On Extensions of Rings with Finite Additive Index

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ON EXTENSIONS OF RINGS WITH FINITE ADDITIVE INDEX

To the memory of Professor Shigeaki Togó

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In [1] we proved that if the additive group of the center $Z$ of a ring $R$ has a finite group-theoretic index in the additive group of $R$, then $R$ has an ideal $I$ contained in $Z$ such that $R/I$ is a finite ring. The purpose of this paper is to extend this result for extensions of rings with finite additive index. As an application of it, we prove that if a derivation $d$ of an infinite simple ring has only finitely many values, then $d = 0$.

For a ring $R$, $R^+$ denotes the additive group of $R$. We shall prove the main theorem of this paper.

Theorem 1. Let $R$ be a subring of a ring $S$. Suppose that $R^+$ has a finite index in $S^+$. Then there exists an ideal $I$ of $S$ contained in $R$ such that $S/I$ is a finite ring.

Proof. Consider the homomorphism $g : R \to \text{End}(S^+/R^+)$ defined by $g(r)(s + R^+) = rs + R^+$ for all $r \in R$ and $s + R^+ \in S^+/R^+$. Since $S^+/R^+$ is a finite group, $\text{End}(S^+/R^+)$ is a finite ring. Hence $\text{Ker}(g) = \{r \in R \mid rS \subseteq R\}$ has a finite index in $R^+$. Similarly, $\{r \in R \mid Sr \subseteq R\}$ has a finite index in $R^+$. Hence $I = \{r \in R \mid Sr \subseteq R\}$ and $rS \subseteq R$ has a finite index in $R^+$. Let $n$ be the index of $R^+$ in $S^+$ and let $S^+/R^+ = \langle a_1 + R^+, a_2 + R^+, \ldots, a_n + R^+ \rangle$. For each $i$, consider the map $f_i : I \to \text{End}(S^+/R^+)$ defined by $f_i(r)(s + R^+) = a_i rs + R^+$ for all $r \in I$ and $s + R^+ \in S^+/R^+$. Then each $f_i$ is an additive map, and so the additive subgroup $\text{Ker}(f_i)$ has a finite index in $I$. Hence $I' = \bigcap_{i=1}^{n} \text{Ker}(f_i)$ has a finite index in $R^+$. Let $r$ be an arbitrary element of $I'$. Then $rS \subseteq R$ and $a_i rS \subseteq R$ for all $i = 1, 2, \ldots, n$, and so $SrS \subseteq R$. Now it is easy to see that $I' = \{r \in R \mid SrS \subseteq R \cap I\}$. Therefore the ideal $J = I' + SI' + I'S + SI'S$ of $S$ is contained in $R$, and $S/J$ is a finite ring.

Corollary 1. Let $R$ be a subring of an infinite simple ring $S$. If $R^+$ has a finite index in $S^+$, then $S = R$.  

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Corollary 2. Let $R$ be an infinite simple ring with identity $e$. If $S$ is an extension of $R$ and if $R^*$ has a finite index in $S^*$, then $S$ is the direct sum of $R$ and a finite ring.

Proof. By Theorem 1, $S$ has an ideal $I$ contained in $R$ such that $S/I$ is a finite ring. Since $R$ is an infinite simple ring, $I$ must coincide with $R$. Thus $R$ is an ideal of $S$, and so $e$ is a central idempotent of $S$. Now our assertion is clear.

Corollary 3. Let $S$ be a ring which has no non-zero finite homomorphic images, and let $d$ be a derivation of $S$. If $d$ has only finitely many values in $S$, then $d = 0$.

Proof. Let $Im(d) = \{s_1, s_2, \ldots, s_n\}$. For each $i = 1, 2, \ldots, n$, take an element $a_i \in S$ such that $d(a_i) = s_i$. Since $d$ is a derivation of $S$, $R = \{a \in S | d(a) = 0\}$ is a subring of $S$. Now we can easily see that $S^*/R^* = \{a_1 + R^*, a_2 + R^*, \ldots, a_n + R^*\}$. Therefore, by Theorem 1, $S$ has an ideal $I$ contained in $R$ such that $S/I$ is a finite ring. Then, by hypothesis, we conclude that $S = R$.

As an immediate consequence of Corollary 3, we have

Corollary 4. Let $S$ be a ring which has no non-zero finite homomorphic images, and let $d$ denote the inner derivation of $S$ induced by an element $x$ of $S$. If $Im(d)$ is a finite subset of $S$, then $x$ is contained in the center of $S$.

Remark. In Corollary 3, $d$ cannot be replaced by an additive map of $S$, and hence, in Theorem 1, $R$ cannot be replaced by an additive subgroup with finite index. For example, let $K = GF(p)$ where $p$ is a prime number, and let $K(x)$ be the field of rational functions in one variable over $K$. Then there exists a $K$-subspace $L$ of $K(x)$ such that $K(x) = K \oplus L$. The projection $p: K(x) \to K$ defined by this decomposition is a non-zero additive map and $Im(p) (= K)$ is a finite subset of $K(x)$.

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