The Taylor coefficients of $\zeta(s),(s-1)\zeta(s)$ and 
$(z/(1-z))\zeta(1/(1-z))$

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THE TAYLOR COEFFICIENTS OF 
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1. Introduction. The coefficients $\Gamma_k$ in the Laurent expansion of the Riemann zeta-function at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \Gamma_k (s-1)^k$$  \hspace{1cm} (1)

have been considered by many mathematicians. (Stieltjes [4], Briggs and Chowla [1], Berndt [2], Israelov [3], recently Matsuoka [5] and other authors. See references in [2, 3] and Ivić [4], p. 49.) Among them Briggs and Chowla, Berndt, Israelov and Matsuoka have made the interesting contributions to the estimation of $|\Gamma_k|$. In this paper, we consider the coefficients $D_k$, $D_k$ and $c_k$ in the Taylor expansions of $\zeta(s)$, $(s-1)\zeta(s)$ and $(z/(1-z))\zeta(1/(1-z))$ at any point $s_0 \neq 1$ and at $z = 0$ respectively:

$$\zeta(s) = \sum_{k=0}^{\infty} D_k (s-s_0)^k, \ (s_0 \neq 1)$$ \hspace{1cm} (2)

$$\ (s-1)\zeta(s) = \sum_{k=0}^{\infty} D_k (s-s_0)^k, \ (s_0 \neq 1)$$ \hspace{1cm} (3)

$$\ (z/(1-z))\zeta(1/(1-z)) = \sum_{k=0}^{\infty} c_k z^k, \ (|z| < 1)$$ \hspace{1cm} (4)

(4) is the transformation of $(s-1)\zeta(s)$ from the complex $s$-plane to the complex $z$-plane by the Möbius transformation $s = 1/(1-z)$ which transforms the right half $s$-plane $|s| > 1/2$ into the $z$-disk $|z| < 1$. Note that the Riemann hypothesis is true if and only if (4) has no zero in the unit disk $|z| < 1$. ([9])

In § 2, we shall give the explicit expressions of $D_k$, $D_k$ and $c_k$ which are our generalizations of the Stieltjes formula [4]. The Stieltjes formula has seemed to be thought as the special case that such a formula exists because the point $s = 1$ is the only pole of $\zeta(s)$. In [2], Berndt states the estimate:

$$|\Gamma_k| \leq 3 + (-1)^k/k \pi^k \leq 4/k \pi^k$$  \hspace{1cm} (5)

but it can be easily seen that his argument leads to the following result:
\[ |\Gamma_k| \leq 4(2\pi)^{-k}N^{-k-1}\prod_{j=1}^{N} (N+j) \]  
(6)

for any integer \( N \) such that \( 1 \leq N \leq k \). In particular, if we take \( N = \lfloor k/2 \rfloor \) where \( \lfloor \cdot \rfloor \) is the Gauss symbol in (6), then

\[ |\Gamma_k| \leq 4\pi^{-\lfloor k/2 \rfloor}[k/2]^{-\lfloor k/2 \rfloor}\prod_{j=1}^{\lfloor k/2 \rfloor} (k/2+j)(2k/2)^{-1} \]  
(7)

\[ \leq 4[k/2]^{-1}\pi[k/2]^{-k/2} \]  
(8)

In § 3, we shall also give the estimation for the value \( |D_k^*|, |D_k| \) and \( |c_k| \) by the argument similar to that of Berndt [2]. As for the estimation of Taylor's coefficients of \( \zeta(s) \), Mitrović [6, 7] showed the following result:

\[ |D_k^*| \leq (\sigma_0-1)^{-k-1}, (k \geq 0) \]  
(9)

where \( \sigma_0 = \text{Re } s_0 \) and \( \sigma_0 > 1 \). Our result in case \( \sigma_0 > 1 \) is sharper than that of Mitrović.

We use hereafter the following notations:

The letter \( a \) always takes the value 1 or 0.

\[ \binom{n}{r} \] denotes the binomial coefficient, \( f_k(u) = (k!)^{-1}u^ke^u = \sum_{r=k}^{\infty} \binom{r}{k}(r!)^{-1}u^r \).

\[ f^m_{\alpha k}(u) = (d^m/du^m)f_k(u), \quad s_0 = 1 + s_1, \quad g_{\alpha k}(u) = u^{-1}f^m_{\alpha k}(-s_1\log u), \quad g^m_{\alpha k}(u) = (d^m/du^m)g_{\alpha k}(u), \quad o(\cdot) \text{ denotes Landau's small } o \text{ symbol, } O(\cdot) \text{ denotes Landau's large } O \text{ symbol, } c_{\nu}(j) \text{ denotes the Stirling number of the first kind which is defined by the relation:} \]

\[ (z-1)(z-2)\ldots(z-\nu) = \sum_{j=0}^{\nu} c_{\nu}(j)z^j, \]
\[ c_{\nu}(0) = 1, \quad c_{\nu}(-1) = c_{\nu}(\nu+1) = 0. \]

Our \( c_{\nu}(j) \) is equal to \( a_{j+1}^{(\nu+1)} \) in Berndt [2] and to \( (-1)^{\nu+j}\nu!b_{\nu+j} \) in Israilov [3]. \( B_n(u) \) denotes the Bernoulli polynomial which is defined by

\[ ze^{zu}/(e^z-1) = \sum_{n=0}^{\infty} (n!)^{-1}B_n(u)z^n, (|z| < 2\pi) \]

\( B_n \) denotes the Bernoulli number which is equal to \( B_n(0) = (n!)^{-1} \).

\( B_n(|u|) \) where \( |u| = u-\lfloor u \rfloor \), \( \lfloor u \rfloor \) is the Gauss symbol which denotes the greatest integer not exceeding \( u \). \( h_r(u) = u^{-1}(\log u)^r \), \( h^{(m)}_r(u) = (d^m/du^m)h_r(u) \).
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$$B_{k,\nu}(n) = \sum_{r=0}^{\min(n,k)} \binom{n}{r} f_{k-r}(-s_1 \log u)$$

$L_k(u)$ denotes the Laguerre polynomial which is defined by

$$L_k(u) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r (r!)^{-1} u^r.$$

$s_0 = 1 + s_1$. \(\mathbb{N}, \mathbb{Z}, \mathbb{C}\) and \(\mathbb{R}\) denote the set of natural numbers, integers, complex numbers and real numbers respectively.

2. The explicit expressions of $D'_k$, $D_k$ and $c_k$. $\zeta(s)$ can be expanded at any point $s = s_0$, $(s_0 \neq 1)$ through (1) as follows:

$$\zeta(s) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k! (s-1)^n} + \sum_{n=k}^{\infty} \frac{n!}{k!} \Gamma_n(s_0-1)^n (s-s_0)^k,$$

so that $D'_k$ in (2) is

$$D'_k = -(1-s_0)^{k-1} + D^{(0)}_{k}$$

where

$$D^{(0)}_{k} = \left\{ \begin{array}{ll}
\sum_{n=k}^{\infty} \binom{n}{k} \Gamma_n s_1^{n-k}, & (k \geq 1) \\
\sum_{n=0}^{\infty} \Gamma_n s_1^n = \zeta(s_0), & (k = 0) 
\end{array} \right. $$

$s_0 = 1 + s_1$

And similarly we have

$$D_k = D^{(1)}_{k} = \left\{ \begin{array}{ll}
\sum_{n=k}^{\infty} \binom{n}{k} \Gamma_{n-1} s_1^{n-k}, & (k \geq 1) \\
1 + \sum_{n=0}^{k} \Gamma_n s_1^{n+1} = (s_0-1) \zeta(s_0), & (k = 0) 
\end{array} \right. $$

and

$$c_k = \left\{ \begin{array}{ll}
\sum_{n=0}^{k-1} \binom{k-1}{n} \Gamma_{k-1-n} = \sum_{n=0}^{k-1} \binom{k-1}{n} \Gamma_n, & (k \geq 1) \\
1, & (k = 0) \end{array} \right. $$

Next we consider the following sum:

$$\sum_{n=1}^{\infty} E_{\alpha,\nu}(n), \ (k \geq 1)$$

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Before the calculation of (14), we need the following two lemmas. As for the Stieltjes formula [4]:

$$\gamma_r = (-1)^r r! \Gamma_r = \lim_{x \to \infty} \left\{ \sum_{n=1}^{\infty} n^{-\ell}(\log n)^{r-1} \int_1^x u^{-\ell}(\log u)^{r} du \right\}$$  \hspace{1cm} (15-a)$$

or

$$\sum_{n=1}^{\infty} n^{-\ell}(\log n)^{r-1} \int_1^x u^{-\ell}(\log u)^{r} du = \gamma_r + E(r, x),$$ \hspace{1cm} (15-b)

$$E(r, x) = o(1)$$

we have

Lemma 1. For the above $E(r, x)$, we have

$$E(r, x) = (1/2)x^{-\ell}(\log x)^r + \sum_{m=2}^{\infty} (-1)^m(B_m/m!)(m-1)_{m-1}^{-1}h_m^{(m-1)}(x)$$

$$+(-1)^N \int_x^\infty P_N(u)h_N^{(r)}(u) du, \quad (r \geq 0, N \geq 1)$$ \hspace{1cm} (16)

where

$$h_m^{(m)}(u) = u^{-m-1} \sum_{j=0}^{m} c_m(j) r(r-1)$$

$$\cdots (r-j+1)(\log u)^{r-j}, \quad (m \geq 0, r \geq 0)$$ \hspace{1cm} (17)

Proof. An easy calculation gives (17) (see Israiov [3]). And the Euler-Maclaurin summation formula yields

$$\sum_{n=1}^{\infty} h_r(n) - \int_1^x h_r(u) du = (1/2)h_r(x) + \sum_{m=2}^{\infty} (-1)^m(B_m/m!)(m-1)_{m-1}^{-1}h_m^{(m-1)}(x)$$

$$- \sum_{m=2}^{\infty} (-1)^m(B_m/m!)(m-1)_{m-1}^{-1}h_m^{(m-1)}(1)$$

$$+(-1)^N \int_1^x P_N(u)h_N^{(r)}(u) du$$ \hspace{1cm} (18)

which shows

$$\gamma_r = - \sum_{m=2}^{\infty} (-1)^m(B_m/m!)(m-1)_{m-1}^{-1}h_m^{(m-1)}(1)$$

$$+(-1)^N \int_1^\infty P_N(u)h_N^{(r)}(u) du$$ \hspace{1cm} (19)
We substitute (19) and (18) for $\gamma_r$ in (15-b) and the left-hand side of (15-b) respectively, then we have the lemma.

**Lemma 2.**

$$g_{m-1}(u) = u^{-m-1} \sum_{r=0}^{\infty} c_m(r) B_{kr}(r+a)(-s_1)^r, \quad (m \geq 0, \quad k \geq 1) \quad (20)$$

**Proof.** This lemma is proved by induction on $m$ by using the properties that $(d/du)f_k(u) = f_k(u) + f_{k-1}(u), \quad (k \geq 1), \quad (d/du)B_{kr}(n) = -s_1 u^{-1} B_{kr}(n + 1), \quad (k \geq 1)$ and $c_j = c_{j-1} + (-m-1)c_{j-1}$.

By an easy calculation and applying (15) to (14), we have

$$\sum_{n=1}^{\infty} g_{\omega_k}(n) = \sum_{n=1}^{\infty} n^{-1} \left[ \sum_{r=0}^{\infty} \binom{r}{k} (-1)^{r-a} (r-a)! (-1)^{s_1 \log n} r-a \right]$$

$$= \sum_{r=0}^{\infty} \binom{r}{k} s_1^{-a} (-1)^{r-a} \log n^{-1} (r-a)! E(r-a, x) + S(a, k) \quad (21)$$

where $S(a, k) = \sum_{r=0}^{\infty} \binom{r}{k} (-1)^{r-a} (r-a)! (-1)^{s_1 \log n} r-a \log n^{-1} E(r-a, x)$.

As for the above $S(a, k)$, we have

**Lemma 3.**

$$S(a, k) = \begin{cases} o(1), \quad (\text{Re } s_0 > 0) \\ (2x)^{-1} \sum_{v=0}^{a} \binom{a}{v} f_{k-v}(-s_1 \log x) + o(1), \\ (-1 < \text{Re } s_0 \leq 0) \\ (2x)^{-1} \sum_{v=0}^{a} \binom{a}{v} f_{k-v}(-s_1 \log x) \\ + \sum_{m=2}^{\infty} (-1)^m (B_m/m!) g^{m-1}_a(x) \quad (\text{Re } s_0 \leq -(M-1)) \end{cases}$$

**Proof.** From Lemma 1 and Lemma 2.

$$S(a, k) = (2x)^{-1} f_{k}^{(-1)}(-s_1 \log x)$$

$$+ \sum_{m=2}^{\infty} (-1)^m (B_m/m!) g^{m-1}_a(x)$$
\[ (+1)^{n} \int_{x}^{\infty} P_{n}(u) g_{x_{a_k}}(u) \, du \]
\[ = (2x)^{-1} \sum_{v=0}^{n} \binom{n}{v} f_{k-v}(-s_{1} \log x) + \sum_{m=2}^{n} (-1)^{m}(B_{m}/m!) \]
\[ x^{-m} \sum_{r=0}^{m-1} c_{m-1}(r) \min_{v=0}^{r} \binom{r+a}{v} f_{k-v}(-s_{1} \log x)(-s_{1})^{r} \]
\[ + (-1)^{n} \int_{x}^{\infty} P_{n}(u) (d^{n}/d\ln u) g_{x_{a_k}}(u) \, du \]

Noting that \( f_{k}(-s_{1} \log x) = O(x^{-\sigma_{0}+\epsilon}) \) (\( \epsilon \) is an arbitrary small positive number, \( \sigma_{0} = \text{Re } s_{1} \)) and
\[
\left| \int_{1}^{\infty} P_{n}(u) g_{x_{a_k}}(u) \, du \right| \leq \int_{1}^{\infty} |g_{x_{a_k}}(u)| \, du < +\infty,
\]
we have
\[
S(a, k) = O(x^{-\sigma_{0}+\epsilon}) + \sum_{m=2}^{m} O(x^{-\sigma_{m}-1-\sigma_{0}+\epsilon}) + o(1), \ (\sigma_{0} = \text{Re } s_{0})
\]
and have the desired result.

From (21) and Lemma 3, we obtain the following

**Theorem 1.** (A generalization of the Stieltjes formula). If \( \text{Re } s_{0} > 0 \), then

\[
D'_{k} = \frac{ (-s_{1})^{-k-1} + D^{(1)}_{k} }{ (2) }
\]
\[
= \frac{ (-s_{1})^{-k-1} + s_{1} \lim_{x \to \infty} \left\{ \sum_{n=1}^{x} n^{-1}f_{k}(-s_{1}, \log n) \right\} }{ (22-a) }
\]
\[
- \int_{1}^{x} u^{-1}f_{k}(-s_{1}, \log u) \, du \right\},
\]
\[
D_{k} = D^{(1)}_{k}
\]
\[
= s_{1}^{k+1} \lim_{x \to \infty} \left\{ \sum_{n=1}^{x} n^{-1}f^{(1)}_{k}(-s_{1}, \log n) \right\}
\]
\[
- \int_{1}^{x} u^{-1}f^{(1)}_{k}(-s_{1}, \log u) \, du \right\}, \quad (22-b)
\]

If \( -1 < \text{Re } s_{0} \leq 0 \), then

\[
D'_{k} = \frac{ (-s_{1})^{-k-1} + D^{(1)}_{k} }{ (2) }
\]
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\[
= -(-s_1)^{-k-1} + s_1^{-k} \lim_{x \to \infty} \left[ \sum_{n=1}^{\infty} n^{-1} f_k(-s_1 \log n) \right. \\
- \left. \int_1^x u^{-1} f_k(-s_1 \log u) du - (2x)^{-1} f_k(-s_1 \log x) \right] \tag{23-a}
\]

and

\[
D_k = D_k^{(1)} \\
= s_1^{-k-1} \lim_{x \to \infty} \left[ \sum_{n=1}^{\infty} n^{-1} f_k^{(1)}(-s_1 \log n) \right. \\
- \left. \int_1^x u^{-1} f_k^{(1)}(-s_1 \log u) du - (2x)^{-1} f_k^{(1)}(-s_1 \log x) \right] \tag{23-b}
\]

And if $-M < \Re s_0 \leq -(M-1)$, $M \in \mathbb{N}$, $M \geq 2$, then

\[
D_k = -(-s_1)^{-k-1} + D_k^{(0)} \\
= -(-s_1)^{-k-1} + s_1^{-k} \lim_{x \to \infty} \left[ \sum_{n=1}^{\infty} n^{-1} f_k(-s_1 \log n) \right. \\
- \left. \int_1^x u^{-1} f_k(-s_1 \log u) du - (2x)^{-1} f_k(-s_1 \log x) \right. \\
- \left. \sum_{m=2}^{M} (-1)^m (B_m/m!) g^{(m-1)}_{-1,0,k}(x) \right] \tag{24-a}
\]

and

\[
D_k = D_k^{(1)} \\
= s_1^{-k-1} \lim_{x \to \infty} \left[ \sum_{n=1}^{\infty} n^{-1} f_k^{(1)}(-s_1 \log n) \right. \\
- \left. \int_1^x u^{-1} f_k^{(1)}(-s_1 \log u) du - (2x)^{-1} f_k^{(1)}(-s_1 \log x) \right. \\
- \left. \sum_{m=2}^{M} (-1)^m (B_m/m!) g^{(m-1)}_{-1,1,k}(x) \right] \tag{24-b}
\]

Note the following

Remark 1. When $\Re s_0 > 0$,

\[
D_k^{(0)} = (-1)^k k !^{-1} \lim_{x \to \infty} \left[ \sum_{n=1}^{\infty} n^{-s_0} (\log n)^k \right. \\
- \left. \int_1^x u^{-s_0} (\log u)^k du \right] \tag{25-a}
\]

and
When \(-1 < \text{Re } s_0 \leq 0\),
\[
D^{(0)}_k = (-1)^k k!^{-1} \lim_{x \to \infty} \left[ \sum_{n=1}^{x} n^{-s_0} (\log n)^k \right. \\
\left. - \int_{1}^{x} u^{-s_0} (\log u)^k du - (2x)^{-1}(\log x)^k \right],
\tag{25-b}
\]

and
\[
D^{(1)}_k = s_1 D^{(0)}_k + D^{(0)}_{k-1}.
\]

We have the similar result for \(\text{Re } s_0 \leq -1\), but in this case, the expression of \(D^{(0)}_k\) is more complicated.

A similar calculation gives the following

**Theorem 2. (Another generalization of the Stieltjes formula).** For \(c_{k+1}\) in (4),
\[
c_{k+1} = \lim_{x \to \infty} \left[ \sum_{n=1}^{x} n^{-1} L_k(\log n) - \int_{1}^{x} u^{-1} L_k(\log u) du \right]
\tag{26}
\]
where \(L_k(\ )\) denotes the Laguerre polynomial.

**Proof.** Instead of (14), we start from the following sum:
\[
\sum_{n=1}^{x} n^{-1} L_k(\log n)
\]
and we omit the calculation which is the same as that of Theorem 1. (In fact the calculation is simpler than that of Theorem 1, for \(L_k\) is the finite sum.) The expressions (22), (23), (24), (26) are the generalizations of the Stieltjes formula (15).

3. The estimation of \(|D^{(0)}_k|\) and \(|c_k|\). To estimate \(|D^{(0)}_k|\), we need the following

**Remark 2.**
\[
g^{(m)}_{\alpha, \epsilon}(x) = O(x^{-m-\alpha+\epsilon}), \quad \sigma = \text{Re } s_0, \quad (m \geq 0, \ k \geq 1)
\]
\[
g^{(m)}_{\alpha, \epsilon}(1) = 0, \quad (0 \leq m \leq k - 1 - \alpha)
\]
which can be seen from Lemma 2.
Theorem 3. For \( s_0 \neq 1, -M < \Re s_0, 1 \leq M \leq N \leq k-a, k \geq 1, M \in \mathbb{N} \)

\[
D^\alpha_k = (-1)^{N+1}s_1^{a-k} \int_1^\infty P_n(u) \frac{u^N}{\alpha \kappa(k)} \, du
\]  
(27)

Proof. We apply the Euler-Maclaurin summation formula to the sum (14) and use Remark 2 and Theorem 1 to obtain (27). Say, for \( s_0 > 0 \).

\[
D^{(a)}_k = s_1^{a-k} \lim_{x \to \infty} \left[ \sum_{n=1}^x g_{\alpha \kappa}(n) - \int_1^x g_{\alpha \kappa}(u) \, du \right]
\]

\[
= s_1^{a-k} \lim_{x \to \infty} \left\{ \frac{1}{2}(g_{\alpha \kappa}(1) + g_{\alpha \kappa}(x)) \right. \\
+ \sum_{m=2}^\infty (-1)^m (B_m/m!) g_{\alpha \kappa}^{m-1}(u) \bigg|_{u=1}^{u=x} \right. \\
\left. + (-1)^{N+1} \int_1^x P_n(u) \frac{u^N}{\alpha \kappa(k)} \, du \right\}
\]

For \(-M < \Re s_0 \leq -(M-1), (M \geq 1)\)

\[
D^{(a)}_k = s_1^{a-k} \lim_{x \to \infty} \left[ \sum_{n=1}^x g_{\alpha \kappa}(n) \right. \\
\left. - \int_1^x g_{\alpha \kappa}(u) \, du - \frac{1}{2} g_{\alpha \kappa}(x) \right. \\
\left. - \sum_{m=2}^M (-1)^m (B_m/m!) g_{\alpha \kappa}^{m-1}(x) \right]
\]

\[
= s_1^{a-k} \lim_{x \to \infty} \left\{ \frac{1}{2} g_{\alpha \kappa}(1) \right. \\
+ \sum_{m=2}^N (-1)^m (B_m/m!) g_{\alpha \kappa}^{m-1}(u) \bigg|_{u=1}^{u=x} \right. \\
\left. - \sum_{m=2}^M (-1)^m (B_m/m!) g_{\alpha \kappa}^{m-1}(x) \right. \\
\left. + (-1)^{N+1} \int_1^x P_n(u) \frac{u^N}{\alpha \kappa(k)} \, du \right\}
\]

Hence, when \( x \) tends to infinity, the desired result follows.

From Theorem 3, we can estimate \(|D^{(a)}_k|\) as follows.

Theorem 4. For every fixed \( s_0 \neq 1, -M < \Re s_0, 1 \leq M \leq N \leq k-a, M \in \mathbb{N}, \)
\[ |D^{(a)}_k| \leq |s_1| + N + \alpha a |4(2 \pi)^{-N} (N + \alpha)^{-k - 1} \prod_{r=1}^k (|s_1| + N + \alpha + r), \] (28)

where \( \alpha = \text{Re } s_0 - 1 \). In particular, if we choose \( N = \lfloor k/2 \rfloor \), then

\[ |D^{(a)}_k| \leq 4\pi^{-k/2}(\lfloor k/2 \rfloor + \alpha)^{-k-1+\lfloor k/2 \rfloor} \prod_{r=1}^{\lfloor k/2 \rfloor} (|s_1| + \lfloor k/2 \rfloor + \alpha + r) \]
\[ \times \frac{|2[\lfloor k/2 \rfloor + 2\alpha]|}{1 + \alpha b}. \] (29-a)

\[ |D_k| \leq 4\pi^{-k/2}(\lfloor k/2 \rfloor + \alpha)^{-k-1+\lfloor k/2 \rfloor} \prod_{r=0}^{\lfloor k/2 \rfloor} (|s_1| + \lfloor k/2 \rfloor + \alpha + r) \]
\[ \times \frac{|2[\lfloor k/2 \rfloor + 2\alpha]|}{1 + \alpha b}. \] (29-b)

where \( b \) is an arbitrary positive number. And if \( k \geq |s_1|/(b-1)+2 \), \( 1 < b < \pi \), then

\[ |D^{(a)}_k| \leq 4(\pi/b)^{-k/2}\lfloor k/2 \rfloor + \alpha)^{-k-1+\lfloor k/2 \rfloor}, \] (30-a)

\[ |D_k| = |D^{(a)}_k| \leq 4(\pi/b)^{-k/2}\lfloor k/2 \rfloor + \alpha)^{-k-1+\lfloor k/2 \rfloor}, \] (30-b)

**Proof.** From (27) and Remark 2,

\[ D^{(a)}_k = (-1)^{N+1} s_1 \alpha^{-k} \sum_{r=0}^{\infty} \sum_{j=0}^{a+r} \binom{a+r}{j} (-s_1)r \cdot \sum_{r=0}^{\infty} \sum_{j=0}^{a+r} \binom{a+r}{j} (-s_1) \int_1^\infty |(k-j)!|^{-1}P_N(u)(-s_1 \log u)^{k-j} u^{-N-s_0} du \]

The above equality leads to the following estimation:

\[ |D^{(a)}_k| \leq |s_1| \sum_{r=0}^{\infty} \sum_{j=0}^{a+r} \binom{a+r}{j} \cdot \sum_{r=0}^{\infty} \sum_{j=0}^{a+r} \binom{a+r}{j} \cdot \int_1^\infty |(k-j)!|^{-1} u^{-N-s_0} du \cdot 4(2 \pi)^{-N}. \]

because \( |P_N(u)| \leq 4(2 \pi)^{-N}. \) Since the integral in the right-hand side of the above inequality is equal to \( (N+\alpha)^{-k-1} \),

\[ |D^{(a)}_k| \leq |s_1| \sum_{r=0}^{\infty} \sum_{j=0}^{a+r} \binom{a+r}{j} \cdot \sum_{r=0}^{\infty} \sum_{j=0}^{a+r} \binom{a+r}{j} \cdot |c_\alpha(r)| \cdot |s_1|^r. \]
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\begin{align*}
&\sum_{j=0}^{\infty} \binom{a+r}{j} |s_1|^{-j(N+a)+j(N+a)^{-k-1}} \\
&= |s_1|^{a} 4(2\pi)^{-N} \sum_{r=0}^{N} |c_N(r) | |s_1|^r. \\
&1 + (N+a)/|s_1|^{a-r}(N+a)^{-k-1} \\
&= (|s_1| + N+a)^{a} 4(2\pi)^{-N}(N+a)^{-k-1} \prod_{r=0}^{N} |s_1| + N+a+r | \\
&\text{by using the definition of } c_N(r). \text{ This completes the proof.}
\end{align*}

Remark 3. If we use the expression in Remark 1, we can have the better estimation of $|D^{|\alpha|}_k|$ applying the Euler-Maclaurin summation formula to $\sum_{n=1}^{N} n^{-sn}(\log n)^k$, say

\begin{equation}
|D^{|\alpha|}_k| \leq 4(2\pi)^{-N}(N+a)^{-k-1} \prod_{j=0}^{N-1} |N+a| + |s_0+j|, \ (N \leq k) \tag{31}
\end{equation}

We also have some expressions of $c_k$ (Theorem 5). For the proof of the theorem, we use the following two lemmas.

Lemma 4.

\begin{align*}
(d^m/du^m)|u^{-1}L_k(\log u)| \\
= u^{-m-1} \sum_{j=0}^{\min\{m,k\}} c_m(j) \sum_{r=j}^{k} \binom{k}{r} (r-j)! |\log u|^{r-j}(-1)^r, \ (m \geq 1)
\end{align*}

Proof. This lemma is also proved by induction on $m$.

Lemma 5.

\begin{equation}
(d^m/dx^m)|x^{-1}L_k(\log x)| = o(1), \ (m \geq 0) \tag{32}
\end{equation}

Proof. Obvious.

From Lemma 5, the following theorem is to be proved.

Theorem 5.

\begin{equation}
c_{k+1} = \sum_{n=1}^{h-1} n^{-1}L_k(\log n) \\
- \int_{0}^{\log h} L_k(t)dt + (1/2)h^{-1}L_k(\log h)
\end{equation}
\[ + \sum_{m=2}^{k} (-1)^{m} (B_{m}/m!) \min_{j=0}^{m-1,k} c_{m-1}(j) \cdot \sum_{r=0}^{k} \binom{k}{r} \left( \frac{1}{r} \right)^{1}(\log h)^{r}(-1)^{r} \]

\[ + (-1)^{N+1} \int_{h}^{\infty} P_{N}(u) u^{-N} \min_{j=0}^{N,k} c_{N}(j) \cdot \sum_{r=0}^{N} \binom{N}{r} \left( \frac{1}{r} \right)^{1}(\log u)^{r}(-1)^{r} du. \quad (N \geq 1, h \geq 1) \quad (32) \]

Especially, for \( h = 1 \),

\[ c_{k+1} = (1/2) + \sum_{m=2}^{k} (-1)^{m} (B_{m}/m!) \min_{j=0}^{m-1,k} c_{m-1}(j) \binom{k}{j}(-1)^{j} \]

\[ + (-1)^{N+1} \int_{1}^{\infty} P_{N}(u) u^{-N} \min_{j=0}^{N,k} c_{N}(j) \cdot \sum_{r=0}^{N} \binom{N}{r} \left( \frac{1}{r} \right)^{1}(\log u)^{r}(-1)^{r} du. \quad (N \geq 1) \quad (33) \]

In particular,

\[ c_{k+1} = (1/2) + \int_{1}^{\infty} |u| u^{-2} L_{k+1}(\log u) du \]

\[ = (1/2) + \int_{0}^{\infty} |e^{t}| e^{-t} L_{k+1}(t) dt. \quad (34) \]

where \(|u| = u - [u], [ ]\) is Gauss's symbol and \( L_{n}'(x) = (d/dx) L_{n}(x) \).

Proof. The proof is the same as that of Theorem 3. (Note that \((d/dx) \cdot \{ L_{n}(x) - L_{n-1}(x) \} = L_{n}(x) \))

As for the estimation of \(|c_{k}|\). the result is not so good as that of \(|D^{(\alpha)}_{n}|\). (From (13), the trivial result that \(|c_{k}| \leq 2^{k}\) is derived.) so we conclude with the following two

Conjecture 1.

\[ c_{k} = O(1) \]

Conjecture 2.

\[ c_{k+1} - c_{k} > 0 \text{ for all integers } k \geq 1 \]

in other words,

\[ \int_{1}^{\infty} |u| u^{-2} L_{k}(\log u) du < 0 \text{ for all integers } k \geq 1. \]
The Taylor coefficients of $\zeta(s)$, $(s-1)\zeta(s)$ and $(z/(1-z))\zeta(1/(1-z))$

The similar results for some other Dirichlet series and sharper results depending on $s_0$, as for the upper estimates of the Taylor coefficients of $\zeta(s)$ by using the complex integral and the saddle point method can be obtained. These results will be treated in the next paper [8].

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