_{m,m-1} & Q_m
\end{pmatrix}
$$

is a QF-ring. So, $R$ is a suitable extension of this QF-ring.

**Theorem 2.** If $R$ is not of type (*), then there are basic left $H$-rings $T_1, T_2, ..., T_n$ and ring epimorphisms $\phi_i: T_i \to T_2, \phi_2: T_2 \to T_3, ..., \phi_n: T_n \to R$ such that $T_i$ is of type (*) and each $\text{Ker } \phi_i$ is a simple ideal of $T_i$.

**Proof.** We prove by induction on $n$. When $m = 1$, $R$ is represented as

$$R = \begin{pmatrix}
Q & \cdots & Q & \bar{Q} & \cdots & \bar{Q} \\
J & \cdots & J & J & \cdots & J \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\bar{J} & \cdots & \bar{J} & \bar{J} & \cdots & \bar{J} \\
J & \cdots & J & J & \cdots & J \\
\bar{Q} & \cdots & \bar{Q} & \bar{Q} & \cdots & \bar{Q}
\end{pmatrix}
$$

where $Q = e_{11}Re_{11}$, $J = J(Q)$ and $\bar{Q} = Q/S(Q)$. We put

$$T_1 = \begin{pmatrix}
Q & \cdots & Q & \cdots & Q \\
J & \cdots & J & \cdots & J \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
J & \cdots & J & \cdots & J \\
J & \cdots & J & J & J
\end{pmatrix}.
$$

Then $T_1$ is a basic left $H$-ring of type (*). We can easily see that

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is a two sided ideal of $T_i$. So, the following $T_i$ canonically become rings:

\[
T_2 = \begin{pmatrix}
Q & \cdots & Q & \bar{Q} \\
J & \ddots & & \\
& & \ddots & \\
& & & \ddots
\end{pmatrix}, \quad T_3 = \begin{pmatrix}
Q & \cdots & Q & \bar{Q} \\
J & \ddots & \ddots & \\
& \ddots & \ddots & \\
& & \ddots & \ddots
\end{pmatrix}
\]

\[
T_i = \begin{pmatrix}
Q & \cdots & Q & \bar{Q} \\
J & \ddots & \ddots & \\
& \ddots & \ddots & \\
& & \ddots & \ddots
\end{pmatrix}, \quad T_s = \begin{pmatrix}
Q & \cdots & Q & \bar{Q} \\
J & \ddots & \ddots & \\
& \ddots & \ddots & \\
& & \ddots & \ddots
\end{pmatrix}
\]

\[s = n(1) + 1\]

\[
T_{s+1} = \begin{pmatrix}
Q & \cdots & \bar{Q} & Q \\
J & \ddots & \ddots & \\
& \ddots & \ddots & \\
& & \ddots & \ddots
\end{pmatrix}, \quad T_u = \begin{pmatrix}
Q & \cdots & Q & \bar{Q} & \bar{Q} \\
J & \ddots & \ddots & \ddots & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots
\end{pmatrix}
\]

\[u = n(1)(n(1) - j)\]

Moreover, we can see that these $T_i$ are basic left $H$-rings and there are canonical ring epimorphisms: $T_1 \to T_2 \to T_3 \to \cdots \to T_n \to R$ whose kernels are simple ideals.

Now, we assume that the proposition is true on $m = k - 1$, and let $m = k$. First, we consider the case $|\sigma(1), \ldots, \sigma(n)| \leq |1, \ldots, n|$. Then we can assume that there exists $i \in \{2, \ldots, k\}$ such that $\sigma(1) = \sigma(2) = \cdots = \sigma(j) \neq \sigma(t)$ for $j < t \leq k$, and $\rho(1) < \rho(2) < \rho(\ell)$ for all $3 \leq \ell \leq j$ if $j \geq 3$. We shall consider the following cases:

1. $\sigma(1) \neq 1, 2$
2. $\sigma(1) = 1$. 

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3) \( \sigma(1) = 2. \)
When the case 1), \( R \) can be represented as

\[
R = \left( \begin{array}{c|ccccc|c}
           & \begin{array}{c}
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots
           \end{array}
           & * \\
\hline
* & \begin{array}{c}
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots
           \end{array}
           & * \\
\hline
B \cdots \cdots B \bar{B} \cdots & * \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\end{array} \right)
\]

where \( A = e_{11} Re_{\sigma(11)} \), \( B = e_{21} Re_{\sigma(11)} \), \( \bar{A} = A/S(A) \) and \( \bar{B} = B/S(B) \).

By replacing \( \begin{pmatrix} \bar{A} \cdots \bar{A} \\ \cdots \\ \bar{A} \cdots \bar{A} \end{pmatrix} \) in

\[
R = \left( \begin{array}{c|ccccc|c}
           & \begin{array}{c}
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots
           \end{array}
           & * \\
\hline
* & \begin{array}{c}
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           A \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots \\
           B \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots
           \end{array}
           & * \\
\hline
B \cdots \cdots B \bar{B} \cdots & * \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\end{array} \right)
\]

by \( \begin{pmatrix} A \cdots A \\ \cdots \\ A \cdots A \end{pmatrix} \) and we denote it by \( T: \)

\[
T = \left( \begin{array}{c|ccccc|c}
           & \begin{array}{c}
           \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           A \bar{A} \cdots \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots \\
           \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots
           \end{array}
           & * \\
\hline
* & \begin{array}{c}
           \cdots A \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           A \bar{A} \cdots \bar{A} \cdots \\
           \cdots \cdots \cdots \\
           \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots \\
           \cdots \cdots B \bar{B} \cdots \\
           \cdots \cdots \cdots
           \end{array}
           & * \\
\hline
B \cdots \cdots B \bar{B} \cdots & * \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\hline
* & B \cdots \cdots B \bar{B} \cdots \\
\end{array} \right)
\]

Let \( f_{id} \) be the element of \( T \) such that \((ij, ij)\) position is the unity of \( Q_i \) and all other positions are zero. Then \( f_{i1}, \ldots, f_{in}, f_{12}, \ldots, f_{2n}, \ldots, f_{m}, \ldots, f_{mn} \) is a complete set of orthogonal primitive idempotents. We can see that \( T \) is a basic left artinian ring such that

a) \( J(f_{id} T_{j}) = f_{i,j+1} T_{j} \) for \( i = 1, \ldots, m \), \( j = 1, \ldots, m(i) - 1 \)
b) \( f_{i1} T \) is injective for \( i \neq 2 \).
As \( T \) is basic and \( S(f_{i1} T) \cong \cdots \cong S(f_{i_n} T) \cong S(f_{i2} T) \cong \cdots \cong S(f_{i_n} T) \), we see that \( J(f_{i_n} T) \cong f_{i2} T \). It is easy to see that both \( J(f_{i_n} T) \) and \( f_{i2} T \) canonically become right \( R \)-modules (note that \( \rho(1) < \rho(2) < \rho(\ell) \) for \( 3 \leq \ell \leq j \) if \( j \geq 3 \)). Since \( J(f_{i_n} T) \) is indecomposable and \( f_{i2} T \) is injective, we have that \( J(f_{i_n} T) \cong f_{i2} T \). Thus \( T \) is a basic left \( H \)-ring.

We see that
\[
\begin{pmatrix}
0 & S(A) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
is a two sided ideal of \( T \). So, as in the proof of the case \( n = 1 \), together with the induction hypothesis, we can obtain desired basic left \( H \)-rings and epimorphisms.

In view of the proof above, the same proof works for 2) and 3), and also for the case \( |\sigma(1), \ldots, \sigma(n)| = |1, \ldots, n| \).

As an immediate corollary of the theorem above, we obtain

**Corollary.** Left \( H \)-rings are (left and) right artinian rings.

By Proposition 1 and Theorems 1, 2, we see that left \( H \)-rings can be constructed by suitable extensions of \( QF \)-rings and their factors.

**References**


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