Lie Algebras Represented as a Sum of Two Subalgebras

Masanobu Honda* Takanori Sakamoto†

*Niigata College of Pharmacy
†Fukuoka University of Education

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Abstract

Let $L$ be a Lie algebra represented as a sum of two subalgebras $A$ and $B$. We prove that if $L$ belongs to a subclass of the class of locally finite Lie algebras over a field of characteristic $\neq 2$ and both $A$ and $B$ are locally nilpotent, then $L$ is locally soluble. We also prove that if $L$ is a serially finite Lie algebra over a field of characteristic zero, then any common serial subalgebra of $A$ and $B$ is serial in $L$. 
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MASANOBU HONDA AND TAKANORI SAKAMOTO

Abstract. Let $L$ be a Lie algebra represented as a sum of two subalgebras $A$ and $B$. We prove that if $L$ belongs to a subclass of the class of locally finite Lie algebras over a field of characteristic $\neq 2$ and both $A$ and $B$ are locally nilpotent, then $L$ is locally soluble. We also prove that if $L$ is a serially finite Lie algebra over a field of characteristic zero, then any common serial subalgebra of $A$ and $B$ is serial in $L$.

1. Introduction

Groups $G$ factorized by two subgroups $A$ and $B$, i.e. $G = AB$, have been investigated by many authors for some decades. Among the works Kegel [8] and Wielandt [15] established the well-known theorem: if $G$ is finite, and $A$ and $B$ are nilpotent, then $G$ is soluble.

In Lie algebras there is a corresponding result: If a finite-dimensional Lie algebra $L$ over a field $\mathfrak{k}$ of characteristic $\neq 2$ is represented as a sum of two nilpotent subalgebras $A$ and $B$, then $L$ is soluble. Goto [4] proved the case of char $\mathfrak{k} = 0$ and Panyukov [10] did the case of char $\mathfrak{k} = p > 2$. On the other hand, Aldosray [2] showed that if $L = A + B$ is an ideally finite Lie algebra over a field of characteristic zero, then any common ascendant subalgebra of both $A$ and $B$ is ascendant in $L$.

In this paper we shall generalize the result of Goto and Panyukov to a certain class of infinite-dimensional Lie algebras and extend the result of Aldosray to a wider class than that of ideally finite Lie algebras.

In Section 2 we shall show that in a locally finite Lie algebra $L$ a common weakly serial subalgebra of each subalgebra $X_i$ of $L$ for $i \in I$ is always a weakly serial subalgebra of $\langle X_i \mid i \in I \rangle$ (Theorem 2). Let $L$ be a Lie algebra represented as a sum of two subalgebras $A$ and $B$. In Section 3 we shall prove that if $L$ is a serially finite Lie algebra (resp. a hyperfinite, serially finite Lie algebra) over a field of characteristic zero, then any common serial (resp. ascendant) subalgebra of $A$ and $B$ is serial (resp. ascendant) in $L$ (Theorem 8 (resp. Corollary 9)). In Section 4 we shall verify that if $L$ belongs to the subclass $L(\text{wser})$ of the class of locally

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finite Lie algebras over a field of characteristic \( \neq 2 \) and both \( A \) and \( B \) are locally nilpotent, then \( L \) is locally soluble (Theorem 15).

2. Notation and terminology

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field \( \mathfrak{k} \) of arbitrary characteristic unless otherwise specified. We mostly follow [3] for the use of notation and terminology.

Let \( L \) be a Lie algebra over \( \mathfrak{k} \) and let \( H \) be a subalgebra of \( L \). For a totally ordered set \( \Sigma \), a series (resp. a weak series) from \( H \) to \( L \) of type \( \Sigma \) is a collection \( \{ L_\sigma, V_\sigma \mid \sigma \in \Sigma \} \) of subalgebras (resp. subspaces) of \( L \) such that

1. \( H \subseteq V_\sigma \subseteq L_\sigma \) for all \( \sigma \in \Sigma \),
2. \( L_\tau \subseteq V_\sigma \) if \( \tau < \sigma \),
3. \( L \setminus H = \bigcup_{\sigma \in \Sigma} (L_\sigma \setminus V_\sigma) \),
4. \( V_\sigma \triangleleft L_\sigma \) (resp. \([ L_\sigma, H ] \subseteq V_\sigma \)) for all \( \sigma \in \Sigma \).

\( H \) is a serial (resp. a weakly serial) subalgebra of \( L \), which we denote by \( H_{\text{ser}} L \) (resp. \( H_{\text{wser}} L \)), if there exists a series (resp. a weak series) from \( H \) to \( L \). For an ordinal \( \sigma \), \( H \) is a \( \sigma \)-step ascendant (resp. weakly ascendant) subalgebra of \( L \), denoted by \( H \vartriangleleft^\sigma L \) (resp. \( H \triangleleft^\sigma L \)), if there exists an ascending chain \( (H_\alpha)_{\alpha \leq \sigma} \) of subalgebras (resp. subspaces) of \( L \) such that

1. \( H_0 = H \) and \( H_\sigma = L \),
2. \( H_\alpha \triangleleft H_{\alpha + 1} \) (resp. \([ H_{\alpha + 1}, H ] \subseteq H_\alpha \)) for any ordinal \( \alpha < \sigma \),
3. \( H_\lambda = \bigcup_{\alpha < \lambda} H_\alpha \) for any limit ordinal \( \lambda \leq \sigma \).

\( H \) is an ascendant (resp. a weakly ascendant) subalgebra of \( L \), denoted by \( H_{\text{asc}} L \) (resp. \( H_{\text{wasc}} L \)), if \( H \triangleleft^\sigma L \) (resp. \( H \triangleleft^\sigma L \)) for some ordinal \( \sigma \). When \( \sigma \) is finite, \( H \) is a subideal (resp. a weak subideal) of \( L \) and denoted by \( H_{\text{si}} L \) (resp. \( H_{\text{wsi}} L \)). For an ordinal \( \alpha \), we denote by \( L^{(\alpha)} \) the \( \alpha \)-th term of the transfinite derived series of \( L \). A subspace \( H \) of \( L \) invariant under all derivations of \( L \) is said to be a characteristic ideal and denoted by \( H_{\text{ch}} L \).

Let \( \mathcal{X}, \mathcal{Y} \) be classes of Lie algebras and let \( \Delta \) be any of the relations \( \leq, <, \triangleleft, \text{ch}, \text{si}, \text{asc}, \text{ser}, \text{wser} \). \( \mathcal{X} \mathcal{Y} \) is the class of Lie algebras \( L \) having an ideal \( I \in \mathcal{X} \) such that \( L/I \in \mathcal{Y} \). A Lie algebra \( L \) is said to lie \( L(\Delta) \mathcal{X} \) if for any finite subset \( X \) of \( L \) there exists an \( \mathcal{X} \)-subalgebra \( H \) of \( L \) such that \( X \subseteq H \Delta L \). In particular we write \( L\mathcal{X} \) for \( L(\leq) \mathcal{X} \). When \( L \in L\mathcal{X} \) (resp. \( L(\text{ser})\mathcal{X} \)), \( L \) is called a locally (resp. a serially) \( \mathcal{X} \)-algebra. \( \mathfrak{Z}, \mathfrak{X}, \mathfrak{N}, \mathfrak{J} \) and \( \mathfrak{E} \mathfrak{A} \) are the classes of Lie algebras which are finite-dimensional, abelian, nilpotent, hypercentral and soluble respectively. The \( \mathcal{X} \)-residual \( \lambda_\mathcal{X}(L) \) of \( L \) is the intersection of the ideals \( I \) of \( L \) such that \( L/I \in \mathcal{X} \). \( \mathcal{T}_\mu(\Delta) \mathcal{X} \) is the class of Lie algebras \( L \) having an ascending series \( (L_\alpha)_{\alpha \leq \mu} \) of \( \Delta \)-subalgebras such that
Thus it follows from Lemma 1 that by induction on such that for any ordinal \( \alpha < \mu \), we may put \( h \). Proof. For any \( i \in I \) we define \( \mathfrak{e}(\Delta)X = \bigcup_{\mu > 0} \mathfrak{e}_\mu(\Delta)X \). In particular we write \( \mathfrak{e}X \) for \( \mathfrak{e}(\leq)X \). When \( L \in \mathfrak{e}(\leq)X \), \( L \) is called a hyper \( X \)-algebra. The Hirsch-Plotkin radical \( \rho(L) \) of \( L \) is the unique maximal locally nilpotent ideal of \( L \). For a locally finite Lie algebra \( L \) the locally soluble radical \( \sigma(L) \) of \( L \) is the unique maximal locally soluble ideal of \( L \). The set of left Engel elements of \( L \) is denoted by \( \mathfrak{e}(L) \).

### 3. Common weakly serial subalgebras

Before considering a common ascendant subalgebra of two permutable subalgebras in Section 4, we shall state more general forms in the following interesting theorem, which is a generalization of [12, Theorem 7]. To do this we need the following useful result.

**Lemma 1** ([5, Theorem 2.12]). Let \( H \) be a subalgebra of a locally finite Lie algebra \( L \). Then \( \mathfrak{h}wserL \) if and only if \( \lambda_{\mathfrak{m}}(H) < L \) and \( H/\lambda_{\mathfrak{m}}(H) \subseteq \mathfrak{e}(L/\lambda_{\mathfrak{m}}(H)) \).

**Theorem 2.** Let \( L \) be a locally finite Lie algebra over any field and let \( \{X_i\}_{i \in I} \) be a collection of subalgebras of \( L \). If \( H \) is a common weakly serial subalgebra of \( X_i \) for any \( i \in I \), then \( H \) is a weakly serial subalgebra of \( \langle X_i \mid i \in I \rangle \).

**Proof.** We may put \( L = \langle X_i \mid i \in I \rangle \). Using Lemma 1 we have \( \lambda_{\mathfrak{m}}(H) < X_i \) for any \( i \in I \), and so \( \lambda_{\mathfrak{m}}(H) < L \). We may also assume that \( \lambda_{\mathfrak{m}}(H) = 0 \) by \( \lambda_{\mathfrak{m}}(H/\lambda_{\mathfrak{m}}(H)) = 0 \) and [5, Proposition 2.5]. Then we get \( H \subseteq \mathfrak{e}(X_i) \) for all \( i \in I \) by using Lemma 1.

On the other hand, \( L \) is spanned by the elements of a form \( \{x_1, x_2, \ldots, x_n\} \), where each \( x_k \) belongs to \( \bigcup_{i \in I} X_i \). For any \( h \in H \), there is an \( m \in \mathbb{N} \) such that \( x_k(ad \ h)^m = 0 \) for \( 1 \leq k \leq n \). Then we can show that

\[
[x_1, x_2, \ldots, x_n](ad \ h)^{nm} = 0
\]

by induction on \( n \), using Leibniz formula. Therefore we have \( H \subseteq \mathfrak{e}(L) \). Thus it follows from Lemma 1 that \( \mathfrak{h}wserL \).

As a direct result of Theorem 2, we have the following:

**Corollary 3.** Let \( L \) be a Lie algebra over any field and let \( \{X_i\}_{i \in I} \) be a collection of subalgebras of \( L \).

1. If \( L \in \mathfrak{e}(\leq)(\mathfrak{A} \cap \mathfrak{F}) \) and \( HascX_i \) for any \( i \in I \), then \( Hasc\langle X_i \mid i \in I \rangle \).
2. If \( L \in \mathfrak{e}(\leq)\overline{\mathfrak{F}} \) and \( HwascX_i \) for any \( i \in I \), then \( Hwasc\langle X_i \mid i \in I \rangle \).
3. If \( L \in L(\mathfrak{E}\mathfrak{A} \cap \mathfrak{F}) \) and \( HserX_i \) for any \( i \in I \), then \( Hser\langle X_i \mid i \in I \rangle \).


Proof. (1) Since \( \mathfrak{e}(\langle \rangle) \leq \mathfrak{L} \) by [7, Corollary 3.3] we obtain \( L \in \mathfrak{L} \). Hence Theorem 2 implies that \( H_{\text{wser}}(X_i \mid i \in I) \). Because \( \langle X_i \mid i \in I \rangle \in \mathfrak{e}(\langle \rangle)(\mathfrak{A} \cap \mathfrak{F}) \) we conclude from [6, Proposition 2] that \( H_{\text{asc}}(X_i \mid i \in I) \).

(2) and (3) follow from [6, Theorem 1] and [5, Theorem 2.7] respectively as in the proof of (1).

4. Common ascendant subalgebras

Let \( L \) be a Lie algebra and let \( A, B \) be subalgebras of \( L \). As in groups we say that \( L \) is factorized by \( A \) and \( B \) if \( L = A + B \).

Let \( L \) be factorized by \( A \) and \( B \), and let \( H \subseteq A \setminus B \). In this section we shall consider some conditions under which \( H_{\text{asc}}A \) and \( H_{\text{asc}}B \) implies \( H_{\text{asc}}L \). First we easily see the following:

**Lemma 4.** Let \( L \in \mathfrak{e}(\langle \rangle) \mathfrak{A} \) and let \( L = A + B \) be the sum of two subalgebras \( A \) and \( B \). If \( H_{\text{asc}}A \) and \( H_{\text{asc}}B \), then \( H_{\text{asc}}L \).

**Proof.** Since \( H_{\text{wasc}}A \) and \( H_{\text{wasc}}B \), it is evident that \( H_{\text{wasc}}L \). Therefore [12, Corollary to Theorem 2] indicates \( H_{\text{asc}}L \).

**Remark.** As in the proof of Lemma 4, we can show the following, which is a generalization of [9, Corollary to Proposition 2] : Let \( L \in \mathfrak{e}(\langle \rangle) \mathfrak{A} \) and let \( H \leq X_i \) \((i = 1, 2, \ldots, n)\) be subalgebras of \( L \) such that \( \langle X_1, \ldots, X_n \rangle = X_1 + \cdots + X_n \). If \( H_{\text{asc}}X_i \) for any \( i \), then \( H_{\text{asc}}\langle X_1, \ldots, X_n \rangle \).

The following is originally due to Tôgô and is a generalization of [9, Remark to Lemma 4], for it is clear that \( \mathfrak{e}(\mathfrak{A}) \cup \mathfrak{Z} \leq \mathfrak{e}(\mathfrak{ch}) \mathfrak{A} \) over any field.

**Lemma 5.** Let \( L \) be a Lie algebra such that \( L = H + K \) with \( H \leq L, K \ll L \) and \( K \in \mathfrak{e}_{\mu}(\langle \rangle) \mathfrak{A} \). If \( H \leq^\lambda L \), then \( H \ll^\lambda \mu L \).

**Proof.** Let \( (K_\alpha)_{\alpha \leq \mu} \) be an ascending abelian series of characteristic ideals of \( K \). We note that \( K_\alpha \ll L \) by [3, Lemma 1.4.4]. Therefore \( K_\alpha \ll H + K_\alpha \leq L \) for all \( \alpha \leq \mu \). Now for any \( \alpha < \mu \) we put \( \overline{H} = (H + K_\alpha)/K_\alpha, \overline{K}_{\alpha+1} = K_{\alpha+1}/K_\alpha \). Then

\[
\overline{K}_{\alpha+1} \ll \overline{H} + \overline{K}_{\alpha+1} \quad \text{and} \quad \overline{K}_{\alpha+1} \in \mathfrak{A}.
\]

On the other hand, we have \( \overline{H} \leq^\lambda \overline{H} + \overline{K}_{\alpha+1} \) as \( H \leq^\lambda H + K_{\alpha+1} \). By virtue of [12, Lemma 3], we obtain \( \overline{H} \ll^\lambda \overline{H} + \overline{K}_{\alpha+1} \). Hence \( H + K_\alpha \ll^\lambda H + K_{\alpha+1} \) for all \( \alpha < \mu \). For any limit ordinal \( \beta < \mu \) it is trivial that \( H + K_\beta = \cup_{\alpha < \beta}(H + K_\alpha) \). Therefore it follows that \( H \ll^\lambda \mu L \). 

Now using Lemma 5 we can generalize [9, Proposition 5] to the following:
**Proposition 6.** Let $L$ be a Lie algebra such that $L = A + B = H + K$ with $A, B, H \leq L, K \not\triangleleft L$ and $K \in \bar{\mathfrak{e}}_{\mu}(\text{ch})\mathfrak{A}$. If $H \leq^\lambda A$ and $H \leq^\lambda B$, then $H <^\lambda \mu L$.

**Proof.** It is evident that $H \leq^\lambda L$. Therefore we conclude the assertion from Lemma 5. \hfill \Box

The following corresponds to [9, Theorem 6].

**Proposition 7.** Let $\mathfrak{X}$ be a class of Lie algebras and suppose that $L = A + B \in \mathfrak{X}$ with $A, B \triangleleft L, H \triangleleft A$ and $H \triangleleft B$, always implies that $H \triangleleft L$. Then $L = A + B \in (\bar{\mathfrak{e}}(\text{ch})\mathfrak{A})\mathfrak{X}$ with $H \triangleleft A$ and $H \triangleleft B$ always implies that $H \triangleleft L$.

**Proof.** Let $L = A + B \in (\bar{\mathfrak{e}}(\text{ch})\mathfrak{A})\mathfrak{X}$ with $H <^\lambda A$ and $H <^\lambda B$. Then there exists an ideal $K$ of $L$ such that $K \in \bar{\mathfrak{e}}_{\mu}(\text{ch})\mathfrak{A}$ and $L/K \in \mathfrak{X}$. Here we denote images under the natural map $L \rightarrow L/K$ by bars. Then

$$\bar{L} = \bar{A} + \bar{B} \in \mathfrak{X}, \quad \bar{H} <^\lambda \bar{A} \quad \text{and} \quad \bar{H} <^\lambda \bar{B}.$$ 

By the hypothesis, there exists an ordinal $\alpha = \alpha(H, \lambda)$ such that $\bar{H} <^\alpha \bar{L}$, so $H + K <^\alpha L$. On the other hand, $H \leq^\lambda L$ since $H \leq^\lambda A$ and $H \leq^\lambda B$. Hence $H \leq^\lambda H + K$. On account of Lemma 5, it follows that $H <^\lambda H + K$. Thus we can reach that $H <^\lambda \mu + \alpha L$. \hfill \Box

Let $L$ be factorized by $A$ and $B$ over a field of characteristic zero and let $H \triangleleft A$ and $H \triangleleft B$. Then Aldosray proved that if $L \in L(\triangleleft)\mathfrak{X}$ then $H \triangleleft L$ ([2, Theorem 6]). We know the facts that $L(\triangleleft)\mathfrak{X} \leq \bar{\mathfrak{e}}(\triangleleft)\mathfrak{X}$ ([14, Lemma 4.1]) and that if $L \in \bar{\mathfrak{e}}(\triangleleft)\mathfrak{X}$, then the notion of serial subalgebras of $L$ coincides with that of ascendant subalgebras of $L$ ([6, Theorem 1]). Now we shall prove the main theorem in this section, which generalize the result of Aldosray.

**Theorem 8.** Let $L$ be a serially finite Lie algebra over a field of characteristic zero and let $H, A, B$ be subalgebras of $L$ such that $L = A + B$ and $H \triangleleft A \cap B$. If $H$ is a common serial subalgebra of both $A$ and $B$, then $H$ is serial in $L$.

**Proof.** From [11, Theorem 5 and Corollary 6] it follows that

$$\lambda_{L,\mathfrak{M}}(H) \triangleleft A \quad \text{and} \quad H/\lambda_{L,\mathfrak{M}}(H) \leq \rho(A/\lambda_{L,\mathfrak{M}}(H));$$

$$\lambda_{L,\mathfrak{M}}(H) \triangleleft B \quad \text{and} \quad H/\lambda_{L,\mathfrak{M}}(H) \leq \rho(B/\lambda_{L,\mathfrak{M}}(H)).$$

Hence we have $\lambda_{L,\mathfrak{M}}(H) \triangleleft L$. Therefore it is enough to show that $H/\lambda_{L,\mathfrak{M}}(H) \leq \rho(L/\lambda_{L,\mathfrak{M}}(H))$. Now since $H \text{wser} L$ by Theorem 2, Lemma 1 indicates

$$H/\lambda_{L,\mathfrak{M}}(H) \subseteq \epsilon(L/\lambda_{L,\mathfrak{M}}(H)).$$
Here we denote images under the natural map $L \rightarrow L/\lambda_{\mathfrak{M}}(H)$ by bars. Then

$$L = \overline{A} + \overline{B} \in \mathfrak{l}(\text{ser})\mathfrak{F}, \quad \overline{H}\text{ser}\overline{A}, \quad \overline{H}\text{ser}\overline{B},$$

$$\overline{H} \leq \rho(\overline{A}) \cap \rho(\overline{B}), \quad \overline{H} \subseteq \varepsilon(\overline{L}),$$

because of [3, Proposition 13.2.4]. Hence we may replace $L, H, A, B$ by $L, H, A, B$.

Then by [13, Theorem 2] $L$ is, so-called, a neoclassical Lie algebra. That is to say, $L = \sigma(L) + \Lambda$, where $\Lambda$ is a direct sum of finite-dimensional, non-abelian simple subalgebras (see [3, Chapter 13]). As the first paragraph of the proof we set $L = L/\sigma(L) = A + B$. Then

$$L \cong \Lambda \in \mathfrak{l}(\prec)\mathfrak{F}, \quad \overline{H}\text{ser}\overline{A}, \quad \overline{H}\text{ser}\overline{B}.$$  

Moreover $\overline{A}, \overline{B} \in \overline{\varepsilon}(\prec)\mathfrak{F}$ owing to [14, Lemma 4.1]. Hence we have $\overline{H}\text{asc}\overline{A}, \overline{H}\text{asc}\overline{B}$ using [6, Theorem 1(1)]. Now we can derive from [2, Theorem 6] that $\overline{H}\text{asc}L$, so $H + \sigma(L)\text{asc}L$. Furthermore $H + \rho(L) \prec H + \sigma(L)$ owing to [3, Corollary 16.3.13]. Hence $H + \rho(L)\text{asc}L$. On the other hand we obtain $H \in \mathfrak{l}\mathfrak{M}$ by $H \leq \rho(A) \cap \rho(B)$. As $H \subseteq \varepsilon(L)$, $H$ acts on $\rho(L)$ by nil derivations, which indicates $H + \rho(L) \in \mathfrak{l}\mathfrak{M}$ by [3, Theorem 16.3.8(b)]. Thus we can reach $H + \rho(L) \leq \rho(L)$ by using [3, Theorem 13.3.7], that is, $H \leq \rho(L)$. This completes the theorem.

By making use of Theorem 8 and [6, Theorem 1(1)], we can obtain a better result than [2, Theorem 6].

**Corollary 9.** Let $L$ be a hyperfinite, serially finite Lie algebra over a field of characteristic zero and be factorized by $A$ and $B$. If $H$ is a common ascendant subalgebra of both $A$ and $B$, then $H$ is ascendant in $L$.

**Remark.** Over any field, $\mathfrak{l}(\prec)\mathfrak{F} < \overline{\varepsilon}(\prec)\mathfrak{F} \cap \mathfrak{l}(\text{ser})\mathfrak{F}$. For, let $X$ be an abelian Lie algebra with basis $\{x_0, x_1, \ldots\}$ and let $\sigma$ be the derivation of $X$ defined by $x_0\sigma = 0$ and $x_{i+1}\sigma = x_i$ ($i \geq 0$). Form the split extension $L = X + \langle \sigma \rangle$. Then $L \in \mathfrak{F} \leq \overline{\varepsilon}(\prec)\mathfrak{F} \cap \mathfrak{l}(\text{ser})\mathfrak{F}$ but $L \notin \mathfrak{l}(\prec)\mathfrak{F}$ (see [6, Remark 1]).

Proposition 7 and Corollary 9 directly lead the following:

**Corollary 10.** Let $L$ be a Lie algebra belonging to $(\varepsilon(\text{ch})\mathfrak{A})(\varepsilon(\prec)\mathfrak{F} \cap \mathfrak{l}(\text{ser})\mathfrak{F})$ over a field of characteristic zero and be factorized by $A$ and $B$. If $H\text{asc}A$ and $H\text{asc}B$, then $H\text{asc}L$.

Using Lemma 5 and Corollary 10, we can easily prove the following corollary, which is a generalization of [1, Corollaries 1 and 2].
Corollary 11. Let \( L \) be a Lie algebra belonging to \((\hat{\text{e}}(\text{ch})\mathfrak{A})(\hat{\text{e}}(\text{ser})\mathfrak{F}) \cap \mathfrak{L})\) over a field of characteristic zero and let \( X_i \) \( (i = 1, 2, \ldots, n) \) be subalgebras of \( L \) such that \( L = X_1 + X_2 + \cdots + X_n \) and \( \langle X_i, X_j \rangle = X_i + X_j \) for all \( i, j = 1, 2, \ldots, n \).

(1) If \( H \text{asc} X_i \) for all \( i \), then \( H \text{asc} L \).

(2) For each \( i \), if \( X_i \text{asc} \langle X_i, X_j \rangle \) for all \( j \), then \( X_i \text{asc} L \).

5. A Generalization for the Result of Goto and Panyukov

In this section we shall generalize the following result.

Lemma 12 (Goto, Panyukov). Let \( L \) be a finite-dimensional Lie algebra over a field of characteristic \( \neq 2 \). If \( L \) is represented as a sum of two nilpotent subalgebras \( A \) and \( B \), then \( L \) is soluble.

For our purpose we need the following two lemmas.

Lemma 13. Let \( H \) be a finitely generated subalgebra of a Lie algebra \( L \).

(1) If \( H \text{wasc} L \), then \( H^{(\omega)} \text{ch} L \).

(2) Assume that \( L \in \mathfrak{L} \). If \( H \text{wser} L \), then \( H^{(\omega)} \prec L \).

Proof. (1) Using [12, Theorem 4] we have \( H \leq^{\omega} L \). Hence [5, Lemma 2.10] leads \( H^{(\omega)} \prec L \). Next form the split extension \( M = L + \text{Der} L \). Then \( H \text{wasc} M \). The argument above indicates that \( H^{(\omega)} \prec M \), so \( H^{(\omega)} \text{ch} L \).

(2) For any \( I \prec H \) such that \( H/I \in \mathfrak{L} \mathfrak{A} \), we have \( H^{(\omega)} \leq I \) since \( H/I \in \mathfrak{E} \mathfrak{A} \). Therefore \( H^{(\omega)} \leq \lambda_{\mathfrak{L} \mathfrak{A}}(H) \). Since, in general, \( \lambda_{\mathfrak{L} \mathfrak{A}}(H) \leq H^{(\omega)} \), it follows from [5, Proposition 2.11] that \( H^{(\omega)} = \lambda_{\mathfrak{L} \mathfrak{A}}(H) \prec L \).

Lemma 14. Let \( L \) be a Lie algebra over a field of characteristic \( \neq 2 \) and let \( L = A + B \) be a sum of Engel subalgebras \( A \) and \( B \).

(1) If \( H \in \mathfrak{F} \) and \( H \text{wasc} L \), then \( H \in \mathfrak{E} \mathfrak{A} \).

(2) Assume that \( L \in \mathfrak{L} \mathfrak{F} \). If \( H \in \mathfrak{F} \) and \( H \text{wser} L \), then \( H \in \mathfrak{E} \mathfrak{A} \).

Proof. (1) Because \( H^{(\omega)} \) is a finite-dimensional ideal of \( L \) by Lemma 13, it follows from [3, Corollary 1.4.3] that

\[
C_L(H^{(\omega)}) \prec L \text{ and } L/C_L(H^{(\omega)}) \in \mathfrak{F}.
\]

Now we denote images under the natural map \( L \rightarrow L/C_L(H^{(\omega)}) \) by bars. Then we have \( \overline{L} \in \mathfrak{F} \) and \( \overline{L} = \overline{A} + \overline{B} \) is a sum of nilpotent subalgebras \( \overline{A} \) and \( \overline{B} \). Therefore Lemma 12 shows \( \overline{L} \in \mathfrak{E} \mathfrak{A} \). In particular, \( \overline{H} \in \mathfrak{E} \mathfrak{A} \), so \( H^{(\omega)} \subseteq C_L(H^{(\omega)}) \). Hence \( H^{(\omega + 1)} = [H^{(\omega)}, H^{(\omega)}] = 0 \). This concludes that \( H \in \mathfrak{E} \mathfrak{A} \).

(2) Since \( H^{(\omega)} \prec L \) by Lemma 13, we can show that \( H \in \mathfrak{E} \mathfrak{A} \) as in the proof of (1).
Now we shall prove the main theorem in the section, which is a generalization of Lemma 12.

**Theorem 15.** Let $L$ be a Lie algebra over a field of characteristic $\neq 2$. If $L \in L(\text{wser})\mathfrak{F}$ and $L$ is represented as a sum of two locally nilpotent subalgebras $A$ and $B$, then $L$ is locally soluble.

**Proof.** Let $X$ be a finite subset of $L$. Then there exists a subalgebra $H$ of $L$ such that $X \subseteq H\text{wser}L$ and $H \in \mathfrak{F}$. Therefore it follows from Lemma 14(2) that $H \in \mathfrak{E}A$. Thus $L \in \mathfrak{LE}A$. \(\square\)

Finally we shall state about any subalgebra of the intersection of permutable two locally nilpotent subalgebras.

**Corollary 16.** Let $L$ be a Lie algebra over a field of characteristic $\neq 2$ and let $L$ be factorized by two locally nilpotent subalgebras $A$ and $B$.

1. If $L \in L(\text{wser})\mathfrak{F}$, then $H\text{ser}L$ for any subalgebra $H$ of $A \cap B$.
2. If $L \in L(<)\mathfrak{F}$, then $H <\omega L$ for any subalgebra $H$ of $A \cap B$.

**Proof.** (1) Using [3, Proposition 13.2.4] we obtain $H\text{ser}A$ and $H\text{ser}B$. Since $L \in \mathfrak{LE}A$ by Theorem 15 we conclude from Corollary 3 that $H\text{ser}L$.

(2) From (1) we have $H\text{ser}L \in \mathfrak{LE}A \cap L(<)\mathfrak{F} = L(<)(\mathfrak{E}A \cap \mathfrak{F})$. Therefore $H <\omega L$ in virtue of [5, Theorem 3.3]. \(\square\)

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**References**


MASANOBU HONDA
NIIGATA COLLEGE OF PHARMACY
NIIGATA 950-2081 JAPAN

TAKANORI SAKAMOTO
DEPARTMENT OF MATHEMATICS
FUJOKA UNIVERSITY OF EDUCATION
MUNAKATA, FUKUOKA 811-4192 JAPAN

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