Quasi KO*-EQUIVALENCES; WOOD SPECTRA AND
ANDERSON SPECTRA

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Yoshimura: Quasi KO*-Equivalences; Wood Spectra and Anderson Spectra


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Dedicated to Professor Yukihiro Kodama on his 60th birthday

**ZEN-ICHI YOSIMURA**

Let $KO$, $KU$ and $KC$ denote the real, complex and self-conjugate $K$-spectrum respectively. Given $CW$-spectra $X$, $Y$ we say that $X$ is quasi $KO$*-equivalent to $Y$ if there exists a map $h: Y \to KO \wedge X$ such that the composite $(\mu \wedge 1)(1 \wedge h): KO \wedge Y \to KO \wedge X$ is an equivalence where $\mu: KO \wedge KO \to KO$ denotes the multiplication of $KO$ (see [Y2]). The $KU$-homology $KU_* X$ is regarded as a $Z/2$-graded abelian group with involution, since the conjugation $t_u: KU \to KU$ gives an involution $t_u*$ on $KU_* X$ for any $CW$-spectrum $X$. Notice that $KU_* X$ and $KU_* Y$ are isomorphic as $Z/2$-graded abelian groups with involution if $X$ is quasi $KO$*-equivalent to $Y$.

Let us denote by $P$ and $Q$ the cofibers of the maps $\eta: \Sigma^i \to \Sigma^0$ and $\eta^2: \Sigma^2 \to \Sigma^0$ respectively where $\eta: \Sigma^i \to \Sigma^0$ denotes the stable Hopf map of order 2. As is well known, $KU_i P = Z \oplus Z$ on which $t_{u^*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $KU_i Q = 0$, and $KU_i Q = Z \leq KU_{i-1} Q$ on both of which $t_{u^*} = 1$. Following [MOY] we call a $CW$-spectrum $X$ a Wood spectrum if $X$ is quasi $KO$*-equivalent to $P$, and an Anderson spectrum if $X$ is quasi $KO$*-equivalent to $Q$ (see [Y2]). For any abelian group $G$ we denote by $SG$ the Moore spectrum of type $G$. Evidently $KU_0 SG \cong G$ on which $t_{u^*} = 1$ and $KU_0 SG = 0$.

Let $X$ be a $CW$-spectrum such that

i) $KU_* X$ is pure projective and 2-torsion free, thus it is a direct sum of a free group and cyclic $p$-groups ($p \neq 2$).

ii) $KU_* X$ is pure injective and 2-divisible, thus it is a direct summand of a direct product of a divisible group and cyclic $p$-groups ($p \neq 2$) (see [F]).

Then $KU_* X$ admits a direct sum decomposition $KU_* X \cong A \oplus B \oplus C \oplus C$ so that the conjugation $t_{u^*}$ on $KU_* X$ behaves as

$$t_{u^*} = 1 \text{ on } A, \quad t_{u^*} = -1 \text{ on } B \quad \text{and} \quad t_{u^*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ on } C \oplus C$$

respectively (use [B. Propositions 3.7 and 3.8] or [CR]).

In [Y2, Theorems 1 and 2] (cf. [MOY]) we have obtained certain results concerning Wood spectra and Anderson spectra. On the other hand,
Bousfield [B, Theorems 3.2 and 3.3] has independently shown the following complete results which contain our partial results, although Bousfield's notation (or statement) is different from ours.

**Theorem 1** (Bousfield). Let $X$ be a CW-spectrum such that $KU_* X$ is pure projective and 2-torsion free. Then there exist abelian groups $A_i (0 \leq i \leq 7)$, $C_j (0 \leq j \leq 1)$ and $G_k (0 \leq k \leq 3)$ with $C_j$ and $G_k$ free so that $X$ is quasi $KO_*$-equivalent to the wedge sum $(\bigvee \Sigma_i S A_i) \vee (\bigvee \Sigma^j P \wedge S C_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge S G_k)$. (Theorem 2.4).

**Theorem 2** (Bousfield). Let $X$ be a CW-spectrum such that $KU_* X$ is pure injective and 2-divisible. Then there exist abelian groups $A_i (0 \leq i \leq 7)$, $C_j (0 \leq j \leq 1)$ and $G_k (0 \leq k \leq 3)$ with $C_j$ and $G_k$ divisible 2-torsion so that $X$ is quasi $KO_*$-equivalent to the wedge sum $(\bigvee \Sigma_i S A_i) \vee (\bigvee \Sigma^j P \wedge S C_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge S G_k)$. (Theorem 3.4).

Strictly speaking, Bousfield has proved that an associative $KO$-module spectrum $W$ is isomorphic as $KO$-module spectra to an extended $KO$-module spectrum $KO \wedge Y$ with $Y = (\bigvee \Sigma_i S A_i) \vee (\bigvee \Sigma^j P \wedge S C_j) \vee (\bigvee_k \Sigma^{k+1} Q \wedge S G_k)$, if $\pi_*(W \wedge P)$ is free or divisible. Our purpose in this note is to give a new proof of Theorems 1 and 2 by applying our method developed in [Y2, Y3]. Our method allows us to prove Bousfield's result for any associative $KO$-module spectrum $W$, although we here give a new proof of his result only for an extended $KO$-module spectrum $KO \wedge X$.

In §1 we recall some properties of $K$-spectra $KO$, $KU$ and $KC$ ([B] or [An1]) and then study the structure of $KC_* X$ for any CW-spectrum $X$ as in Theorem 1 or 2. In §2 and §3 we will only deal with CW-spectra $X$ as in Theorems 1 and 2 respectively. After giving a refined decomposition of $KU_* X$ in each case, we will prove Theorem 1 (Theorem 2.4) along the line adopted in [Y2, Y3] and Theorem 2 (Theorem 3.4) by a dual argument. In the proof of Theorem 2 we use the Anderson universal coefficient sequences (see [An2] or [Y1]), as was implicitly suggested in [B].

In this note we will work in the stable homotopy category of CW-spectra [Ad].

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1. The real, complex and self-conjugate $K$-spectrum.

1.1. Let $KO$, $KU$ and $KC$ denote the real, complex and self-conjugate $K$-spectrum respectively. All of these $K$-spectra are associative and commutative ring spectra with unit. As relations among these $K$-spectra we have the following cofiber sequences ([An1], [B]):

\begin{align*}
(1.1) \text{i) } & \Sigma^1 KO \xrightarrow{\eta \wedge 1} KO \xrightarrow{\varepsilon_u} KU \xrightarrow{\varepsilon_o \pi_u^{-1}} \Sigma^2 KO \\
& \text{ii) } \Sigma^2 KO \xrightarrow{\eta^2 \wedge 1} KO \xrightarrow{\varepsilon_c} KC \xrightarrow{\tau \pi_c^{-1}} \Sigma^3 KO \\
& \text{iii) } KC \xrightarrow{\xi} KU \xrightarrow{\pi_u(1-t_u)} \Sigma^2 KU \xrightarrow{\gamma \pi_u} \Sigma^1 KC \\
& \text{iv) } \Sigma^1 KC \xrightarrow{(-\tau, \tau \pi_c^{-1})} KO \vee \Sigma^1 KO \xrightarrow{\varepsilon_u \vee \pi_c \varepsilon_u} KU \xrightarrow{\varepsilon_c \varepsilon_o \pi_u^{-1}} \Sigma^3 KC \\
& \text{v) } \Sigma^2 KU \xrightarrow{(-\varepsilon_o \pi_u, \varepsilon_o \pi_u^{-1})} KO \vee \Sigma^1 KO \xrightarrow{\varepsilon_c \vee \pi_c \varepsilon_c} KC \xrightarrow{\varepsilon_u \tau \pi_c^{-1}} \Sigma^3 KU.
\end{align*}

The maps involved in (1.1) admit several properties as follows. The stable Hopf map $\eta: \Sigma^1 \to \Sigma^0$ has order 2. The maps $\varepsilon_u: KO \to KU$, $\varepsilon_c: KO \to KC$ and $\xi: KC \to KU$ are ring maps with $\xi \varepsilon_c = \varepsilon_u$, and the maps $\varepsilon_o: KU \to KO$, $\tau: \Sigma^1 KC \to KO$ and $\gamma: KU \to \Sigma^1 KC$ are merely $KO$-module maps with $\tau \gamma = \varepsilon_o$. The periodicity maps $\pi_u: \Sigma^3 KU \to KU$ and $\pi_c: \Sigma^1 KC \to KC$ satisfy $\xi \pi_c = \pi_c \xi$ and $\pi_c \gamma = \gamma \pi_c$ respectively. The conjugation maps $t_u: KU \to KU$ and $t_c: KC \to KC$ are ring maps satisfying $t_u^2 = 1$, $t_c^2 = 1$, $t_u \pi_u = -\pi_u t_u$ and $t_c \pi_c = -\pi_c t_c$, and besides

\begin{align*}
(1.2) \quad & t_c \varepsilon_c = \varepsilon_c, \tau t_c = -\tau, t_u \xi = \xi t_c = \xi \text{ and } t_c \gamma = -\gamma t_u = -\gamma.
\end{align*}

Moreover there hold the following equalities among these maps (see [B, 1.9]):

\begin{align*}
(1.3) \text{i) } & \varepsilon_o \varepsilon_u = 2, \tau \varepsilon_c = \eta \wedge 1, \tau \pi_c \varepsilon_c = 0, \pi_c \varepsilon_c \tau \pi_c^{-1} = \varepsilon_c \tau + \eta \wedge 1, \\
& \zeta \gamma = 0 \text{ and } \gamma \pi_u \zeta = \eta \wedge 1. \text{ and also} \\
& \text{ii) } \varepsilon_u \varepsilon_o = 1 + t_u, \gamma \varepsilon_u \tau = 1 - t_c \text{ and } \varepsilon_c \varepsilon_o \zeta = 1 + t_c.
\end{align*}

Let $K$ denote the $K$-spectrum $KO$, $KU$ or $KC$. To any map $f: Y \to K \wedge X$ we assign a $K$-module map $\chi_k(f) = (\mu \wedge 1)(1 \wedge f): K \wedge Y \to K \wedge X$ where $\mu: K \wedge K \to K$ denotes the multiplication of $K$. The assignment $\chi_k: [Y, K \wedge X] \to [K \wedge Y, K \wedge X]$ gives a right inverse of the induced homomorphism $\iota(\wedge 1)^*: [K \wedge Y, K \wedge X] \to [Y, K \wedge X]$ where $\iota: \Sigma^0 \to K$ denotes the unit of $K$. This homomorphism $\chi_k$ induces a homomorphism

\begin{align*}
(1.4) \quad & \chi^k: [Y, K \wedge X] \to \text{Hom}(K_1 Y, K_1 X).
\end{align*}
assigning any map $f$ to its induced homomorphism $\chi_i(f)_*$ in dimension $i$, which is often abbreviated as $\chi_i$.

Let $\mathcal{V} K(\mathbb{G})$ denote the Anderson dual spectrum of $\mathbb{G}$ with coefficients in $\mathbb{G}$ (see [An2] or [Y1, I and II]). The CW-spectra $\mathbb{G}$ and $\mathcal{V} K(\mathbb{G})$ are related by the following universal coefficient sequence

$$0 \to \text{Ext}(\mathbb{G}_{*-1}, X, \mathbb{G}) \to \mathcal{V} K(\mathbb{G}) X \to \text{Hom}(\mathbb{G}_{*}, X, \mathbb{G}) \to 0.$$

Recall that $\mathcal{V} KU(\mathbb{G}) \cong \mathbb{G} \wedge \mathcal{V} \mathbb{G}$, $\mathcal{V} KO(\mathbb{G}) \cong \Sigma^{-1} KO \wedge \mathcal{V} \mathbb{G}$ and $\mathcal{V} KC(\mathbb{G}) \cong \Sigma^{-1} KC \wedge \mathcal{V} \mathbb{G}$ where $\mathcal{V} \mathbb{G}$ denotes the Moore spectrum of type $\mathbb{G}$ ([An2] or [Y1, I]). So we may rewrite the above universal coefficient sequence as follows:

\begin{enumerate}
  \item \[0 \to \text{Ext}(\mathbb{G}_{*-1}, X, \mathbb{G}) \to [X, \mathbb{G} \wedge \mathcal{V} \mathbb{G}] \xrightarrow{x_{\mathbb{G}}^*} \text{Hom}(\mathbb{G}_{*}, X, \mathbb{G}) \to 0\]
  \item \[0 \to \text{Ext}(\mathbb{G}_{*}, X, \mathbb{G}) \to [X, \mathbb{G} \wedge \mathcal{V} \mathbb{G}] \xrightarrow{x_{\mathbb{G}}^*} \text{Hom}(\mathbb{G}_{*}, X, \mathbb{G}) \to 0\]
  \item \[0 \to \text{Ext}(\mathbb{G}_{*}, X, \mathbb{G}) \to [X, \mathbb{G} \wedge \mathcal{V} \mathbb{G}] \xrightarrow{x_{\mathbb{G}}^*} \text{Hom}(\mathbb{G}_{*}, X, \mathbb{G}) \to 0.\]
\end{enumerate}

1.2. In this note we will only deal with a CW-spectrum $X$ such that

\begin{enumerate}
  \item $\mathbb{G}_{*} X$ is pure projective and 2-torsion free, thus it is written as a direct sum of a free group and cyclic $p$-groups ($p \neq 2$), or
  \item $\mathbb{G}_{*} X$ is pure injective and 2-divisible, thus it is written as a direct summand of a direct product of a divisible group and cyclic $p$-groups ($p \neq 2$) (see [F]).
\end{enumerate}

Given such a CW-spectrum $X$, $\mathbb{G}_{*} X$ and $\mathbb{G}_{*} X$ are respectively decomposed into the forms of

\begin{equation} K\mathbb{G}_{*} X \cong A \oplus B \oplus C \oplus C \text{ and } K\mathbb{G}_{*} X \cong D \oplus E \oplus F \oplus F \end{equation}

on which the conjugation $t_{u*}$ behaves as follows:

\begin{equation} t_{u*} = 1 \text{ on } A \text{ or } D, \quad t_{u*} = -1 \text{ on } B \text{ or } E, \quad \text{and} \end{equation}

\begin{equation} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \text{ on } C \oplus C \text{ or } F \oplus F. \end{equation}

Here $C$ and $F$ may be taken to be free in the (1.6) i) case, and to be divisible 2-torsion in the (1.6) ii) case (see [B, Propositions 3.7 and 3.8] or [CR]).

In order to compute $K\mathbb{G}_{*} X$ we use the short exact sequence
induced by the cofiber sequence (1.1) iii) (see [Y2, Lemma 2.1 i]). Since
the composite homomorphism \((\zeta \varepsilon_c \varepsilon_0)_*: KU_i X \to KU_i X\) restricted to the
image \(\zeta_*(KC_i X)\) is just multiplication by 2, the above short exact sequence
is split after tensored with \(\mathbb{Z}\left[\frac{1}{2}\right]\). Under our assumption (1.6) it is a pure
exact sequence, and actually a split exact sequence. Thus \(KC_* X\) admits
the following direct sum decomposition:

\[
\begin{align*}
KC_0 X &\cong (A \oplus B \ast Z/2 \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F), \\
KC_1 X &\cong (D \oplus E \ast Z/2 \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C), \\
KC_2 X &\cong (A \ast Z/2 \oplus B \oplus C) \oplus (D \otimes Z/2 \oplus E \oplus F), \\
KC_3 X &\cong (D \ast Z/2 \oplus E \oplus F) \oplus (A \otimes B \otimes Z/2 \oplus C).
\end{align*}
\]

(1.9)

Since \(\eta \wedge 1 = \gamma \pi_u \xi : \Sigma^1 KC \to KC\), the induced homomorphisms \(\eta_* : KC_0 X \to KC_1 X\) restricted to \(A\) and \(B \ast Z/2\) are respectively identified with
the canonical projection \(A \to A \otimes Z/2\) and the canonical inclusion \(B \ast Z/2
\to B\), and the one \(\eta_*\) restricted to the other components \(C \oplus (D \oplus E \otimes Z/2
\oplus F)\) is trivial. Thus \(\eta_* : KC_0 X \cong A \oplus B \ast Z/2 \oplus C \oplus D \otimes E \otimes Z/2 \oplus F
\to KC_1 X \cong D \oplus E \ast Z/2 \oplus F \oplus A \otimes Z/2 \oplus B \oplus C\) is given by

\[
\eta_*(a, b, c, d, [e], f) = (0, 0, 0, [a], b, 0)
\]

(1.10) \(o\) where \([\ ]\) stands for the mod 2 reduction. For \(\eta_* : KC_i X \to KC_{i+1} X\), \(1 \leq
i \leq 3\), we can obtain similar expressions (1.10) \(i\) to (1.10) \(o\), which will be
used later.

On the other hand, the composite homomorphism \((\gamma \xi)_*: KC_i X \cong D \oplus
E \ast Z/2 \oplus F \oplus A \otimes Z/2 \oplus B \oplus C \to KC_0 X \cong A \oplus B \ast Z/2 \oplus C \oplus D \otimes E
\otimes Z/2 \oplus F\) is given by

\[
(1.11) \_0 \quad (\gamma \xi_*)=(d, e, f, [a], b, c) = (0, 0, 0, d, 0, 2f).
\]

For \((\gamma \xi)_* : KC_{i-1} X \to KC_i X\), \(1 \leq i \leq 3\), we can also obtain similar ex-
pressions (1.11) \(i\) to (1.11) \(o\).

The conjugation \(t_{\varepsilon}^*\) on \(KC_i X \cong \zeta_*(KC_i X) \oplus (\gamma \pi_u)_*(KU_{i-1} X)\) can be
represented by the following matrix

\[
(1.12) \begin{pmatrix}
1 & 0 \\
\varepsilon_i & -1
\end{pmatrix} (0 \leq i \leq 3)
\]

for a certain homomorphism \(t_\varepsilon : \zeta_*(KC_i X) \to (\gamma \pi_u)_*(KU_{i-1} X)\). In particu-
lar, take $X = \Sigma^i Q \land SG$ when $G$ is free or divisible. Here $Q$ denotes the cofiber of the square $\eta^i : \Sigma^i \to \Sigma^0$. Use the following commutative diagram

$$
\begin{array}{cccc}
KU_{i-1}(Q \land SG) & \to & KO_{i-1}(Q \land SG) & \leftarrow 0 \\
\downarrow & & \downarrow & \\
KO_{i-1}(Q \land SG) & \leftarrow KCl_{i-1}(Q \land SG) & \downarrow & \\
0 & & & KU_{i-1}(Q \land SG)
\end{array}
$$

in which the diagonal exact sequences are induced by the cofiber sequences (1.1) ii) and iii). Recall that $KU_{i-1}(Q \land SG) \cong G \cong KU_0(Q \land SG)$ on both of which $t_{\ast} = 1$, and besides $KO_{i-1}(Q \land SG) \cong G, G, G \otimes Z/2$ or $G \ast Z/2$ according as $i \equiv 0, 1, 2$ or $3$ mod $4$. By parallel discussions to [Y2, (2.3)] we then observe that

$$
t_{c\ast} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \text{ on } KCl_{i-1}(Q \land SG) \cong G \oplus G
$$

(1.13) $t_{c\ast} = 1$ on $KC_0(Q \land SG) \cong G \oplus G \otimes Z/2$ and $\quad KC_1(Q \land SG) \cong G \ast Z/2 \oplus G \otimes Z/2$

$t_{c\ast} = -1$ on $KC_2(Q \land SG) \cong G \ast Z/2 \oplus G$.

1.3. The cofiber sequence $\Sigma^2 \xrightarrow{\eta^2} \Sigma^0 \xrightarrow{i_0} Q \xrightarrow{j_0} \Sigma^3$ gives the following commutative diagram

$$
\begin{array}{cccc}
0 & \to & \text{Hom}(KC_{i-1}, SG, KC_0X) & \to \text{Hom}(KC_{i-1}(Q \land SG), KC_0X) \\
\uparrow x_i & & \uparrow x_i & \\
0 & \to & \text{Hom}(KC_{i-1}(Q \land SG), KC_0X) & \to \text{Hom}(KC_{i-1}, SG, KC_0X)
\end{array}
$$

in which the vertical arrows $x_i$ ($i = 0, 1$) are abbreviated $x_i^{\text{kc}}$ of (1.4). Since $Q \land Q \cong Q \lor \Sigma^3 Q$ and $KC \cong KO \land Q$, all of the three rows are split short exact sequences. Notice that their splittings are compatible with $x_i$ ($i = 0, 1$) because the assignment $x_{kc} : [Y, KC \land X] \to [KC \land Y, KC \land X]$ admits the induced homomorphism $(\lor \land 1)^{\ast}$ as a left inverse. Obviously the right lower arrow $x_1$ and the left upper one $x_0$ become both epimorphisms, because they are isomorphisms if the abelian group $G$ is free.

We here assume that $KU_{\ast}X$ is pure projective and $2$-torsion free, and hence $KC_{\ast}X$ is written into the form of (1.9). For each map $g : \Sigma^i Q \land$
$SG \to KC \wedge X$ we can choose unique maps $g_0 : \Sigma^1 SG \to KC \wedge X$ and $g_1 : \Sigma^1 SG \to KC \wedge X$ with $g_0 = g(i_0 \wedge 1)$, under the direct sum decomposition $[\Sigma^1 Q \wedge SG, KC \wedge X] \cong [\Sigma^1 SG, KC \wedge X] \oplus [\Sigma^1 SG, KC \wedge X]$. Express the induced homomorphisms $x_1(g_0) : KC_0 SG \to KC_0 X$ and $x_0(g_1) : KC_{-1} SG \to KC_{-1} X$ as $x_1(g_0) = u + v + w : G \to (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C)$ and $x_0(g_1) = x + y + z : G \to (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F)$ respectively, where $u : G \to D$, $v : G \to F$, $w : G \to A \otimes Z/2 \oplus B \oplus C$, $x : G \to A$, $y : G \to C$ and $z : G \to D \oplus E \otimes Z/2 \oplus F$.

The induced homomorphism $x_0(g_0) : KC_{-1} SG \to KC_0 X$ is identified with the composite $(\gamma_0)_* x_1(g_0) : KC_0 SG \to KC_0 X$ because $(\gamma_0)_* : KC_0 SG \to KC_{-1} SG$ is regarded as the identity on $G$. On the other hand, the induced homomorphism $x_1(g_1) : KC_{-1} SG \to KC_{-1} X$ coincides with the mod 2 reduction of the composite $\eta_* x_0(g_1) : KC_{-1} SG \to KC_{-1} X$ because $\eta_* : KC_{-1} SG \to KC_{-1} SG$ is just the canonical projection $G \to G \otimes Z/2$. By means of (1.10) and (1.11) we then observe that

(1.14) i) $x_0(g_0) : KC_{-1} SG \to KC_0 X$ is expressed as the sum $u + 2v : G \to D \oplus F \subset (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F)$, and

ii) $x_1(g_1) : KC_{-1} SG \to KC_{-1} X$ is expressed as the mod 2 reduction $[x] : G \otimes Z/2 \to A \otimes Z/2 \subset (G \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C)$.

This result implies that the induced homomorphisms $x_0(g) : KC_{-1}(Q \wedge SG) \to KC_0 X$ and $x_1(g) : KC_0(Q \wedge SG) \to KC_{-1} X$ are respectively represented by the following matrices

\[
\begin{pmatrix} x+y & 0 \\
0 & z \\
u + u + 2v \\
w & x \\
\end{pmatrix} : G \oplus G \to (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F)
\]

where $KC_{-1}(Q \wedge SG) \cong KC_{-1} SG \oplus KC_{-1} SG \cong G \oplus G$ and $KC_0(Q \wedge SG) \cong KC_0 SG \oplus KC_{-1} SG \cong G \oplus (G \otimes Z/2)$.

The abelian group $G$ is now assumed to be free. In this situation the assignment $(x_0, x_1) : \Sigma^1 Q \wedge SG, KC \wedge X] \to Hom(KC_{-1}(Q \wedge SG), KC_0 X) \oplus Hom(KC_0(Q \wedge SG), KC_{-1} X)$ is obviously a monomorphism. As in (1.12) we represent the conjugations $t_c \cdot$ on $KC_c X (i = 0, 1)$ by matrices $t_i = \begin{pmatrix} 1 \\
0 \\
t_i \\
-t_1 \\
\end{pmatrix}$ for certain homomorphisms $t_0 : A \oplus C \to D \oplus E \otimes Z/2 \oplus F$ and $t_1 : D \oplus F \to A \otimes Z/2 \oplus B \oplus C$. In particular, (1.13) asserts that $t_0 = 1 : G \to G$ and $t_1 = 0 : G \to G \otimes Z/2$ when $X = \Sigma^1 Q \wedge SG$. Since $x_i((t_c \wedge 1) g) = \ldots$
for any map $g : \Sigma^I Q \wedge SG \to KC \wedge X$, we can easily check that
\[(1.16) \quad (t_c \wedge 1)g = g \text{ if and only if } t_s(x+y) = 2z + u + 2v \text{ and } t_l(u+v) = 2w, \]
where $x_0(g) = \begin{pmatrix} x+y & 0 \\ z & u+2v \end{pmatrix}$ and $x_1(g) = \begin{pmatrix} u+v & 0 \\ w & [x] \end{pmatrix}$ as in (1.15).

1.4. We next consider the following commutative diagram
\[
\begin{align*}
0 & \to \text{Hom}(KC_0X, KC_{-1}SG) \to \text{Hom}(KC_0X, KC_{-1}(Q \wedge SG)) \to \text{Hom}(KC_0X, KC_{-1}SG) \to 0 \\
\uparrow x_0 & \quad \quad \uparrow x_0 & \quad \quad \uparrow x_0 \\
0 & \to [X, \Sigma^I KC \wedge SG] \quad \to [X, \Sigma^I KC \wedge Q \wedge SG] \quad \to [X, \Sigma^I KC \wedge SG] \to 0 \\
\uparrow x_1 & \quad \quad \downarrow x_1 & \quad \quad \downarrow x_1 \\
0 & \to \text{Hom}(KC_0X, KC_0SG) \to \text{Hom}(KC_0X, KC_0(Q \wedge SG)) \to \text{Hom}(KC_0X, KC_{-1}SG) \to 0
\end{align*}
\]
induced by the cofiber sequence $\Sigma^2 \to \Sigma^0 \to \Sigma^1 \to Q \to \Sigma^3$, where the vertical arrows $x_i (i = 0, 3)$ are abbreviated $x_i^{kc}$ of (1.4). All of the three rows are split short exact sequences, and their splittings are compatible with $x_i (i = 0, 3)$. From (1.5) iii) it follows that the right lower arrow $x_3$ and the left upper one $x_0$ are both epimorphisms, and in particular they become isomorphisms if the abelian group $G$ is divisible.

Assume that $KU_\ast X$ is pure injective and 2-divisible, and hence $KC_\ast X$ is written into the form of (1.9). For each map $g : X \to \Sigma^I KC \wedge Q \wedge SG$ we can choose unique maps $g_0 : X \to \Sigma^I KC \wedge SG$ and $g_1 : X \to \Sigma^I KC \wedge SG$ with $g_0 = (j_0 \wedge 1)g$, under the direct sum decomposition $[X, \Sigma^I KC \wedge Q \wedge SG] \cong [X, \Sigma^I KC \wedge SG] \oplus [X, \Sigma^I KC \wedge SG]$. Express the induced homomorphisms $x_0(g_0) : KC_0X \to KC_{-1}SG$ and $x_3(g_3) : KC_3X \to KC_{-1}SG$ as $x_0(g_1) = w + u + v : (A \oplus B \ast Z/2 \oplus C) \oplus (D \oplus F) \to G$ and $x_3(g_3) = z + x + y : (D \ast Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to G$ respectively, where $u : D \to G$, $v : F \to G$, $w : A \oplus B \ast Z/2 \oplus C \to G$, $x : A \to G$, $y : C \to G$ and $z : D \ast Z/2 \oplus E \oplus F \to G$. By a similar argument to (1.14) we obtain
\[(1.17) \quad \text{i) } x_0(g_0) : KC_0X \to KC_{-1}SG \text{ is expressed as the sum } x + 2y : (A \oplus B \ast Z/2 \oplus C) \oplus (D \oplus F) \to A \oplus C \to G, \text{ and}
\]
\[
\text{ii) } x_3(g_3) : KC_3X \to KC_{-1}SG \text{ is expressed as the mod 2 restriction } u : (D \ast Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to D \ast Z/2 \to G \ast Z/2.
\]
This result implies that the induced homomorphisms $x_0(g) : KC_0X \to KC_{-1}(Q \wedge SG)$ and $x_3(g) : KC_3X \to KC_{-1}(Q \wedge SG)$ are respectively represented by the following matrices.
\[
\begin{pmatrix}
x + 2y \\
w \\
u + v
\end{pmatrix} : (A \oplus B \ast Z/2 \oplus C) \oplus (D \oplus F) \to G \oplus G
\]
(1.18)
\[
\begin{pmatrix}
u \\
z \\
x + y
\end{pmatrix} : (D \ast Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to (G \ast Z/2) \oplus G
\]
where \(KC_{-1}(Q \land SG) \cong KC_{-1}SG \oplus KC_{-1}SG \cong G \oplus G\) and \(KC_{1}(Q \land SG) \cong KC_{1}SG \oplus KC_{-1}SG \cong (G \ast Z/2) \oplus G\).

The abelian group \(G\) is now assumed to be divisible. Then the assignment \((x_0, x_3) : [X, \Sigma^1 KC \land Q \land SG] \to \text{Hom}(KC_0X, KC_{-1}(Q \land SG)) \oplus \text{Hom}(KC_1X, KC_{1}(Q \land SG))\) is obviously a monomorphism. The conjugations \(t_i\) on \(KC_iX\) \((i = 0, 3)\) are represented by matrices \(\begin{pmatrix} 1 & 0 \\ t_i & -1 \end{pmatrix}\) for certain homomorphisms \(t_0 : A \oplus B \ast Z/2 \oplus C \to D \oplus F\) and \(t_3 : D \ast Z/2 \oplus E \oplus F \to A \oplus C\). As a result corresponding to (1.16) we can similarly show that

\[
(x_0 \circ g) = x_0 + 2y - 2w \quad \text{and} \quad (x_3 \circ g) = \begin{pmatrix} u \\ z \\ x + y \end{pmatrix}
\]
for any map \(g : X \to \Sigma^1 KC \land Q \land SG\) as in (1.18).

1.5. When an abelian group \(G\) is free, we consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(KU_0(Q \land SG), KU_0X) & \xrightarrow{(i_0)_*} & \text{Hom}(KU_0SG, KU_0X) \\
\downarrow x_0 & & \uparrow x_1 \\
[0 \to \Sigma^1 SG, KU \land X] & \xrightarrow{(\lambda \land 1)_*} & [Q \land SG, KU \land X] \\
\downarrow x_1 & & \uparrow x_1 \\
\text{Hom}(KU_{-1}SG, KU_X) & \xrightarrow{(j_0)_*} & \text{Hom}(KU_1(Q \land SG), KU_1X)
\end{array}
\]

in which the right vertical arrow \(x_0 = x_0^{ku}\) and the left one \(x_1 = x_1^{ku}\) are both isomorphisms, and the top and the bottom horizontal arrows \((i_0)_*\) and \((j_0)_*\) are also isomorphisms. The induced homomorphism \((\lambda \land 1)_* : [\Sigma^3 SG, KU \land X] \to [Q \land SG, KU \land X]\) admits as a left inverse the composite \(((j_0)_* x_1)^{-1} x_1\), which is compatible with the conjugations \((t_u \land 1)_*\) because \(x_1((t_u \land 1)f) = t_u x_1(f) t_u\). In other words, there exists a homomorphism

\[
\lambda : [SG, KU \land X] \to [Q \land SG, KU \land X]
\]

satisfying \((i_0 \land 1)_* \lambda = 1\) and \((t_u \land 1)_* \lambda = \lambda(t_u \land 1)_*\).
Lemma 1.1. Let $G$ be a free abelian group and $g' : \Sigma^1 Q \wedge SG \to KC \wedge X$ be a map satisfying $(t_c \wedge 1)g' = g$. If the composite $(\eta \wedge 1)(\tau \pi_c^{-1} \wedge 1)g'(i_0 \wedge 1) : SG \to \Sigma^1 KO \wedge X$ is trivial, then there exist maps $h_1 : \Sigma^1 SG \to KO \wedge X$ and $g : \Sigma^1 Q \wedge SG \to KC \wedge X$ such that $h_1(j_q \wedge 1) = (\tau \pi_c^{-1} \wedge 1)g$, $(t_c \wedge 1)g = g$ and $(\xi \wedge 1)g = (\xi \wedge 1)g'$ (cf. [Y3, Lemma 1.1]).

Proof. First choose a map $h_0 : \Sigma^1 SG \to KO \wedge X$ satisfying $(\varepsilon_u \wedge 1)h_0 = (\xi \wedge 1)g'(i_0 \wedge 1)$, and then a map $\ell : SG \to KU \wedge X$ such that $(\varepsilon_c \wedge 1)h_0 = g'(i_0 \wedge 1) + (\gamma \pi_u \wedge 1)\ell$. Composing the conjugation map $t_c \wedge 1$ after the second equality, we see that $2(\gamma \pi_u \wedge 1)\ell = 0$ because $t_c \varepsilon_c = \varepsilon_c$ and $t_c \gamma = -\gamma$. So there exists a map $k : SG \to KU \wedge X$ satisfying $2\ell = (1 + t_u \wedge 1)k'$. Applying the right inverse of $(i_0 \wedge 1)^*$ obtained in (1.20) onto the above equality, we show that the composite $(\gamma \pi_u \wedge 1)\lambda(\ell') : \Sigma^1 Q \wedge SG \to KC \wedge X$ has order 2. Setting $g = g' + (\gamma \pi_u \wedge 1)\lambda(\ell')$, its map satisfies the equalities $(\varepsilon_c \wedge 1)h_0 = g(i_0 \wedge 1)$, $(t_c \wedge 1)g = g$ and $(\xi \wedge 1)g = (\xi \wedge 1)g'$. Using the first equality we can then find a map $h_1 : \Sigma^1 SG \to KO \wedge X$ with $h_1(j_q \wedge 1) = (\tau \pi_c^{-1} \wedge 1)g$.

When an abelian group $G$ is divisible, we next consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(KU,X, KU_1(Q \wedge SG)) & \xymatrix{\rightarrow \ar[r]^{j_*} & \text{Hom}(KU,X, KU_1(Q \wedge SG))} \\
0 \ar[r] & [X, KU \wedge SG] & \xymatrix{[X, KU \wedge Q \wedge SG] \ar[r]^{(1 \wedge i_k \wedge 1)^*} \ar[d]_{\chi_1} & [X, \Sigma^1 KU \wedge SG] \ar[r] & 0} \\
& & \xymatrix{\text{Hom}(KU_0X, KU_0SG) \ar[r]^{i_{**}} & \text{Hom}(KU_0X, KU_0(Q \wedge SG))} \\
\end{array}
\]

Here the left vertical arrow $\chi_0 = \chi_{0\text{KU}}$ and the right one $\chi_1 = \chi_{1\text{KU}}$ are both isomorphisms because of (1.5) (i), and the top and the bottom horizontal arrows $j_{**}$ and $i_{**}$ are also isomorphisms. Then a similar discussion to (1.20) shows that there exists a homomorphism

(1.21) $\rho : [X, \Sigma^1 KU \wedge SG] \to [X, KU \wedge Q \wedge SG]$

satisfying $(1 \wedge j_q \wedge 1)^*\rho = 1$ and $(t_u \wedge 1)^*\rho = \rho(t_u \wedge 1)^*$.

As a dual of Lemma 1.1 we have

Lemma 1.2. Let $G$ be a divisible abelian group and $g' : X \to \Sigma^1 KC \wedge Q \wedge SG$ be a map satisfying $(t_c \wedge 1)g' = g'$. If the composite $(\eta \wedge 1)(\tau \pi_c^{-1})$
\( \wedge 1 \)\((1 \wedge j_0 \wedge 1)g' : X \to \Sigma^4 KO \wedge SG \) is trivial, then there exist maps \( h_1 : X \to \Sigma^4 KO \wedge SG \) and \( g : X \to \Sigma^4 KC \wedge Q \wedge SG \) such that \((1 \wedge i_0 \wedge 1)h_1 = (\gamma \pi_\Sigma^{-1} \wedge 1)g, (tc \wedge 1)g = g \) and \((\xi \wedge 1)g = (\xi \wedge 1)g' \).

**Proof.** Choose a map \( h_0 : X \to \Sigma^4 KO \wedge SG \) satisfying \((\varepsilon u \wedge 1)h_0 = (\xi \wedge 1)(1 \wedge j_0 \wedge 1)g' \), and hence a map \( \varrho : X \to \Sigma^4 KU \wedge SG \) such that \((\varepsilon c \wedge 1)h_0 = (1 \wedge j_0 \wedge 1)g' + (\gamma \pi_\Sigma \wedge 1)\varrho \). Then it follows that \( 2(\gamma \pi_\Sigma \wedge 1)\varrho = 0 \) because \( t_c \varepsilon c = \varepsilon c \) and \( t_c \gamma = -\gamma \). By applying the right inverse \( \rho \) of \((1 \wedge j_0 \wedge 1)\) obtained in (1.21) we verify that the composite \((\gamma \pi_\Sigma \wedge 1)\rho(\varrho) : X \to \Sigma^4 KC \wedge Q \wedge SG \) has order 2. Set \( g = g' + (\gamma \pi_\Sigma \wedge 1)\rho(\varrho) \). then its map satisfies the equalities \((\varepsilon c \wedge 1)h_0 = (1 \wedge j_0 \wedge 1)g, (tc \wedge 1)g = g \) and \((\xi \wedge 1)g = (\xi \wedge 1)g' \). Obviously the first equality implies that there exists a map \( h_1 : X \to \Sigma^4 KO \wedge SG \) with \((1 \wedge i_0 \wedge 1)h_1 = (\gamma \pi_\Sigma^{-1} \wedge 1)g \).

2. Pure projective and 2-torsion free.

2.1. In this section we will only deal with a CW-spectrum \( X \) such that \( KU_X \) is pure projective and 2-torsion free, and hence \( KC_X \) is expressed as in (1.9). We denote by \( A_k \) the image of the induced homomorphism \((\varepsilon c \tau \eta)_* : KC_{-1}X \to KC_X \) where \( KC_X \cong (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C) \). Then

\[
(2.1) \quad A_k = (\varepsilon c \tau \eta)_*(KC_{-1}X) = (\varepsilon c \tau)_*(E \otimes Z/2) \subset A \otimes Z/2 \subset KC_X
\]

because \( \eta_*(KC_{-1}X) = E \otimes Z/2 \) and \( \eta_*(KC_X) = A \otimes Z/2 \) by (1.10), \( i = 3, 0 \).

Since \( t_c \varepsilon c = \eta : \Sigma^4 KO \to KO \), it follows immediately that \( t_* A_k = 0 = (\tau \pi^{-1}_\Sigma)_* A_k \). Choose subgroups \( A'_0 \) and \( A' \) in \( A \otimes Z/2 \) so that \( A \otimes Z/2 \cong A_k \oplus A_0 \oplus A' \) and \( \text{Ker}(\varepsilon c \tau)_{|_{\otimes Z/2}} \cong A_k \oplus A' \). Thus the subgroups \( A_0 \) and \( A' \) satisfy

\[
(2.2) \quad (\varepsilon c \tau)_* : A_0 \overset{\approx}{\longrightarrow} (\varepsilon c \tau)_*(A \otimes Z/2) = D_\lambda \text{ and } (\varepsilon c \tau)_* A' = 0.
\]

We moreover put

\[
(2.3) \quad A'_0 = (\varepsilon c \eta)_*(KO_0X) \text{ and } A'_i = (\pi_c \varepsilon c \eta)_*(KO_{-i}X)
\]

both of which are subgroups of \( A \otimes Z/2 \). It is obvious that

\[
(2.4) \quad A'_k \subset A_0 \cap A'_i \text{ and } (\varepsilon c \tau)_* A'_i = 0 = (\varepsilon c \tau)_* A'_i
\]

because \( \varepsilon c \tau \eta = \pi_c \varepsilon c \eta \pi^{-1}_\Sigma : \Sigma^3 KC \to KC \) and \( \tau \pi_c \varepsilon c = 0 : \Sigma^3 KO \to KO \).

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More precisely we have

\[ (2.5) \quad \begin{align*}
&i) \quad \tau_* A_\varepsilon^i = \eta^i_\varepsilon(KO_\varepsilon X), \quad (\tau \pi^{-1})_* A_i^\varepsilon = \eta^i_\varepsilon(KO_{-\varepsilon} X), \text{and} \\
&\quad ii) \quad (\tau \pi^{-1})_* A_\varepsilon^i \equiv 0 = \tau_* A_i^\varepsilon.
\end{align*} \]

**Lemma 2.1.** There exists a direct sum decomposition

\[ A \otimes Z/2 \cong A_{\varepsilon} \oplus A_\varepsilon \oplus A_i \oplus A_i \varepsilon \]

with \[ A_{\varepsilon} \oplus A_\varepsilon \cong (\varepsilon \cdot \eta)_*(KO_\varepsilon X) \]
and \[ A_{\varepsilon} \oplus A_\varepsilon \cong (\pi \varepsilon \cdot \eta)_*(KO_{-\varepsilon} X). \]

Similarly there exist direct sum decompositions

\[ D \otimes Z/2 \cong D_\varepsilon \oplus D_{\bar{\varepsilon}} \oplus D_i \oplus D_{\bar{i}}, \quad B \otimes Z/2 \cong B_\varepsilon \oplus B_{\bar{\varepsilon}} \oplus B_i \oplus B_{\bar{i}} \]

and

\[ E \otimes Z/2 \cong E_\varepsilon \oplus E_i \oplus E_{\bar{\varepsilon}} \oplus E_{\bar{i}} \]

with suitable isomorphisms as above.

**Proof.** We will prove only the \( A \otimes Z/2 \) case. Choose subgroups \( A_\varepsilon \) and \( A_\varepsilon \) in \( A \otimes Z/2 \) so that \( A_{\varepsilon} \oplus A_\varepsilon \cong A_\varepsilon \) and \( A_{\varepsilon} \oplus A_\varepsilon \cong A_\varepsilon \). It is sufficient to show that \( \text{Ker}(\varepsilon \cdot \tau)_* |_{A \otimes Z/2} \cong A_{\varepsilon} \oplus A_\varepsilon \oplus A_\varepsilon \). First, take an element \( x \in A \otimes Z/2 \) with \( (\varepsilon \cdot \tau)_* x = 0 \). Using the equality \( \eta^2 = \tau \varepsilon \cdot \eta : \Sigma^2 KO \to KO \), we get elements \( u \in KO_\varepsilon X \) and \( v \in KO_\varepsilon X \) such that \( x = (\varepsilon \cdot \eta)_* u + (\pi^{-1} \varepsilon \cdot \eta)_* v \). Moreover we notice that \( \varepsilon \cdot \tau \cdot v = 0 \) because \( \varepsilon \cdot \tau \cdot Z/2 = 0 \). This implies that the element \( x \) is contained in \( A_{\varepsilon} \oplus A_\varepsilon \oplus A_{\varepsilon} \). Thus it is verified that \( \text{Ker}(\varepsilon \cdot \tau)_* |_{A \otimes Z/2} \cong A_{\varepsilon} \oplus A_\varepsilon \oplus A_\varepsilon \).

Next we take elements \( a \in A_\varepsilon \), \( b \in A_\varepsilon \) and \( c \in A_{\varepsilon} \) satisfying \( a + b + c = 0 \). Then it follows from (2.5) \( ii \) that \( \tau_* a = 0 = (\tau \pi^{-1})* b \). Since \( a \in (\varepsilon \cdot \eta)_*(KO_\varepsilon X) \) and \( b \in (\pi \varepsilon \cdot \eta)_*(KO_{-\varepsilon} X) \) we use (2.5) \( i \) to find elements \( x \) and \( y \) in \( KC_{-\varepsilon} X \) such that \( a = (\varepsilon \cdot \eta)_* x \) and \( b = (\pi \varepsilon \cdot \eta \cdot \pi^{-1})_* y \). Since the elements \( a \) and \( b \) are both belonging to \( A_{\varepsilon} \), they must be zero, thus \( a = b = c = 0 \). Consequently it is shown that \( \text{Ker}(\varepsilon \cdot \tau)_* |_{A \otimes Z/2} \cong A_{\varepsilon} \oplus A_\varepsilon \oplus A_\varepsilon \).

2.2. We now choose a direct sum decomposition

\[ (2.6) \quad A \cong A_{\varepsilon} \oplus A_\varepsilon \oplus A_\varepsilon \oplus A_\varepsilon \text{ with } A_{\varepsilon}, A_\varepsilon \text{ and } A_\varepsilon \oplus A_\varepsilon \text{ free,} \]

which after tensored with \( Z/2 \) gives the direct sum decomposition \( A \otimes Z/2 \cong A_{\varepsilon} \oplus A_\varepsilon \oplus A_\varepsilon \oplus A_\varepsilon \) obtained in Lemma 2.1 (use [Ku]). Similarly we choose direct sum decompositions

\[ (2.7) \quad D \cong D_\varepsilon \oplus D_\varepsilon \oplus D_i \oplus D_{\bar{i}}, \quad B \cong B_\varepsilon \oplus B_\varepsilon \oplus B_i \oplus B_{\bar{i}} \text{ and} \]

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\[ E \cong E_a \oplus E_b \oplus E_3 \oplus E_7, \]

which after tensored with \( Z/2 \) are respectively the direct sum decompositions of \( D \otimes Z/2, B \otimes Z/2 \) and \( E \otimes Z/2 \) obtained in Lemma 2.1.

Set \( G = A_\nu \cong D_\lambda \), which is free. We denote by \( i_\lambda : G \to A \) and \( i_\nu : G \to D \) the canonical inclusions with \( i_\lambda (G) = A_\nu \) and \( i_\nu (G) = D_\lambda \). Let \( t_0 : A \oplus C \to D \oplus E \otimes Z/2 \oplus F \) and \( t_1 : D \oplus F \to A \otimes Z/2 \oplus B \oplus C \) be the homomorphisms given in (1.12), which are determined by the conjugations \( t_c \) on \( KC_iX \) \((i = 0, 1)\) respectively.

**Lemma 2.2.** There exist unique homomorphisms \( r : G \to D \oplus F \) and \( s : G \to B \oplus C \) satisfying \( t_0 i_\lambda - i_\nu = 2r \) and \( t_1 i_\nu = 2s \).

**Proof.** First we use the canonical inclusion \( i_\lambda : G \to KC_0X \cong (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F) \). By use of (1.10), we observe that the composite \( \eta \epsilon \iota \lambda : G \to KC_1X \cong (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C) \) is identified with the canonical projection \( G = A_\nu \to A_\nu \otimes Z/2 \cong A_\nu \). So the composite \( (\eta \epsilon \iota \lambda) \circ i_\lambda : G \to KC_1X \cong (B \oplus C) \oplus (D \otimes Z/2 \oplus E \oplus F) \) is factorized through \( D_i \) because \( (\epsilon \iota \lambda) \circ i_\lambda : A_\nu \to D_i \) by (2.2). Therefore the composite \( (\epsilon \iota \lambda) \circ i_\lambda : G \to KC_1X \) is written into the form of a sum \(-i_\nu + 2u + v + w\) for some homomorphisms \( u : G \to D, v : G \to F \) and \( w : G \to A \otimes Z/2 \oplus B \oplus C \) (use (1.10)). Then it follows from (1.11) that the composite \( (\eta \epsilon \iota \lambda) \circ i_\lambda : G \to KC_0X \) coincides with the sum \(-i_\nu + 2u + v \). Since \( \gamma \xi \epsilon \iota \lambda v = 1 - t_0 : KC \to KC \), it is easily checked that \( t_0 i_\lambda = i_\nu + 2r \) setting \( r = -(u + v) : G \to D \oplus F \).

Next, use the canonical inclusion \( i_\nu : G \to KC_1X \cong (D \oplus F) \oplus (A \otimes Z/2 \oplus B \oplus C) \) in place of \( i_\lambda : G \to KC_0X \). The composite \( (\eta \epsilon \iota \lambda \circ i_\nu : G \to KC_1X \cong (E \oplus F) \oplus (A \oplus B \otimes Z/2 \oplus C) \) is trivial because \( \tau \epsilon \iota \lambda i_\nu = 0 \) by use of (2.2) and (1.10). By a parallel discussion to the above we can find a homomorphism \( s : G \to B \oplus C \) such that \( (\gamma \xi \epsilon \iota \lambda \circ i_\nu = -2s : G \to KC_1X \). Use the equality \( \gamma \xi \epsilon \iota \lambda v = 1 - t_1 \) again to obtain the desired one \( t_1 i_\nu = 2s \).

Let \( f_c : \Sigma^i Q \wedge SG \to KU \wedge X \) be the map whose induced homomorphisms \( \chi_{\epsilon \circ f_c} : KU_{i-1}(Q \wedge SG) \to KU_iX \) \((i = 0, 1)\) are given by the canonical inclusions \( i_\lambda : G \to A \oplus B \oplus C \oplus C \) and \( i_\nu : G \to D \oplus E \oplus F \oplus F \) respectively. Since \( (t_\nu \circ 1) f_c = f_c \), we obtain a map \( g_c : \Sigma^i Q \wedge SG \to KC \wedge X \) with \( (\xi \wedge 1) g_c = f_c \) by use of the cofiber sequence (1.1) (iii). According to (1.15) the induced homomorphisms \( \chi_{\epsilon \circ g_c} : KC_{i-1}(Q \wedge SG) \to KC_iX \) and

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\( x_1(g_c) : KC_6(Q \land SG) \to KC_1X \) are respectively given by the following matrices

\[
\begin{pmatrix}
  i_4 & 0 \\
  z & i_p
\end{pmatrix} : G \oplus G \to (A \oplus C) \oplus (D \oplus E \oplus Z/2 \oplus F)
\]

\[
(2.8)
\begin{pmatrix}
  i_p & 0 \\
  w & i_4
\end{pmatrix} : G \oplus (G \oplus Z/2) \to (D \oplus F) \oplus (A \oplus Z/2 \oplus B \oplus C)
\]

for some homomorphisms \( z : G \to D \oplus E \oplus Z/2 \oplus F \) and \( w : G \to A \oplus Z/2 \oplus B \oplus C \). In particular, take \( z = r \) and \( w = s \), both of which are obtained in Lemma 2.2. Then (1.16) shows that the given map \( g_c \) satisfies \((t_c \land 1)g_c = g_c\). Thus we have

**Corollary 2.3.** There exists a map \( g_c : \Sigma^1 Q \land SG \to KC \land X \) such that \((\xi \land 1)g_c = f_c\) and \((t_c \land 1)g_c = g_c\).

2.3. We will now prove one of our main theorems.

**Theorem 2.4.** Let \( X \) be a CW-spectrum such that \( KU_\bullet X \) is pure projective and 2-torsion free, thus it is a direct sum of a free group and cyclic \( p \)-groups \((p \neq 2)\). Then there exist abelian groups \( A_i (0 \leq i \leq 7) \), \( C_j (0 \leq j \leq 1) \) and \( G_k (0 \leq k \leq 3) \) so that \( X \) is quasi \( KO_\bullet \)-equivalent to the wedge sum \((\vee_A \Sigma^i S A_i) \cup (\vee_j \Sigma^j P \land S C_j) \cup (\vee_k \Sigma^{k+1} Q \land S G_k)\) where \( C_j \) and \( G_k \) are taken to be free (cf. [B. Theorem 3.2]).

**Proof.** Using the abelian groups chosen in (2.6) and (2.7) we set \( A_1 = D_1, A_2 = B_2, A_3 = E_3, A_4 = D_4, A_5 = B_5, A_6 = E_7, C_0 = C \) and \( C_1 = F \), and moreover \( G_0 = A_0 \cong D_6, G_1 = D_9 \cong B_9, G_2 = E_{11} \cong E_{11} \) and \( G_3 = E_{14} \cong A_8 \). Abbreviate by \( Y \) the required wedge sum \((\vee_A \Sigma^i S A_i) \cup (\vee_j \Sigma^j P \land S C_j) \cup (\vee_k \Sigma^{k+1} Q \land S G_k)\). It is obvious that \( KU_\bullet Y \cong KU_\bullet X \) on both of which the conjugations \( t_{u, s} \) behave as the same action. For each component \( Y_n \) of the wedge sum \( Y \) we choose a map \( f_n : Y_n \to KU \land X \) whose induced homomorphism \( \chi_{KU}(f_n)_* : KU_\bullet Y_n \to KU_\bullet X \) is the canonical inclusion. Here \( H \) is taken to be \( A_1 (0 \leq i \leq 7) \), \( C_j (0 \leq j \leq 1) \) and \( G_k (0 \leq k \leq 3) \). Since \((t_u \land 1)f_n = f_n\) by virtue of [Y3. Lemma 1.2], there exists a map \( g_n : Y_n \to KC \land X \) satisfying \((\xi \land 1)g_n = f_n\) for each \( H \). Along the line adopted in [Y2, Y3] we will find a map \( h_n : Y_n \to KO \land X \) such that \((e_u \land 1)h_n = f_n\), and then apply [Y2. Proposition 1.1] to show that the map \( h = \ldots \)
\( \vee h_h : Y = \vee Y_h \to KO \wedge X \) is a quasi \( KO^* \)-equivalence.

We will only find such maps \( h_h \) in the cases \( H = A_0, C_0 \) and \( G_0 \). The other cases are similarly done.

i) The \( H = A_0 \) case: The induced homomorphism \( (\eta \tau \pi c^{-1})_* : KC_0 X \to KO_0 X \) restricted to \( A_0 \) is trivial since \((\tau \pi c^{-1})_* A_0 = 0 \) by (2.5) ii). Hence the composite \((\eta \wedge 1)(\tau \pi c^{-1} \wedge 1)g_{A_0} = (\varepsilon_0 \pi u^{-1} \wedge 1)f_{A_0} : SA_0 \to \Sigma^2 KO \wedge X \) becomes trivial because \( A_0 \) is written as a direct sum of a free group and a uniquely 2-divisible group. So we get a required map \( h_{A_0} : SA_0 \to KO \wedge X \) with \((\varepsilon_0 \wedge 1)h_{A_0} = f_{A_0} \).

ii) The \( H = C_0 \) case: Since \( \eta \wedge 1 : \Sigma^1 KO \wedge P \to KO \wedge P \) is trivial, it is immediate that the composite \((\eta \wedge 1)(\tau \pi c^{-1} \wedge 1)g_{C_0} = (\varepsilon_0 \pi u^{-1} \wedge 1)f_{C_0} : P \wedge SC_0 \to \Sigma^2 KO \wedge X \) becomes trivial. So we get a required map \( h_{C_0} : P \wedge SC_0 \to KO \wedge X \) with \((\varepsilon_0 \wedge 1)h_{C_0} = f_{C_0} \).

iii) The \( H = G_0 \) case: For simplicity we put \( G = G_0, f = f_{G_0} \) and \( g = g_{G_0} \) where \( G = A_0 \cong D_3 \) and it is free. By Corollary 2.3 the map \( g : \Sigma^1 Q \wedge SG \to KC \wedge X \) can be chosen to satisfy \((t_\varepsilon \wedge 1)g = g \) as well as \((\xi \wedge 1)g = f \). The induced homomorphism \((\eta \tau \pi c^{-1})_* : KC_1 X \to KO_1 X \) restricted to \( D_3 \) is trivial since \((\tau \pi c^{-1})_* D_3 = 0 \). So the composite \((\eta \wedge 1)(\tau \pi c^{-1} \wedge 1)g_1 : SG \to \Sigma^1 KO \wedge X \) becomes trivial. By applying Lemma 1.1 we then get a map \( h_1 : \Sigma^1 SG \to KO \wedge X \) such that \( h_1(j_0 \wedge 1) = (\tau \pi c^{-1} \wedge 1)g_1 \) although the map \( g_1 \) with \((t_\varepsilon \wedge 1)g = g \) and \((\xi \wedge 1)g = f \) might be changed slightly for the new one.

In order to observe that the composite \((\varepsilon_0 \pi u^{-1} \wedge 1)f = (\eta \wedge 1)(\tau \pi c^{-1} \wedge 1)g = (\eta \wedge 1)h_1(j_0 \wedge 1) : Q \wedge SG \to \Sigma^1 KO \wedge X \) is trivial, we will next show that there exists a map \( k : SG \to KO \wedge X \) satisfying \((\eta \wedge 1)k = (\eta \wedge 1)h_1 \) as in the proof of [Y2, Theorem 3.4]. Denote by \( i_A : G \to A \) and \( i_D : G \to D \) the canonical inclusions with \( i_A(G) = A_0 \) and \( i_D(G) = D_3 \) respectively. Moreover we note that the conjugation \( t_{\vee} = \left( \begin{array}{cc} 1 & 0 \\ t_0 & -1 \end{array} \right) \) on \( KC_0 X = (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F) \) for a certain homomorphism \( t_0 : A \oplus C \to D \oplus E \otimes Z/2 \oplus F \) (see (1.12)). By (2.8) we may express as \( \chi_{KC}(g)_* = \chi_{K}(g) = \left( \begin{array}{cc} i_A & 0 \\ z & i_D \end{array} \right) : KC_0(Q \wedge SG) \cong G \oplus G \to KC_1 X \cong (A \oplus C) \oplus (D \oplus E \otimes Z/2 \oplus F) \). Here the homomorphism \( z : G \to D \oplus E \otimes Z/2 \oplus F \) satisfies \( t_0 i_A = 2 z + i_D \) by virtue of (1.16) because \((t_\varepsilon \wedge 1)g = g \). Recall [Y2, (3.5)] that the induced homomorphism \( \varepsilon_{\vee} : KO_0(Q \wedge SG) \to KC_0(Q \wedge SG) \) is represented by the column \( \left( \begin{array}{c} 2 \\ 1 \end{array} \right) : G \to G \oplus G \) (cf. (1.13)). Then
an easy computation shows that the composite \( x_{Kc}(g) \ast \varepsilon_c \ast \) : \( KO_3(Q \wedge SG) \to KC_3X \) coincides with the composite \((1 + t_c) \ast h_\lambda : G \to (A \oplus C) \oplus (D \oplus E \oplus Z/2 \oplus F)\). Since \( \tau \pi_c^{-1} t_c = - \pi_c \), it is easily checked that the composite \((\tau \pi_c^{-1}) \ast x_{Kc}(g) \ast \varepsilon_c \ast \) : \( KO_3(Q \wedge SG) \to KO_1X \) is trivial. Thus \( x_{K0}(\tau \pi_c^{-1} \wedge 1)g \ast = x_{K0}(h_\lambda(j_0 \wedge 1)) \ast : KO_3(Q \wedge SG) \to KO_1X \) is trivial.

Use the commutative diagram

\[
\begin{array}{ccc}
[SG, \Sigma^{-1}KC \wedge X] & \xrightarrow{[\eta^\wedge 1] \ast} & [SG, \Sigma^{-1}KC \wedge X] \\
\downarrow \chi & & \downarrow \chi \\
\text{Hom}(KO_3SG, KC_1X) & \xleftarrow{\eta \ast} & \text{Hom}(KO_3SG, KC_1X) \\
\end{array}
\]

where the left two vertical arrows \( x_0 = x_0^\ast \) are isomorphisms since \( G \) is free. Notice that \( x_0((\varepsilon_c \wedge 1)h_\lambda(j_0 \wedge 1)) : KO_3(SG) \to KC_1X \) is trivial. So we see that \( x_0((\varepsilon_c \wedge 1)h_\lambda) : KO_3SG \to KC_1X \) has order 2 since the bottom right horizontal arrow \((\eta \ast) \ast \) is just multiplication by 2 on Hom\((G, KC_1X)\). In other words, \( x_0((\varepsilon_c \wedge 1)h_\lambda) : G \to D \oplus F \oplus A \oplus Z/2 \oplus B \oplus C \) is factorized through \( A \oplus Z/2 \). Then \( x_0((\eta \varepsilon_c \wedge 1)h_\lambda) : KO_3SG \to KC_1X \) becomes trivial because \( \eta \ast (A \oplus Z/2) = 0 \). Therefore the composite \((\eta \varepsilon_c \wedge 1)h_\lambda : \Sigma^1SG \to KC_1X \) is trivial since the left vertical arrow \( x_0 \) is an isomorphism. So we get a map \( k : SG \to KO \wedge X \) satisfying \((\eta^\wedge 1)k = (\eta \wedge 1)h_\lambda \). Consequently there exists a map \( h : \Sigma^1Q \wedge SG \to KO \wedge X \) with \((\varepsilon_u \wedge 1)h = f \) as desired.

3. Pure injective and 2-divisible.

3.1. In this section we will only deal with a \( CW \)-spectrum \( X \) such that \( KU \ast X \) is pure injective and 2-divisible, and hence \( KC \ast X \) is expressed as in (1.9). Denote by \( A_\varepsilon \) the image of the induced homomorphism \((\varepsilon \tau \eta) \ast : KC_1X \to KC_3X \) where \( KC_3X \cong (D \ast Z/2 \oplus E \oplus F) \oplus (A \oplus C) \). Thus

\[ A_\varepsilon = (\varepsilon \tau \eta) \ast (KC_1X) = (\varepsilon \tau) \ast (E \ast Z/2) \subset A \ast Z/2 \subset KC_3X \]

because \( \eta \ast (KC_1X) = E \ast Z/2 \) and \( \eta \ast (KC_2X) = A \ast Z/2 \) by (1.10) \((i = 1, 2)\).

Since \( \tau \ast A_\varepsilon = 0 \), we can choose subgroups \( A_0 \) and \( A' \) in \( A \ast Z/2 \) so that \( A \ast Z/2 \cong A_\varepsilon \oplus A_0 \oplus A' \) and \( \text{Ker}(\varepsilon \tau) \ast |_{A \ast Z/2} \cong A_\varepsilon \oplus A' \). Thus the subgroups \( A_0 \) and \( A' \) satisfy

\[ (\varepsilon \tau) \ast : A_0 \cong (\varepsilon \tau) \ast (A \ast Z/2) = D_i \] and \( (\varepsilon \tau) \ast A' = 0 \).
As a dual of Lemma 2.1 we have

**Lemma 3.1.** There exists a direct sum decomposition

\[ A \ast Z/2 \cong A' \oplus A' \oplus A' \oplus A' \]

with \( A' \oplus A' \cong (\varepsilon_c \eta) \ast (KO_2 X) \) and \( A' \oplus A' \cong (\pi_c \varepsilon \eta) \ast (KO_{-2} X) \).

Similarly there exist direct sum decompositions

\[ D \ast Z/2 \cong D' \oplus D' \oplus D' \oplus D', \quad B \ast Z/2 \cong B' \oplus B' \oplus B' \oplus B' \quad \text{and} \quad E \ast Z/2 \cong E' \oplus E' \oplus E' \oplus E' \]

with suitable isomorphisms as above.

**Proof.** Choose subgroups \( A' \) and \( A' \) in \( A \ast Z/2 \) so that \( A' \oplus A' \cong (\varepsilon \eta) \ast (KO_2 X) \) and \( A' \oplus A' \cong (\pi_c \varepsilon \eta) \ast (KO_{-2} X) \). Then we can easily show that \( \text{Ker}(\varepsilon \ast) \mid A' \ast Z/2 \cong A' \oplus A' \oplus A' \) by the quite same argument as in the proof of Lemma 2.1.

We now choose a direct sum decomposition

\[ A \cong A' \oplus A' \oplus A' \oplus A' \oplus A' \] \hspace{1cm} (3.3)

with \( A' \) and \( A' \) divisible 2-torsion.

which restricted to the torsion subgroups of order 2 is just the direct sum decomposition \( A \ast Z/2 \cong A' \oplus A' \oplus A' \oplus A' \) obtained in Lemma 3.1. Similarly we choose direct sum decompositions

\[ D \cong D' \oplus D' \oplus D' \oplus D' \] \hspace{1cm} (3.4)

\[ B \cong B' \oplus B' \oplus B' \oplus B' \] \hspace{1cm} and

\[ E \cong E' \oplus E' \oplus E' \oplus E' \]

which induce respectively the direct sum decompositions of \( D \ast Z/2 \), \( B \ast Z/2 \) and \( E \ast Z/2 \) obtained in Lemma 3.1.

Set \( G = A_0 \cong D_1 \), which is divisible 2-torsion. We denote by \( p_A : A \to G \) and \( p_B : D \to G \) the canonical projections with \( p_A(A_0) = G \) and \( p_B(D_1) = G \). Let \( t_0 : A \oplus B \ast Z/2 \oplus C \to D \oplus F \) and \( t_1 : D \ast Z/2 \oplus E \oplus F \to A \oplus C \) be the homomorphisms given in (1.12). which are determined by the conjugations \( t_0 \) on \( KC_i X (i = 0, 3) \) respectively. By a dual argument to the proof of Lemma 2.2 we show

**Lemma 3.2.** There exist unique homomorphisms \( r : A \oplus C \to G \) and \( s : E \oplus F \to G \) satisfying \( p_B t_0 p_A = 2r \) and \( p_A t_1 = 2s \).

**Proof.** First we use the canonical projection \( p_0 : KC_0 X = (A \oplus B \ast Z/2 \oplus C) \oplus (D \oplus F) \to G \). By means of (1.10), \( (i = 2, 3) \) and (3.2) we
observe as in the proof of Lemma 2.2 that the composite \( p_0(\epsilon \tau \gamma)_\ast : KC_{-1}X \cong (D \star Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to G \) is written into the form of a sum \( p_1 + 2x + y + z \) for some homomorphisms \( x : A \to G \), \( y : C \to G \) and \( z : D \star Z/2 \oplus E \oplus F \to G \). Then (1.11)_3 shows that the composite \( p_0(\epsilon \tau \gamma \xi)_\ast : KC_X \to G \) is identified with the sum \( p_1 + 2(x + y) \). Since \( \epsilon \tau \gamma \xi = 1 + t_c \), \( KC \to KC \), it is easily checked that \( p_0t_0 = p_1 + 2r \) setting \( r = x + y : A \oplus C \to G \).

Using the canonical projection \( p_A : KC_X \cong (D \star Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to G \) in place of \( p_0 : KC_X \to G \), we can similarly find a homomorphism \( s : E \oplus F \to G \) such that \( p_A(\epsilon \tau \gamma \xi)_\ast = 2s : KC_X \to G \). This equality implies the desired one \( p_A t_3 = 2s \) because \( \epsilon \tau \gamma \xi = 1 + t_c \).

Let \( f_c : X \to \Sigma^i KU \wedge Q \wedge SG \) be the map whose induced homomorphisms \( \kappa_i(f_c)_\ast : KU_iX \to KU_{i-1}(Q \wedge SG) \) \((i = 0, 1)\) are given by the canonical projections \( p_A : A \oplus B \oplus C \oplus C \to G \) and \( p_D : D \oplus E \oplus F \oplus F \to G \) respectively. Since \( (t_u \wedge 1)f_c = f_c \), there exists a map \( g_c : X \to \Sigma^i KC \wedge Q \wedge SG \) with \((\xi \wedge 1)g_c = f_c\). According to (1.18) the induced homomorphisms \( \kappa_i(g_c) : KU_iX \to KU_{i-1}(Q \wedge SG) \) and \( \kappa_3(g_c) : KC_X \to KC_2(Q \wedge SG) \) are respectively given by the following matrices

\[
\begin{pmatrix}
p_A & 0 \\
w & p_D
\end{pmatrix} : (A \oplus B \star Z/2 \oplus C) \oplus (D \oplus F) \to G \oplus G \\
\begin{pmatrix}
p_D & 0 \\
z & p_A
\end{pmatrix} : (D \star Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to (G \star Z/2) \oplus G
\]

for some homomorphisms \( w : A \oplus B \star Z/2 \oplus C \to G \) and \( z : D \star Z/2 \oplus E \oplus F \to G \). In particular, take \( w = -r \) and \( z = -s \) by using the homomorphisms \( r \) and \( s \) obtained in Lemma 3.2. Then (1.19) asserts that the given map \( g_c \) satisfies \((t_c \wedge 1)g_c = g_c\). Thus we have

**Corollary 3.3.** There exists a map \( g_c : X \to \Sigma^i KC \wedge Q \wedge SG \) such that \((\xi \wedge 1)g_c = f_c\) and \((t_c \wedge 1)g_c = g_c\).

3.2. We will finally prove another main result, which is a dual of Theorem 2.4.

**Theorem 3.4.** Let \( X \) be a CW-spectrum such that \( KU_aX \) is pure injective and 2-divisible, thus it is a direct summand of a direct product of a divisible group and cyclic \( p \)-groups \((p \neq 2)\). Then there exist abelian groups \( A_i \) \((0 \leq i \leq 7)\), \( C_j \) \((0 \leq j \leq 1)\) and \( G_k \) \((0 \leq k \leq 3)\) so that \( X \) is

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quasi $KO_*\text{-equivalent to the wedge sum } (\bigvee_i \Sigma^i SA_i) \vee (\bigvee_j \Sigma^j S \land SC_j) \vee (\bigvee_k \Sigma^{k+1} Q \land SG_k)$ where $C_j$ and $G_k$ are taken to be divisible 2-torsion (cf. [B. Theorem 3.3]).

Proof. As in the proof of Theorem 2.4 we take $A_i$, $C_j$ and $G_k$ to be the abelian groups chosen in (3.3) and (3.4). For each component $Y^u$ of the required wedge sum $Y = (\bigvee_i \Sigma^i SA_i) \vee (\bigvee_j \Sigma^j S \land SC_j) \vee (\bigvee_k \Sigma^{k+1} Q \land SG_k)$, we choose a map $f^u : X \rightarrow KU \land Y^u$ whose induced homomorphism $x_{\eta^u}(f^u)_* : KU_*X \rightarrow KU_*Y^u$ is the canonical projection. Since $(t_u \land 1)f^u = f^u$, there exists a map $g^u : X \rightarrow KC \land Y_u$ satisfying $(\xi \land 1)g^u = f^u$ for each $H$. By a dual argument to the proof of Theorem 2.4 we will only show that there exist maps $h^u : X \rightarrow KO \land Y_u$ such that $(\epsilon_u \land 1)h^u = f^u$ in the cases $H = A_0$, $C_0$ and $G_0$.

i) The $H = A_0$ case: Use the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \text{Ext}(KO_0X, A_0) \\
\downarrow{(\eta^u)_*} & & \downarrow{(\eta \land 1)_*} \\
0 & \rightarrow & \text{Hom}(KO_0X, A_0)
\end{array}
$$

involving the Anderson universal coefficient sequences (1.5) ii), where $x_i = x_i^{ko}$. The induced homomorphism $x_{\eta^u}(\eta \land 1)g_{A_0}_* : KO_0X \rightarrow KC_\tau SA_0$ is trivial because $(\pi\tau^{-1} \epsilon \eta)_*(KO_0X) \cong A_2 \oplus A_4 \subset A_4 \oplus A_4$ by Lemma 3.1. This implies that $x_i((\eta \land 1)(\pi\tau^{-1}\epsilon \eta)g_{A_0}) = 0$ in $\text{Hom}(KO_0X, A_0)$. Obviously the composite $(\eta \land 1)(\pi\tau^{-1}\epsilon \eta)g_{A_0}$ has order 2. However Ext$(KO_0X, A_0)$ is uniquely 2-divisible since $A_0$ is a direct sum of a divisible group and a uniquely 2-divisible group. So we see that the composite $(\eta \land 1)(\pi\tau^{-1}\epsilon \eta)g_{A_0}$ is in fact trivial. Hence there exists a required map $h_{A_0} : X \rightarrow KO \land SA_0$ with $(\epsilon_u \land 1)h_{A_0} = f_{A_0}$.

ii) The $H = C_0$ case: The composite $(\eta \land 1)(\pi\tau^{-1}\epsilon \eta)g_{C_0} = (\epsilon \pi\tau^{-1}\epsilon \eta)g_{A_0}$ is uniquely 2-divisible. So we get a required map $h_{C_0} : X \rightarrow KO \land P \land SC_0$ with $(\epsilon_u \land 1)h_{C_0} = f_{C_0}$.

iii) The $H = G_0$ case: For simplicity we put $G = g_{C_0}, f = f_{C_0}, g = g_{C_0}$ where $G = A_0 \cong D_4$ and it is divisible 2-torsion. By virtue of Corollary 3.3 the map $g : X \rightarrow \Sigma^i KC \land Q \land SG$ can be chosen to satisfy $(t_c \land 1)g = g$ as well as $(\xi \land 1)g = f$. Denote by $p_A : A \rightarrow G$ and $p_D : D \rightarrow G$ the canonical projections with $p_A(A_0) = G$ and $p_D(D_4) = G$. According to
(3.5), \( \kappa_{\kappa c}(g) \ast = \kappa_0(g) = \begin{pmatrix} p_1 & 0 \\ w & p_\beta \end{pmatrix} : KC_0X \to KC_{-1}(Q \land SG) \) and \( \kappa_{\kappa c}(g) \ast = \kappa_0(g) = \begin{pmatrix} p_1 & 0 \\ z & p_\beta \end{pmatrix} : KC_0X \to KC_{-1}(Q \land SG) \) for some homomorphisms \( w : A \oplus B \ast Z/2 \oplus C \to G \) and \( z : D \ast Z/2 \oplus E \oplus F \to G \). As is easily checked, the induced homomorphism \( \kappa_{\kappa o}((\eta \land 1)g) \ast = \kappa_{\kappa c}(g) \ast (\varepsilon_c \eta) \ast : KO_2X \to KC_2(Q \land SG) \) is trivial because \( (\varepsilon_c \eta) \ast (KO_2X) \cong A_F \oplus A_i \subset A_E \oplus A_o \) and \( p_{\alpha}(A_F \oplus A_o \oplus A_i) = 0 \). Hence the composite \( (\eta \land 1)((\tau \pi_{c}^{-1} \land 1)(1 \land j_o \land 1)) : X \to \Sigma^k KO \land SG \) becomes trivial since \( \kappa_{\kappa c}(g) : [\Sigma^{-k} X, KO \land SG] \to \text{Hom}(KO_2X, G) \) is an isomorphism by (1.5) ii). By applying Lemma 1.2 we then get a map \( h_i : X \to \Sigma^k KO \land SG \) such that \( (1 \land i_o \land 1)h_i = (\tau \pi_{c}^{-1} \land 1)g \) although the map \( g \) with \( (t_c \land 1)g = g \) and \( (\xi \land 1)g = f \) might be changed slightly for the new one.

As in the latter part of the proof iii) of Theorem 2.4 we will next show that there exists a map \( k : X \to \Sigma^k KO \land SG \) satisfying \( (\eta \land 1)k = (\eta \land 1)h_i \), in order to observe that the composite \( (\varepsilon_c \pi_{o}^{-1} \land 1)f = (\eta \land 1) \cdot (\tau \pi_{c}^{-1} \land 1)g = (1 \land i_o \land 1)(\eta \land 1)h_i : X \to \Sigma^k KO \land Q \land SG \) is trivial. By (1.12) we note that the conjugation \( t_c^* = \begin{pmatrix} 1 & 0 \\ t_o & -1 \end{pmatrix} \) on \( KC_0X \cong (A \oplus B \ast Z/2 \oplus C) \oplus (D \oplus F) \) for a certain homomorphism \( t_o : A \oplus B \ast Z/2 \oplus C \to D \oplus F \). Then (1.19) says that the homomorphism \( w : A \oplus B \ast Z/2 \oplus C \to G \) satisfies \( p_{\beta}t_o = p_1 - 2w \) where \( \kappa_{\kappa c}(g) \ast = \kappa_0(g) = \begin{pmatrix} p_1 & 0 \\ w & p_\beta \end{pmatrix} \), because \( (t_c \land 1)g = g \). Recall that \( (\tau \pi_{c}^{-1})^\ast : KC_{-1}(Q \land SG) \to KO_{-1}(Q \land SG) \) is represented by the row \( (\varepsilon_c \pi_{o}^{-1} \ast (\tau \pi_{c}^{-1} \ast (\tau \pi_{c}^{-1} \ast (\tau \pi_{c}^{-1} \ast \kappa_{\kappa c}(g) \ast : KC_0X \to KO_{-1}(Q \land SG) \) coincides with the composite \( p_{\beta}(1 - t_c^*) : (A \oplus B \ast Z/2 \oplus C) \oplus (D \oplus F) \to G \). So the composite \( (\tau \pi_{c}^{-1})^\ast \kappa_{\kappa c}(g) \ast : KO_0X \to KO_{-1}(Q \land SG) \) becomes trivial because \( t_c \varepsilon_c = \varepsilon_c \). Thus \( \kappa_{\kappa o}((\tau \pi_{c}^{-1} \land 1)g) \ast = \kappa_{\kappa o}((1 \land i_o \land 1)h_i) \ast : KO_0X \to KO_{-1}(Q \land SG) \) is trivial.

Since the induced homomorphism \( i_{\omega} : KO_{-4}SG \to KO_{-4}(Q \land SG) \) is multiplication by 2 on G, it follows immediately that \( \kappa_{\kappa o}(h_i) \ast : KO_0X \to KO_{-1}SG \) has order 2. Hence the composite \( \kappa_{\kappa o}(h_i) \ast (\tau \pi_{c}^{-1})^\ast : KC_3X \cong (D \ast Z/2 \oplus E \oplus F) \oplus (A \oplus C) \to KO_{-1}SG \cong G \) is factorized through \( D \ast Z/2 \). Since \( \eta \ast (KC_2X) = A \ast Z/2 \), the composite \( \kappa_{\kappa o}((\eta \land 1)h_i) \ast (\tau \pi_{c}^{-1})^\ast : KC_2X \to KO_{-1}SG \) is trivial, too. We here use the commutative diagram
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$$\begin{align*}
\left[\Sigma^{-1}X, \Sigma^sKO \wedge SG\right] & \xrightarrow{(\eta^*\wedge 1)^*} \left[\Sigma^sX, \Sigma^sKO \wedge SG\right] \\
\downarrow_{x_0} & \downarrow_{x_0} \\
\text{Hom}(KO,X, KO_{*}, SG) & \xrightarrow{(\eta^*)^*} \text{Hom}(KO_{*}, X, KO_{*}, SG) \xrightarrow{(\epsilon^*\wedge 1)^*} \text{Hom}(KC,X, KO_{*}, SG)
\end{align*}$$

where the two vertical arrows $x_0 = x_0^{ko}$ are isomorphisms by (1.5) ii). As is easily seen, there exists a map $k : X \to \Sigma^sKO \wedge SG$ satisfying $(\eta^2 \wedge 1)^*k = (\eta \wedge 1)^*h_1$. Consequently we obtain a map $h : X \to \Sigma^sKO \wedge Q \wedge SG$ with $(\epsilon_u \wedge 1)^*h = f$ as desired.

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