On artinian rings of right local type

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ON ARTINIAN RINGS OF RIGHT LOCAL TYPE

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Let $A$ be an artinian ring with identity. Then $A$ is called of right local type if any finitely generated indecomposable right $A$-module is local and a ring of left colocal type is defined as the dual version. Yoshii [9] and Tachikawa [8] studied finite dimensional algebras of right local type (or equivalently of left colocal type). In the following theorems, for a module $L$, $\ell_L$ denotes the composition length of $L$, $\overline{L}$ the top of $L$ and $p(A)$ the set of primitive idempotents of $A$.

Theorem A (Tachikawa [8], Yoshii [9]). For a finite dimensional algebra $A$ with Jacobson radical $J$, the following conditions (a) $\sim$ (c) are equivalent.

(a) $A$ is of right local type.
(b) $A$ is of left colocal type.
(c) (1) $A$ is left serial.
(2) $eJ$ is a serial module (i.e. a direct sum of uniserial modules) with $\ell_{eJ} \leq 2$ for each $e \in p(A)$.

But in [8] and [9] they essentially used the property “$A$ has the self duality” (and in [9] there are some mistakes; see the footnote of [8, p. 227]). On the other hand in [7]. Tachikawa studied artinian rings of left colocal type (see [6]). Hence in this paper we study artinian rings of right local type as the dual notion of rings of left colocal type and in particular we prove the following (see Section 4):

Theorem B. Let $A$ be an artinian ring with Jacobson radical $J$. Consider the following conditions:

(a) $A$ is of right local type.
(b) $A$ is of left colocal type.
(c) (1) $A$ is left serial.
(2) $eJ$ is serial with $\ell_{eJ} \leq 2$ for any $e \in p(A)$.
(d) (1) $A$ is left serial.
(2) $J$ is serial as a right $A$-module.
(3) If $eJ$ is not homogeneous. then $\ell_{eJ} \leq 2$, where $e \in p(A)$.
If $U$ is a uniserial left $A$-module with $|U| \geq 2$, then $[D(U) : D_r(U)]_r \leq 2$.

Then it holds the following (i) and (ii) (see Section 4 for the definition $[D(U) : D_r(U)]_r$ and the condition (D)).

(i) (a) implies (d) and under the condition (D) (d) implies (a).
(ii) (a) and (b) are satisfied if and only if so are (c) and (D).

In Section 3 we give necessary and sufficient conditions for a left serial ring $A$ to have the right serial Jacobson radical. Moreover, in Section 5 we give a simple proof of [6, Example 3] using the dual assertion to one obtained by Asashiba.

Throughout this paper $A$ is a left and right artinian ring with identity. $J$ is the Jacobson radical of $A$ and all modules are finitely generated unitary right $A$-modules (except for Section 5) unless otherwise stated. We call a module $M$ serial if $M$ is a direct sum of uniserial modules, and call an element $a$ of $A$ right local (resp. local) if $a = af$ (resp. $a = eaf$) for some $e, f \in p(A)$. For a module $M$ and a subset $I$ of $A$ we use the following notations (refer [6] for the other definitions and notations):

- $\overline{M}$: the top of $M$ (i.e. $\overline{M} = M/MJ$).
- $\text{Soc}(M)$: the socle of $M$.
- $|M|$: the composition length of $M$.
- $p(A)$: the set of primitive idempotents of $A$.
- $t(eJ^rf) = eJ^rf \backslash eJ^{r+1}f$, where $r$ is an integer $\geq 0$ and $e$ and $f$ are idempotents of $A$.
- $\text{loc}(I)$ (resp. $\text{loc}(I)$): the set of local (resp. right local) elements in $I$.
- $[D : D_r]_r$ (resp. $[D : D_r']_r$): the dimension of the left (resp. right) vector space $D$ over $D_r$ where $D$ is a division ring and $D_r'$ is a division subring of $D$.

The author has useful communication (in particular for Proposition 1.3 and Lemma 1.4) with Doctor H. Asashiba. In [2] Baba and Harada have independently obtained the same results as some ones in Section 2.

1. Preliminaries. Let $M_i$ be a module and $T_i$ its submodule; $i = 1, 2$. If a homomorphism $\theta : T_1 \to T_2$ has some extension map $\phi : M_1 \to M_2$ (i.e. $\phi(a) = \theta(a)$ for any $a \in T_1$), then $\theta$ is called $(M_1, M_2)$-extendible. On the other hand $\theta$ is called $(M_1, M_2)$-maximal if $\theta$ is not $(M_1', M_2)$-extendible for any submodule $M_i'$ with $T_1 \cong M_i' \subset M_i$.

Let $\alpha_1, \alpha_2 \in T : T \to M_1 \oplus M_2$ be a monomorphism with a monomorphism $\alpha_1$ (where $(\alpha_1, \alpha_2)^T$ denotes the transposed matrix of $(\alpha_1, \alpha_2)$) and put $T_i = \text{Im} \alpha_i$. 

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Clearly we have the map \( \theta : T_1 \to T_2 \) with \( \theta a_1 = a_2 \). Then we say that \([a_1, a_2]\) is extendible (resp. \([a_1, a_2]\) is maximal) provided \( \theta \) is \((M_1, M_2)\)-extendible (resp. \((M_1, M_2)\)-maximal). The following lemma is immediate from [6. Lemma 1.2 (1)] (see [6. Remark 1]).

**Lemma 1.1.** Let \( 0 \to T \xrightarrow{\alpha} M_1 \oplus M_2 \to M \to 0 \) be an exact sequence of modules with \( \alpha = (a_1, a_2)^T \) and \( a_1 \) a monomorphism. If \([a_1, a_2]\) is extendible, then \( M \cong (M_1/Im a_1) \oplus M_2 \). In case \( a_2 \) is also a monomorphism, the converse holds.

Let \( M \) be a module and \( S \) a non-empty subset of \( M \). If \( S \neq 0 \) (resp. \( S = 0 \)), we put \( d_M(S) = \max\{n \mid S \subseteq MJ^n\} \) (resp. \( d_M(S) = \infty \)) and we call \( d_M(S) \) the depth of \( S \) in \( M \). In particular for \( M = A \), we simply denote \( d_M(S) \) by \( d(S) \) and call \( d(S) \) the depth of \( S \) (cf. [3, p. 76] for the definition of "depth").

We define a linear order \( "<" \) for the set \( \mathbb{Z} \times \mathbb{Z} \) (where \( \mathbb{Z} \) is the set of integers) by the following:

\((r, s) < (r', s')\) if (1) \( r < r' \) or (2) \( r = r' \) and \( s \geq s' \).

For a local module \( L \) with a simple submodule \( S \), we call \((L, S)\) a local-simple set. We denote the isomorphism class of \( S \) by \( C(S) \). Then for a local-simple set \((L, S)\) we define its degree \( \deg(L, S) \) by setting \( \deg(L, S) = (C(S), d_i(S), \left| L \right|) \) and give a partial order \( "\leq" \) to the set \( \{\deg(L, S) \mid (L, S) \text{ is a local-simple set}\} \) as follows:

\( \deg(L, S) \leq \deg(L', S') \) if \( C(S) = C(S') \) and \( (d_i(S), \left| L \right|) \prec (d_i(S'), \left| L' \right|) \).

For convenience we define \( \deg(L, S) < \deg(L', 0) \) for any local module \( L' \).

**Lemma 1.2.** Let \((L_i, S_i)\) be a local-simple set: \( i = 1, 2, \) and \( \theta : S_1 \to S_2 \) an isomorphism. If \( \theta \) is extended to a map \( \phi : L_1 \to L_2 \), then the following hold.

1. \( \deg(L_1, S_1) \leq \deg(L_2, S_2) \).
2. In the case \( d_{L_1}(S_1) = d_{L_2}(S_2) \), \( \phi \) is an epimorphism.
3. In the case \( \deg(L_1, S_1) = \deg(L_2, S_2) \), \( \phi \) is an isomorphism.

**Proof.** (1) and (2). Put \( n = d_{L_1}(S_1) \). Then \( S_1 \subseteq L_1J^n \), so \( S_2 = \phi(S_1) \subseteq \phi(L_1)J^n \subseteq L_2J^n \). Thus we have \( d_{L_1}(S_1) \leq d_{L_2}(S_2) \). If \( \phi \) is not an epimorphism, then we have \( \phi(L_1) \subseteq L_2J^r \) for \( L_1 \) is local. Hence \( S_2 \subseteq \phi(L_1)J^n \subseteq L_2J^{n+1} \), which implies \( d_{L_2}(S_2) \geq n+1 \). It follows (2) and (1).
(3) $\phi$ is an epimorphism by (2) and we have $|L_i| = |L_j|$. Thus $\phi$ is an isomorphism.

**Remark 1.** Let $(L_i, S_i); i = 1, 2$, be local-simple sets with $\deg(L_i, S_i) = \deg(L_j, S_j)$. If an isomorphism $\theta: S_i \to S_j$ is not $(L_i, L_j)$-extendible, then $\theta^{-1}: S_j \to S_i$ is not $(L_j, L_i)$-extendible by the above lemma.

**Proposition 1.3.** Let $A$ be an artinian ring. Then the following conditions are equivalent.

1. $A$ is of right local type.
2. For any local-simple sets $(L_i, S_i); i = 1, 2$, with $\deg(L_i, S_i) \leq \deg(L_j, S_j)$, any isomorphism $\theta: S_i \to S_j$ is $(L_i, L_j)$-extendible.

**Proof.** (1) $\Rightarrow$ (2). It is immediate from [6, Lemma 2.1].

(2) $\Rightarrow$ (1). Let $C$ be the class of modules which are direct sums of local modules. Every module is a homomorphic image of a module in $C$. Hence we show that $C$ is closed under taking factor modules. In order to do it, as easily seen, it suffices to show the following: If $L = \bigoplus_{i=1}^{n} L_i$, each $L_i$ is a local module, i.e., $L$ is in $C$, and $S$ is a simple submodule of $L$, then $L/S$ is also in $C$. Let $L = \bigoplus_{i=1}^{n} L_i$ and $S$ be as above and $\alpha = (a_1, \ldots, a_n)^{\top}: S \to L_1 \oplus \cdots \oplus L_n$ the inclusion map and put $S_i = \alpha_i(S)$. If we pick out $j$ such that $\deg(L_j, S_i) \leq \deg(L_i, S_i)$ for any $i = 1, \ldots, n$, then $[a_j, a_i]$ is extendible: $i = 1, \ldots, n$. Therefore $[a_j, a_j]$ is extendible where $a_j: S \to \bigoplus_{i=1}^{n} L_i$ is the map induced from $\alpha: S \to \bigoplus_{i=1}^{n} L_i$, and by Lemma 1.1, $L/S$ is in $C$.

**Remark 2.** For any local-simple sets $(L_i, S_i); i = 1, 2$, with $S_1 \cong S_2$, we have $\deg(L_1, S_1) \leq \deg(L_2, S_2) \text{ or } \deg(L_1, S_1) \geq \deg(L_2, S_2)$. Hence Proposition 1.3 implies the following: An artinian ring is of right local type if and only if for any monomorphism $(a_1, a_2)^{\top}: S \to L_1 \oplus L_2, [a_1, a_2]$ or $[a_2, a_1]$ is extendible. This proposition was independently proved by H. Asashiba.

Let $L_i$ be a local module satisfying the following condition (i) for each $i = 1, 2, 3$:

1. $|L_iJ| = 1, L_iJ = U_i \oplus U_i$, where $U_i$ ($i = 1, 2$) are simple modules.

2. $L_2J = V_1 \oplus V_2 \oplus V_3$, where $V_i$ ($i = 1, 2, 3$) are simple modules.
with $V_2 \not\cong V_3$.

(3) $L_2J = W_1 \oplus W_2 \oplus W_3$, where $W_i$ ($i = 1, 2$) are simple modules and $W_3$ is a uniserial module with $|W_3| = 2$.

Then the local module $L_i$ induces local-simple sets $(L_i', S_i')$ and $(L_i'', S_i'')$ with a canonical isomorphism $\theta_i : S_i' \to S_i''$ for each $i = 1, 2, 3$ as follows (see the diagram below):

(1') $L_1' = L_1J/U_2$, $S_1' = (U_1 \oplus U_2)/U_2 (\cong U_1)$; $L_1'' = L_1$, $S_1'' = U_1$,

(2') $L_2' = L_2/V_3$, $S_2' = (V_1 \oplus V_3)/V_3 (\cong V_1)$; $L_2'' = L_2/V_2$, $S_2'' = (V_1 \oplus V_2)/V_2 (\cong V_1)$.

(3') $L_3' = L_3/W_2J$, $S_3' = (W_1 \oplus W_3J)/W_3J (\cong W_1)$; $L_3'' = L_3/W_2$, $S_3'' = (W_1 \oplus W_2)/W_2 (\cong W_1)$.

and $\theta_i : S_i' \to S_i''$ ($i = 1, 2, 3$) are canonical isomorphisms.

As easily seen it holds

(1'') $d_{L_1'}(S_1') < d_{L_1''}(S_1'')$ and the uniserial module $L_i'$ can not be embedded in $L_i''$.

(2'', 3'') $\deg(L_i', S_i') = \deg(L_i'', S_i'')$ and $L_i' \not\cong L_i''$ for each $i = 2, 3$.

Hence $\theta_i$ is not $(L_i', L_i'')$-extendible for each $i = 1, 2, 3$ (see Lemma 1.2).

For each $i = 1, 2, 3$, we say a module $L$ is of type-$i$ if $L$ is a local module which has the same form as the above $L_i$. Now summarizing the above

**Lemma 1.4.** If there exists a module $L_i$ of type-$i$, then $L_i$ induces local-simple sets $(L_i', S_i')$, $(L_i'', S_i'')$ and an isomorphism $\theta_i : S_i' \to S_i''$ such that $\theta$ is not $(L_i', L_i'')$-extendible and

1. $d_{L_i'}(S_i') < d_{L_i''}(S_i'')$ (in the case $i = 1$).
2. $\deg(L_i', S_i') = \deg(L_i'', S_i'')$ (in the case $i = 2, 3$).

**Remark 3.** The assertion for $i = 3$ in the above lemma was communicated by H. Asashiba. He has independently given results [1, Lemmas 1.4 and 1.7] which include Lemma 1.4. We represent the above modules $L_i$, $L_i'$, $S_i'$ etc. and the isomorphisms $\theta_i : S_i' \to S_i''$ as the following diagrams:

![Diagram](image-url)
Lemma 1.5. If $A$ is of right local type, then $A$ is left serial.

Proof. Let $u_i \in l(e_iJ^{-1}f)$ ($e_i, f \in p(A)$); $i = 1, 2$ for $r \geq 2$, and $e_i : \overline{fA} \rightarrow e_iA/e_iJ$ the monomorphism such that $a(x+fJ) = u_i x + e_iJ^{-1}f$ for each $x+fJ \in \overline{fA}$. By Remark 2 and [6, Lemma 1.4] $\overline{u_i} \in \overline{A\overline{u_i}}$, or $\overline{u_i} \in \overline{A\overline{u_i}}$ where $\overline{u_i} = u_i + J^{-1}f \in J^{-1}f/J^{-1}$. This shows the left $A$-module $\overline{A}$ is uniserial.

2. Left serial rings. In this section we prove some results for left serial rings which are used later.

If $u$ is a right local element with $u = u(f \in p(A))$, then $uA$ is a local right $A$-module with $uA \cong \overline{fA}$, and conversely if $L$ is a local module with $\overline{L} \cong \overline{fA}$, then we can take such a right local element $u$ as a generator of $L$. Hence for every local module $L$, we always take such a generator, and moreover for a several local modules $L_i (i = 1, \ldots, n)$ with $\overline{L_i} \cong \overline{fA}$ we take generators $u_i = u_i f$ on a common primitive idempotent $f$. Let $a = ae$ and $b = bf$ be elements in $A : e, f \in p(A)$. If we say that $c$ is a local element with $a = bc$, then we mean $c = fce$, and in a similar case we take such elements. The following lemma is obvious.

Lemma 2.1. Let $u$ and $v$ be elements of $A$.

(1) If $Au \supset Av$ (resp. $Au = Av$), then there exists an epimorphism (resp. isomorphism) $\overline{a} : uA \rightarrow vA$ which is a left multiplication map by some element $a$ of $A$.

(2) If $Au$ and $Av$ are submodules of a uniserial left ideal of $A$ with $d(u) \leq d(v)$ (resp. $d(u) = d(v)$), then it holds $Au \supset Av$ (resp. $Au = Av$).

Remark 4. Let $A$ be left serial. If $L_1$ and $L_2$ are local right ideals with $\overline{L_1} \cong \overline{L_2}$ and $d(L_1) = d(L_2)$, then $L_1 \cong L_2$ by Lemma 2.1. As a
generalization of this assertion we have (2) in Lemma 2.2.

Lemma 2.2. Let A be a left serial ring.
(1) If \((L_i, S_i)\) are local-simple sets \((i = 1, 2)\) with \(S_1 \cong S_2\) and \(d_{L_i}(S_i) = d_{L_2}(S_2) = r\), then \(L_i \cong L_2\).
(2) Let \(u_i \in \text{loc}(A)\) with \(d(u_i) = d(u_2)\). If \(u_i J^r\) \((i = 1, 2)\) have simple submodules \(S_i\) such that \(S_1 \cong S_2\) for some \(r \geq 0\), then \(u_i A \cong u_2 A\).

Proof. (1) Let \(L_i \cong e_i A\) \((i = 1, 2)\) and \(S_1 \cong S_2 \cong \overline{f A}\) where \(e_i, f \in p(A)\). Then we have an epimorphism \(\phi_i : e_i A \rightarrow L_i\) for each \(i\). Since \(S_1 \subseteq L_i J^r = \phi(e_i J^r)\) and \(S_2 \subseteq L_2 J^{r+1} = \phi(e_2 J^{r+1})\), there exists a local element \(x_i = e_i x_i f\) with \(d(x_i) = r\). Hence \(A e_i x_i f = A e_2 x_2 f\), which implies \(A e_i \cong A e_2\), i.e., \(e_1 A \cong e_2 A\).
(2) is immediate from (1) and Remark 4.

The following lemma is a slight modification of [6, Lemma 4.1].

Lemma 2.3. Let A be a left serial ring, M a direct summand of the right ideal \(J^r\) for an integer \(r \geq 0\). Let \(u_1, \ldots, u_n\) be right local elements such that \(u_i \in M\) and \(d(u_i) = r\) for each \(i = 1, \ldots, n\), and denote by \(\overline{a}\) the residue class of \(a\) in \(M/MJ\) for any \(a \in M\). Then the following hold.
(1) If \(M = \overline{u}_1 A \oplus \cdots \oplus \overline{u}_n A\), then \(M = u_1 A \oplus \cdots \oplus u_n A\).
(2) If \(\overline{u}_1 A, \ldots, \overline{u}_n A\) are independent, then \(M = \overline{u}_1 A \oplus \cdots \oplus \overline{u}_n A \oplus \overline{v}_1 A \oplus \cdots \oplus \overline{v}_m A\) with some right local elements \(v_i\) in \(M\); \(i = 1, \ldots, m\).

Proof. (1) The canonical epimorphism \(\sigma : J^r \rightarrow J^r/J^{r+1}\) induces the isomorphism \(\overline{M} \rightarrow \sigma(M)\), and so \(\overline{M} = \overline{u}_1 A \oplus \cdots \oplus \overline{u}_n A\) if and only if \(\sigma(M) = \sigma(u_1) A \oplus \cdots \oplus \sigma(u_n) A\). Hence if \(\overline{M} = \overline{u}_1 A \oplus \cdots \oplus \overline{u}_n A\), then \(u_1, \ldots, u_n A\) are independent by [6, Lemma 4.1] and \(M = u_1 A \oplus \cdots \oplus u_n A\), for \(MJ\) is small in \(M\). (2) It is clear from semi-simplicity of \(\overline{M}\).

Let \(M\) be a direct summand of the right module \(J^r\) and \(L\) a submodule of \(M\). Then \(L\) is a submodule of \(MJ^s\) which is a direct summand of \(J^{r+s}\), and \(d(L) = d(MJ^s) = r+s\), where \(s = d(L)\). Hence it suffices to assume \(d(L) = d(M)\) when we consider such a submodule \(L\) of \(M\). As an immediate consequence of the above lemma we have

Corollary 2.4. Let \(A, M\) and \(r\) be as in Lemma 2.3 and \(L_i (i = 1, 2)\) local submodules of \(M\) with \(d(L_1) = d(L_2) = r\).
(1) \(L_i\) is a direct summand of \(M\).
(2) If \( L_1 \cap L_2 \neq 0 \) and \( M = L_1 \oplus N \) for a module \( N \), then \( M = L_2 \oplus N \), in particular \( L_1 \cong L_2 \).

**Lemma 2.5.** Let \( A \) be a left serial ring and \( M \) a direct summand of the right ideal \( J^r \) for an integer \( r \geq 0 \). If \( w \) is a right local element in \( M \) with \( d_u(w) = s \), then there exists a right local element \( u \) in \( M \) such that \( d(u) = r \) and \( w \in uJ^s \).

**Proof.** Let \( w = wf \); \( f \in p(A) \). Since \( J^sf \) is a uniserial left \( A \)-module, we have \( J^sf = Aa \) for some \( a \in t(eJ^sf) \); \( e \in p(A) \). Hence \( w \in M J^sf = Ma \), and so \( w = ua \in uJ^s \) for some \( u = ue \in M \). Then \( d(u) = r \) is clear, for \( w \in J^{r+s} \).

**Lemma 2.6.** Let \( u \) and \( v \) be elements of \( A \). Then the following conditions hold.

1. \( d(u+v) \geq \min \{ d(u), d(v) \} \).
2. If \( d(u) < d(v) \), then \( d(u+v) = d(u) \).
3. If \( J^r = M_i \oplus M_2 \), \( u \in M_i \) and \( v \in M_2 \) for some integer \( r \geq 0 \) and some modules \( M_i \) and \( M_2 \), then \( d(u+v) = \min \{ d(u), d(v) \} \).

**Proof.** (1) and (2) are clear.

(3) If \( u+v \in J^{r+s} \), then \( u+v \in M_i J^s \oplus M_2 J^s \). Hence \( u \in M_i J^s \subset J^{r+s} \) and similarly \( v \in J^{r+s} \).

**Lemma 2.7.** Let \( A \) be a left serial ring, \( u \in t(J^re) \) and \( v \in t(eJ^sf) \), where \( e, f \in p(A) \). Then it holds \( Auw = J^{r+s}f \). In particular

1. If \( uv \neq 0 \), then \( uv \in t(J^{r+s}e) \) (i.e. \( d(uv) = d(u) + d(v) \)).
2. If \( uv = 0 \), then \( u'v' = 0 \) for any \( u' \in J^re \) and \( v' \in eJ^sf \).

**Proof.** Since \( A \) is left serial, \( w \in t(J^ng) \) if and only if \( Aw = J^ng \neq 0 \) for any \( g \in p(A) \). Thus it follows from \( Au = J^re \) and \( Av = J^sf \) that \( Auw = J^{r+e}f \). The other assertions are immediate from this.

**Corollary 2.8.** Let \( A \) be a left serial ring and assume \( eJ^r = M_i \oplus \cdots \oplus M_n \) for an integer \( r \geq 1 \), where each \( M_i \) is a right ideal and \( e \in p(A) \). Then for any element \( a \in A \), it holds \( aeJ^r = aM_i \oplus \cdots \oplus aM_n \).

**Proof.** We may clearly assume that \( a = ae \) and \( n \geq 2 \). If \( ax_j + \cdots + ax_n = 0 \) and \( ax_j \neq 0 \) for some \( j \) \((1 \leq j \leq n)\), where \( x_i = ex_i \in M_i \) \((i = 1, \ldots, n)\), then we have \( a(x_1f + \cdots + x_nf) = ax_1f + \cdots + ax_nf = 0 \) and \( ax_if \neq \ldots \)
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0 for some \( f \in p(A) \). But by (3) in Lemma 2.6, \( d(x_1f + \cdots + x_nf) \leq d(x_if) \). This is a contradiction by (2) in Lemma 2.7.

3. Left serial rings with right serial Jacobson radicals. The following lemma is immediate from Corollary 2.4 and Lemma 2.2.

**Lemma 3.1.** Let \( A \) be a left serial ring, and \( M \) a serial direct summand of the right module \( J \).

1. Any local submodule of \( M \) is uniserial.
2. If \( u \) and \( v \) are right local elements of \( M \) such that \( d(u) = d(v) \) and \( uJ^r = vJ^r = 0 \) for some integer \( r \geq 0 \), then \( uA = vA \).

**Lemma 3.2.** Let \( A \) be a left serial ring such that the right ideal \( J \) is serial, and let \( eJ = u_1A \oplus \cdots \oplus u_nA \) (\( n \geq 2 \)) with local elements \( u_i \) and \( e \in p(A) \).

1. If \( a \in J \) and \( au_i = 0 \), then \( au_i = 0 \) for any \( i = 2, \ldots, n \).
2. If \( u_iA \cong u_2A \), then \( Ju_i = 0 \). Therefore it holds \( JeJ = 0 \) in the case \( u_iA \cong u_iA \) for each \( i = 2, \ldots, n \).
3. Assume \( u_iA \cong u_2A \). If \( u_iJ^{r-1} = fA \) for some \( f \in p(A) \), then \( J^{r+1}f = 0 \) (i.e. \( r = |Jf| \)). In particular \( u_iJ^s = u_iJ^s (\cong u_iJ^s) = 0 \) implies \( s = s' \).

**Proof.** (1) We may assume \( a = ae \). By Corollary 2.8 \( aeJ = au_iA \oplus \cdots \oplus au_nA \). But by Lemma 3.1 \( aeA \) is uniserial and so is \( aeJ \). It follows (1).

(2) If \( u_iA \cong u_2A \cong fA \), we may assume \( u_i = u_i f \) for some \( f \in p(A) \); \( i = 1, 2 \). Then \( Au_i = Ju_i = Jf \). Suppose \( Ju_i = 0 \). Then \( au_i = 0 \) for some local element \( a \) of \( A \), so \( au_2 = 0 \) by (2) of Lemma 2.7. This is a contradiction to (1).

(3) We have \( J^{r+1}f = Ju_iJ^{r-1}f = 0 \) by (2) since \( Jrf = Au_iJ^{r-1}f \).

**Lemma 3.3.** Let \( A \) be a left serial ring, \( M \) a direct summand of \( J_A \), and \( I \) a submodule of \( M \). If \( M \) is serial, then there exist right local elements \( u_1, \ldots, u_n \) such that \( M = u_1A \oplus \cdots \oplus u_nA \) and \( I = u_1J^r \oplus \cdots \oplus u_nJ^n \) for some integers \( r_1, \ldots, r_n \geq 0 \).

**Proof.** Assume \( I \neq 0 \). Then we have \( I \subseteq MJ^r \) and \( I \subseteq MJ^{r+1} \) for some \( r \geq 0 \), and we can choose a right local element \( w \) in \( I \) such that \( w \in MJ^{r+1} \).
It is immediate from Lemma 2.5 and Corollary 2.4 that \( M = uA \oplus L \) and 
\( wA = wJ \) for some \( u \in \text{loc}'(A) \) and some right module \( L \). It follows \( I = wJ \oplus (L \cap I) \) from \( I \subset MJ = wJ \oplus LJ \subset wJ \oplus L \). Since \( L \) is a direct summand of \( M \), we can apply the same argument to \( L \) and \( (L \cap I) \). Hence by iterating this, our assertion is verified.

**Lemma 3.4.** Let \( A \) be a left serial ring, \( M \) a direct summand of \( J_A \), and \( I \) and \( K \) submodules of \( M \) with a simple module \( K/I \). If \( M \) is serial, then there exist right local elements \( u_1, \ldots, u_n \) such that \( M = u_1A \oplus \cdots \oplus u_nA \), 
\( K = u_1J^{r_1-1} \oplus u_2J^{r_2} \oplus \cdots \oplus u_nJ^{r_n} \) and 
\( I = u_1J^{r_1} \oplus u_2J^{r_2} \oplus \cdots \oplus u_nJ^{r_n} \) for some integers \( r_1, \ldots, r_n \).

**Proof.** By Lemma 3.3 there exist \( u_i \in \text{loc}'(A) \) (i.e., \( d(x_i) = r_i - 1 \)) if and only if \( i \in \{1, \ldots, m\} \). Put \( v' = u_1x_1 + \cdots + u_nx_n, M' = u_1A \oplus \cdots \oplus u_mA, I' = u_1J^{r_1} \oplus \cdots \oplus u_mJ^{r_m} \) and \( K' = v'A + I' \). Considering \( M', I', \) and \( K' \) instead of \( M, I, \) and \( K \) respectively, we may assume \( u_ix_i \notin I \) for any \( i; 1 \leq i \leq n \), since \( K = K' \oplus \left( \bigoplus_{i=r+1}^{n} u_iJ^{r_i} \right) \).

If \( d(x_j) \leq d(x_i) \) for any \( i \), say \( j = 1 \), then \( x_i \) is in the uniserial module \( Ax_1 \), so \( x_i = u_ix_i \) for \( i \in \text{loc}(A) \) where \( i = 2, \ldots, n \). Putting \( u = u_1 + u_2a_2 + \cdots + u_na_n \), we have \( d(u) = 1 \) and 
\( v = u_1x_1 + \cdots + u_nx_n = (u + u_2a_2 + \cdots + u_na_n)x_i = u_1x_i \). If \( w_1y_1 + u_2y_2 + \cdots + u_ny_n = 0 \) for some \( y_i \in A \), then \( u_1y_1 + \sum_{i=2}^{n} u_iy_i = 0 \) and by Lemma 2.7 \( w_1y_1 = 0 \). This shows \( M = uA \oplus u_2A \oplus \cdots \oplus u_nA \). Since \( v = u_1x_1 + \cdots + u_nx_n = u_1x_i and \( d(x_i) = r_i - 1 \) for any \( i (1 \leq i \leq n) \), we have \( I = u_1J^{r_1} \oplus \cdots \oplus u_nJ^{r_n} \) and \( K = vA + I = u_1J^{r_1} \oplus u_2J^{r_2} \oplus \cdots \oplus u_nJ^{r_n} \).

Let \((L_i, S_i) \) (i = 1, 2) be local-simple sets with \( L_i \neq S_i \) and assume 
\( \theta : S_1 \to S_2 \) is an isomorphism. Then by Lemma 3.4, we may assume the following situation for \( i = 1, 2 \):

(St1) \( L_i = e_iA/I_i \) (\( e_i \in p(A) \)), \( S_i = K_i/I_i \), \( e_iJ = u_iA \oplus N_i \), \( K_i = x_iA \oplus M_i \) and \( I_i = x_iJ \oplus M_i \), where \( u_i \in t(e_iJ_i) (f_i \in p(A)) \), \( x_i = e_ix_i \) and \( u_iA (g \in p(A)) \), \( M_i \subset N_i \).
Under the situation, we have \( \theta(x_i + I_i) = x_i c + I_2 \) for some unit element \( c \) in \( gAg \), since \( \theta(S_i) = S_2 \) is simple. Then \( x_i c A = x_i A \), so we may assume the following situation by exchanging \( x_i c \) with \( x_i :
\)

\( \text{St2) } \theta(x_i + I_i) = x_i + I_2 \) for \( x_i + I_i \in K_i/I_i (i = 1, 2) \).

Since \( e_i A \) is the projective cover of \( L_i \) for \( i = 1, 2 \), \( \theta \) is \((L_1, L_2)\)-extendible if and only if there exists an element \( a \in e_i A e_i \) such that \( ax_i - x_i \in I_2 \) and \( aI_i \subset I_2 \). But if \( ax_i - x_i \in I_2 \), then \( ax_i J \subset I_2 \). Hence \( \theta \) is \((L_1, L_2)\)-extendible if and only if the following condition holds:

(E) There exists an element \( a \in e_i A e_i \) such that \( ax_i - x_i \in I_2 \) and \( aM_i \subset I_2 \).

Therefore we have the following lemma.

**Lemma 3.5.** Let \( A \) be a left serial ring, \((L_i, S_i) (i = 1, 2)\) local-simple sets with \( L_i \neq S_i \), and \( \theta : S_1 \to S_2 \) be an isomorphism, and assume the situations \((\text{St1})\) and \((\text{St2})\). Then \( \theta \) is \((L_1, L_2)\)-extendible if and only if it holds the condition \((E)\).

**Theorem 3.6.** For a left serial ring \( A \), the following conditions are equivalent.

1. \( J \) is serial as a right module.
2. For any local module \( L_i \), \( LJ \) is serial.
3. There exist no modules of type-1.
4. If \((L_i, S_i) (i = 1, 2)\) are local-simple sets with \( d_{i_1}(S_1) < d_{i_2}(S_2) \) and \( S_1 \cong S_2 \), then any isomorphism \( \theta : S_1 \to S_2 \) is \((L_1, L_2)\)-extendible.
5. For any uniserial left \( A \)-modules \( U_i (i = 1, 2) \) with socles \( S_i \) and \( |U_1| \leq |U_2| \), any isomorphism \( \theta : S_1 \to S_2 \) is \((U_1, U_2)\)-maximal or \((U_1, U_2)\)-extendible.

**Proof.** The equivalence of (1) and (5) follows from [6, Lemma 4.3]. The implication from (4) to (3) is immediate from Lemma 1.4. (1) is a special case of (2) and the converse is obtained by Lemma 3.3.

(3) \( \Rightarrow \) (1). Assume \( eJ \) is not serial for some \( e \in p(A) \). Since \( eJ \) is a direct sum of local modules, there exist a local module \( K \) and a module \( M \) such that \( eJ = K \oplus M, K/KJ^r \) is uniserial and \( KJ^r \) is not simple for some \( r \geq 1 \). Thus we have a module \( L \) of type-1 as a factor module of \( KJ^{r-1}/KJ^{r+1} \) (resp. \( eA/(KJ^r \oplus M) \)) in the case \( r > 1 \) (resp. in the case \( r = 1 \)).

(1) \( \Rightarrow \) (4). Let \((L_i, S_i) (i = 1, 2)\) be as in (4). Then as easily seen we may assume that \((L_i, S_i) (i = 1, 2)\) are fulfilled the situations \( L_i \neq S_i \).
(St1) and (St2) in Lemma 3.5 since \( \mathcal{d}_1(S_1) < \mathcal{d}_2(S_2) \). Then \( Ax_1 \) and \( Ax_2 \) are submodules of the uniserial module \( Ag \) with \( d(x_1) = \mathcal{d}_1(S_1) < \mathcal{d}_2(S_2) = d(x_2) \), which implies \( Jx_1 \supset A x_2 \). Hence there exists an element \( a \in e_2 J e_1 \) with \( ax_1 = x_2 \). Then \( a u_1 A \supset ax_1 A = x_2 A = 0 \), and by Lemma 3.2 \( a M_1 \cap aN_1 = 0 \). Thus by Lemma 3.5, \( \theta \) is \((L_1, L_2)\)-extendible.

4. Artinian rings of right local type. For any ring \( B \) with Jacobson radical \( J(B) \), we denote \( B / J(B) \) and the residue class \( b + J(B) \in B / J(B) \) by \( \bar{B} \) and \( \bar{b} \), respectively.

Let \( v \in t(eJf) \); \( e, f \in p(A) \). Put \( D_1(v) = eAe, D_2(v) = fAf \) and \( D_i(v) = \bar{a} \in eAe \), \( av \in vA \) \( D_i(v) = \bar{b} \in fAf \), \( vb \in Av \). Then we have a ring isomorphism \( \phi: D_1(v) \to D_2(v) \); \( \phi(a) \to (\bar{b}) \) such that \( av = vb ; a \in eAe \) and \( b \in fAf \) (see [8, Section 3] and [4, Section 1] for the definition of division rings \( D_i(v) \) and \( D_i(v) \); \( i = 1, 2 \)). We identify \( D_i(v) \) with \( D_i(v) \) through the isomorphism \( \phi: D_i(v) \to D_i(v) \).

Assume \( A \) is left serial. Then it is clear \( D_2(v) = D_2(v) \) and in particular we have \( D_2(v) (= D_2(v) = D_1(v)) \supset D_1(v) \). Moreover, putting \( U = A f / J, \) we can identify \( D_1(v) \) and \( D_2(v) \) with division rings \( D_1(U) \) and \( D_2(U) \) defined in [6], respectively (see [6], Section 3). We quote the following lemma from [6, Lemma 3.2].

**Lemma 4.1.** Let \( A \) be a left serial ring and \( v \in t(eJf) \); \( e, f \in p(A) \). Then \( [D_1(v) : D_2(v)] \geq n \) if and only if \( eJ \) contains \( fA^n \) (i.e., the direct sum of \( n \)-copies of \( fA \)).

**Remark 5.** Let \( A \) be a left serial ring and \( e \) a primitive idempotent of \( A \). Consider the following conditions:

(1) \( |eJ| \leq 2 \).

(2) \( |eJ| \leq 2 \) provided \( eJ \) is not homogeneous.

(3) \( [D_1(U) : D_2(U)] \geq 2 \) for any uniserial left \( A \)-module \( U \) with \( |U| \leq 2 \) and \( \text{Soc}(U) \cong A e \).

Then (1) is satisfied if and only if so are (2) and (3) by Lemma 4.1 (see [6, Lemma 3.2]).

**Remark 6.** Let \( v \in t(eJf) \); \( e, f \in p(A) \). If \( \bar{a} \in D_2(v) \) for an element \( a \in eAe \), then \( av \in vA + J^p \) by the definition of \( D_2(v) \). In case \( A \) is left serial, conversely we assume \( av \in vA + J^p \), i.e., \( av = vb + c \) for some \( b = bf \in A \) and \( c \in J^p \). Since \( d(vb) \geq d(v) \) and \( Avb \) and \( Av \) are submodules of \( a \).
uniserial module $Af$, we have $a'v = vb$ for some $a' \in eAe$. Thus $(a-a')v = c \in J^2$, and so $(a-a') \in J$. Hence $\overline{a} = \overline{a} \in D_2(v)$. This shows that we may assume $J^2 = 0$ when we consider $[D_1(v) : D_2(v)]_t$ (or $[D_1(v) : D_2(v)]_r$) for any left serial ring $A$.

The following lemma is essentially shown in the proof of [6, Example 4] (cf. Examples in Section 5).

**Lemma 4.2.** Let $A$ be a left serial ring with $J^2 = 0$, and $v \in t(eJf)$; $e,f \in p(A)$. Then the following are equivalent.

1. $[D_1(v) : D_2(v)]_t \leq n$.

2. If $L_i = e/A/vA$ ($i = 1, \ldots, n$) and $a = (a_1, \ldots, a_n) : vA \to \bigoplus_{i=1}^n L_i$ is a monomorphism, then $[a', a_j]$ is extendible for some $j$, where $a_j : vA \to \bigoplus_{i \neq j} L_i$ is the map induced from $a$.

**Proof.** Since $A$ is left serial, the map $a_i : vA \to L_i$ is a left multiplication map by some element $b_i \in eA : a_i(v) = b_i v + vA \subseteq L_i$. Therefore it suffices to show the following equivalence: $\overline{b}_n \in D_2(v) + D_2(v)\overline{b}_1 + \cdots + D_2(v)\overline{b}_{n-1}$, where $\overline{b}_i = b_i + eJe \in D_2(v)$ (i.e. $\overline{b}_n - (\overline{a}_i\overline{b}_1 + \cdots + \overline{a}_{n-1}\overline{b}_{n-1}) \in D_2(v)$ for some $\overline{a}_i \in D_2(v); i = 1, \ldots, n-1$) if and only if $[a', a_n]$ is extendible. But this is easily seen, since we can identify $D_2(v)$ with $\text{End}(eA/vA)$.

**Proposition 4.3.** Let $A$ be an artinian ring of right local type. Then the following hold:

1. $A$ is a left serial ring.

2. $J$ is serial as a right $A$-module.

3. If $eJ$ is not homogeneous, then $|eJ| \leq 2$, where $e \in p(A)$.

4. If $U$ is a uniserial left $A$-module with $|U| \geq 2$, then $[D_1(U) : D_2(U)]_t \leq 2$.

**Proof.** (1) follows from Lemma 1.5. For any local-simple sets $(L_i, S_i)$ ($i = 1, 2$) with $\text{deg}(L_i, S_i) \leq \text{deg}(L_2, S_2)$, any isomorphism $\theta : S_1 \to S_2$ is $(L_1, L_2)$-extendible by Proposition 1.3. Hence (2) follows from Theorem 3.6. Moreover (3) is immediate from Lemma 1.4. Next $A/J^2$ is also of right local type, and so (4) follows from Lemma 4.2.

**Proposition 4.4.** Let $A$ be an artinian ring with the Jacobson radical $J$. If $A$ satisfies the following conditions (1) ~ (4), then $A$ is of right local type.
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(1) $A$ is a left serial ring.
(2) $J$ is serial as a right $A$-module.
(3) $|eJ| \leq 2$ for any $e \in p(A)$.
(4) If $U$ is a uniserial left $A$-module with $|U| \geq 2$, then $[D_i(U) : D_2(U)]_1 \leq 2$.

Proof. Let $(L_i, S_i) \ (i = 1, 2)$ be local-simple sets with $\deg(L_1, S_1) \leq \deg(L_2, S_2)$ and $\theta : S_1 \rightarrow S_2$ an isomorphism. Then it suffices to show that $\theta$ is $(L_1, L_2)$-extendible by Proposition 1.3. We assume $d_i(S_1) = d_{i,t}(S_2)$ since in the other case the assertion is verified by Theorem 3.6. Moreover we may assume $L_i \neq S_i \ (i = 1, 2)$, and the situations (St1) and (St2) in Lemma 3.5. Then we have $N_i = v_iA$ and $M_i = y_iA$ for some $v_i, y_i \in \text{loc}(A)$ by (3). Now $Ax_1$ and $Ax_2$ are submodules of the uniserial left module $Ag$ with $d(x_1) = d_{i,t}(S_1) = d_{i,t}(S_2) = d(x_2)$. Hence there exists an element $a$ in $e_2Ae_1$ such that $ax_1 = x_2$ and $d(a) = 0$ by (1) in Lemma 2.7. Moreover we have $e_1A \cong e_2A$ and $u_jA \cong u_2A$ by Lemma 2.2 and we may assume $e_1 = e_2$. Put $e = e_1 = e_2$. It suffices to show the condition (E) by Lemma 3.5. Since $d(x_1) = d(x_2)$ i.e., $|u_iA/x_1A| = |u_iA/x_2A|$ by (1) in Lemma 3.1 and $|L_1| \geq |L_2|$, we have $|v_1A/y_1A| \geq |v_2A/y_2A|$ i.e., $d(y_1) \geq d(y_2)$. Put $r = d(x_1)$ and $s = d(y_i)$. Then $u_iJ^r = x_2J^r, v_iJ^{r-1} = y_1A$ and $v_2J^{s-1} \subset y_2A$. If $r < s$, then $ax_1 = x_2$ and $ay^i_e \in eJ^s \subset x_2J \oplus y_2A = I_2$, and (E) is satisfied. Thus we assume $r \geq s$.

Now suppose the condition

(E') $c x_1 - x_2 \in x_2J \oplus v_2A$ and $c y^i_e \in x_2J \oplus v_2A$ for some element $c \in eAe$. (i.e. (E') in the case $I_i = x_2J \oplus v_2A; \ i = 1, 2$).

Then $c x_1 - x_2 \in (x_2J \oplus v_2A) \cap eJ^r = (x_2J \oplus v_2A) \cap (u_iJ^{r-1} \oplus v_2J^{r-1}) = x_2J \oplus v_2J^{r-1} \subset x_2J \oplus y_2A = I_2$ and $cy^i_e \in c v_1J^{s-1} \subset x_2J^s \oplus v_2J^{s-1} \subset x_2J \oplus y_2A = I_2$, so (E') is satisfied. Thus it suffices only to show (E'), and so we may assume that $I_i = x_2J \oplus v_iA \ (i = 1, 2)$, i.e., $L_i$ are uniserial.

(i) In the case $u_iA \cong v_iA \ (i.e. \ u_2A \ncong v_1A)$ : Write $av_i = u_2p + v_2q$ for some $p \in f_2A_h$, $q \in h_2A_h$ ; where $h_1, h_2 \in p(A)$. If $p \in J$, then $av_iA \cong \overline{u_2pA}$ and $d(av_i) = d(u_2p) = 1$. Hence $u_2pA \cong v_1A \cong v_1A$ by Remark 4, which is a contradiction. Thus we have $p \notin J$. If $u_2p = 0$, (E') is clearly satisfied. For $ax_1 = x_2$ and $av_i = v_2q \in I_2$. Hence we assume $u_2p \neq 0$. We have $c v_i = u_2p \neq 0$ for some $c \in eJ^e$, since $Au_2 \subset Jv^i_e (\subset Ah_i)$. Then by (1) in Lemma 3.2 $c v_i = 0$ and consequently $c x_1 = 0$, for $x_1A \subset u_1A$. Thus we have $(a-c)x_1 = ax_1 = x_2$ and $(a-c)v_i = av_i - u_2p = v_2q \in I_2$.

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(ii) In the case $u_1A \cong v_1A$ (i.e. $u_1A \cong u_2A \cong v_1A \cong v_2A$): For some unit $b$ of $eAe$, we have $bv_1 = v_2$ and $bI_1 = beJ^{r+1} + bv_1A = eJ^{r+1} + v_2A = I_2$, so $L_1 \cong L_2$. Thus it suffices to prove our assertion under the condition $L_1 = L_2$, $u_1A \cong v_1A$, and $J^{r+1} = 0$, since $L$ is an $A/J^{r+1}$-module. In this case our assertion is verified from the following lemma.

**Lemma 4.5.** Let $A$ be as in the above proposition and $eJ = uA \oplus vA$ and $uA \cong vA$, where $e \in p(A)$ and $u, v \in \text{loc}(A)$. Put $L = eA/vA$ and $S = \text{Soc}(L)$. Then any isomorphism $\theta : S \to S$ is $(L, L)$-extendible.

**Proof.** As in the proof of the above lemma, we may assume $S = (x_1A + vA)/vA$ ($i = 1, 2$), $\theta(x_i + vA) = x_i + vA$ and $ax_i = x_i$, where $x_i, a \in \text{loc}(A)$ with $x_i \in uA$. $d(x_i) = r, d(a) = 0$. Since $\text{Soc}(auA) = ax_iA = x_iA = \text{Soc}(uA)$ and $d(au) = d(u)$, we have $eJ = auA \oplus vA$ by Corollary 2.4 (more explicitly we can show $auA = uA$ by (3) in Lemma 3.2 and Lemma 2.5). Put $u_1 = u$ and $u_2 = au_i (= au_i)$. Since $x_i \in u_iA$, for some $z \in \text{loc}(J^{r+1})$ we have $x_i = u_i z$, so $x_i = ax_i = au_i z = u_i z$. Put $L = LJ/LJ^2$ and $S' = LJ/LJ^2$ and consider the isomorphism $\theta' : S' \to S'$ with $\theta'(\overline{u_i}) = \overline{au_i} (= \overline{u_i})$, where $\overline{u_i} = u_i + LJ^2 \in S'$. Since $L$ is a module over the ring $A/J^2$, $\theta'$ is $(L, L')$-extendible by Lemma 4.2. Hence there exists a unit $c$ in $eAe$ such that $cu_i - u_i \in eJ^2 + vA$ and $cv \in eJ^2 + vA$. On the other hand $d(z) = r - 1$ and $eJ^{r+1} = 0$. Therefore $cx_i - x_i = (cu_i - u_i) z \in eJ^{r+1} + vJ^{r+1} \subset vA$. Moreover since $d(cv) = 1$ and $cv \in u_iJ \oplus vA$, we have $cv \in vA$ by (3) in Lemma 3.2. Thus $\theta$ is $(L, L)$-extendible by Lemma 3.5.

We consider the following condition (D) introduced in [6, Section 3]:

(D) $[D_1(U) : D_2(U)]_r = [D_1(U) : D_2(U)]_s$ for any uniserial left $A$-module $U$ with $|U| \geq 2$.

If the conditions (b-1, 3-4) in the following theorem and (D) are satisfied, then it holds $|eJ| \leq 2$ for any $e \in p(A)$ (see Remark 5). Thus by Propositions 4.3 and 4.4 we have

**Theorem 4.6.** Let $A$ be an artinian ring with the Jacobson radical $J$. Consider the following conditions:

(a) $A$ is of right local type.
(b) (1) $A$ is a left serial ring.
(2) $J$ is serial as a right $A$-module.
(3) If $eJ$ is not homogeneous, then $|eJ| \leq 2$, where $e \in p(A)$.
(4) If $U$ is a uniserial left $A$-module with $|U| \geq 2$, then $[D_1(U) : D_2(U)]_s = [D_1(U) : D_2(U)]_r$. 

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\[ D_{l}(U) \mid_{l} \leq 2. \]

Then (a) implies (b), and under the condition (D) the converse holds.

Representing [6, Theorem 4.5] as the dual version of Theorem 4.6, we have (see Remark 5)

**Theorem 4.7** ([7, Theorem 5.3] and [6, Lemma 4.3]). Let \( A \) be an artinian ring with the Jacobson radical \( J \). Consider the following conditions:

(c) \( A \) is of left colocal type.

(d) (1) \( A \) is a left serial ring.

(2) \( J \) is serial as a right \( A \)-module.

(3) If \( eJ \) is not homogeneous, then \( |eJ| \leq 2 \), where \( e \in p(A) \).

(4) If \( U \) is a uniserial left \( A \)-module with \( |U| \geq 2 \), then \([D_{l}(U) : D_{r}(U)]_{r} \leq 2\).

Then (c) implies (d), and under the condition (D) the converse holds.

We note that in (b) and (d) of the above theorems there is only one difference between indeces \( l \) and \( r \) in \([D_{l}(U) : D_{r}(U)]_{l} \) and \([D_{l}(U) : D_{r}(U)]_{r} \).

Moreover when \( b-4 \) and \( d-4 \) are satisfied, \( D_{l}(U) \neq D_{r}(U) \) implies \([D_{l}(U) : D_{r}(U)]_{l} = [D_{l}(U) : D_{r}(U)]_{r} (= 2) \). Thus by the theorems above we have

**Theorem 4.8.** Let \( A \) be an artinian ring with the Jacobson radical \( J \). Consider the following conditions:

(a) \( A \) is of right local type.

(b) \( A \) is of left colocal type.

(c) (1) \( A \) is left serial.

(2) \( eJ \) is a right serial module with \( |eJ| \leq 2 \) for any primitive idempotent \( e \) of \( A \).

Then the following statements hold.

(i) Under the condition (D), the conditions (a), (b) and (c) are equivalent.

(ii) (a) and (b) are satisfied if and only if so are (c) and (D).

5. **Some properties on local modules.** In this section, all modules are (finitely generated unitary) left \( A \)-modules unless otherwise stated. Any homomorphism between left \( A \)-modules operates on the right.

Let \( A \) be of right local type. H. Asashiba showed that there exist no modules of type-3 (see Remark 3). As easily seen this implies
Proposition 5.1 (Asashiba). If $A$ is of right local type, then $|\overline{eJ}| \leq 2$ for any $e \in p(A)$ with $eJ^2 \neq 0$.

Let $e \in p(A)$. Then $eJ^2 \neq 0$ if and only if there exists a uniserial module $U$ such that $|U| = 3$ and $\text{Soc}(U) = eJU \cong \overline{Ae}$ (i.e. $eJ^2g \neq 0 \iff J^2(\text{Ag}/J^2g) \cong \overline{Ae}$ for some $g \in p(A)$). Hence the following conditions (1) and (2) are equivalent provided the conditions (b-1, 3) in Theorem 4.6 are fulfilled (see Remark 5).

(1) $|\overline{eJ}| \leq 2$ provided $eJ^2 \neq 0$.

(2) $[D_1(U): D_2(U)]_C \leq 2$ for any uniserial module $U$ with $|U| \geq 3$ and $\text{Soc}(U) \cong \overline{Ae}$.

Thus as an immediate consequence of Proposition 5.1, we have

Proposition 5.2. If $A$ is of right local type, then $[D_1(U): D_2(U)]_C \leq 2$ for any uniserial module $U$ with $|U| \geq 3$.

Now we show the dual version to Proposition 5.2. In order to do it, we consider the dual notions to some ones in Section 1. As the dual notions to "extension", "extendible" and "maximal" we define "coextension", "coextendible" (or "liftable") and "comaximal", respectively. That is, for epimorphisms $\alpha_i: L_i \rightarrow K$ $(i = 1, 2)$, $[\alpha_1, \alpha_2]$ has a coextension $\phi: L_1 \rightarrow L_2$ (i.e. $[\alpha_1, \alpha_2]$ is coextendible) if and only if $\alpha_i = \phi \alpha_2$, and $[\alpha_1, \alpha_2]$ is comaximal if and only if $[\alpha_1, \overline{\alpha_2}]$ has no coextension for any submodule $N$ with $N \cong \text{Ker} \alpha_2$ and the map $\overline{\alpha_2}: L_2/N \rightarrow K$ which is induced from $\alpha_2$.

Let $\alpha_i: L_i \rightarrow K$ be epimorphisms and $M = \text{Ker} \alpha$, where $\alpha$ is an epimorphism $\alpha = (\alpha_1, \alpha_2)^T: L_1 \oplus L_2 \rightarrow K$. If $\beta = (\beta_1, \beta_2): M \rightarrow L_1 \oplus L_2$ is the inclusion map, then we have the exact sequence

$$(E) \quad 0 \rightarrow M \xrightarrow{\beta} L_1 \oplus L_2 \xrightarrow{\alpha} K \rightarrow 0.$$

Then we should note that $\beta_1$ and $\beta_2$ are also epimorphisms and the sequence (E) induces the following exact sequences: $0 \rightarrow M \cap L_i \rightarrow L_i \xrightarrow{\alpha_i} K \rightarrow 0$ and $0 \rightarrow M \cap L_j \rightarrow M \xrightarrow{\beta_i} L_i \rightarrow 0$, where $|i, j| = |1, 2|$. The following lemma is the dual one to [6, Lemma 1.2] but we give a proof for the sake of readers ((1) in Lemma 5.3 is the dual one to Lemma 1.1 and we only use it in this paper).

Lemma 5.3. Let $0 \rightarrow M \xrightarrow{\beta} L_1 \oplus L_2 \xrightarrow{\alpha} K \rightarrow 0$ be an exact sequence with epimorphisms $\alpha_i$ and an inclusion map $\beta$, where $\alpha = (\alpha_1, \alpha_2)^T$ and $\beta = $
(\beta_1, \beta_2). Then the following hold.

(1) \(|a_1, a_2|\) is coextensible if and only if \(M = L' \oplus (L_2 \cap M)\) for some module \(L'\). In this case \(L' \cong L_1\).

(2) \(|a_1, a_2|\) is comaximal if and only if \(\overline{M} \cong \overline{L_1}\).

Proof. (1). Assume \(|a_1, a_2|\) has a coextension \(\phi\). Then putting \(L' = l(x, -x\phi) | x \in L_1\), it holds \(M \subset L' \oplus L_2\) and \(L' \subset M\), so \(M = L' \oplus (L_2 \cap M)\). Conversely, if \(M = L' \oplus (L_2 \cap M)\) for some module \(L'\), then the epimorphism \(\beta_1\) and the projection \(\phi: M = L' \oplus (M \cap L_2) \to L'\) induce the isomorphisms \(\overline{\beta_1}: M / (M \cap L_2) \to L_1\) and \(\overline{\phi}: M / (M \cap L_2) \to L'\). As easily seen the composition map \(\phi = \overline{\beta_1}^{-1} \overline{\phi}\) is an coextension of \([a_1, a_2]\), where \(\gamma: L' \to M\) is the map with \(x\gamma = -x: x \in L'\).

(2) For any submodule \(N\) of \(\text{Ker } a_2 = M \cap L_2\), the above exact sequence induces the exact sequence \(0 \to M/N \to L_1 \oplus (L_2/N) \to K \to 0\) by identifying \((L_1 \oplus N)/N\) with \(L_1\). Then we should note that \([a_1, \overline{a_2}]\) is coextensible if and only if \(M/N = M'/N \oplus (M \cap L_2)/N\) and \(M'/N \cong L_1\) for some submodule \(M'\) of \(M\) by (1).

"If" part: Assume \(M/N = M'/N \oplus (M \cap L_2)/N\) for some modules \(N\) and \(M'\) as above. Since \(\overline{M}/\overline{N}\) is a homomorphic image of \(\overline{M}\), \(\overline{M} \cong \overline{L}_1\) (i.e. \(\overline{M}\)) implies \((M \cap L_2)/N = 0\), so \(N = \text{Ker } a_2\). This shows \([a_1, a_2]\) is comaximal.

"Only if" part: The exact sequence \(0 \to M \cap L_2 \to M \to L_1 \to 0\) induces the exact sequence \(0 \to \sigma(M \cap L_2) \to \overline{M} \to \overline{L}_1 \to 0\), where \(\sigma: M \to \overline{M}\) is a canonical epimorphism. Since \(\overline{M}\) is semi-simple, there exists a submodule \(M'\) of \(M\) such that \(M = M' + (M \cap L_2)\) and \(M' \cap (M \cap L_2) \subset MJ\). Putting \(N = M' \cap (M \cap L_2), M/N = M'/N \oplus (M \cap L_2)/N\) and \(N \subset MJ\). Hence the comaximality of \([a_1, a_2]\) implies \(M \cap L_2 = N \subset MJ\) and consequently \(\sigma(M \cap L_2) = 0\). Thus we have \(\overline{M} \cong \overline{L}_1\).

For a colocal module \(C\) and a simple module \(S\) which is a component of \(\overline{C}\), we call \((C, S)\) a colocal-simple set.

Let \(M\) be a module with socle \(S_1 \oplus \cdots \oplus S_n\). Then \(M\) is embedded in \(E(S_1) \oplus \cdots \oplus E(S_n)\), where \(E(S_i)\) is the injective hull of \(S_i\). Since \(M\) is finitely generated, we have \(M \subset C_1 \oplus \cdots \oplus C_n\) for some finitely generated submodules \(C_i\) of \(E(S_i)\) (i.e. some finitely generated and colocal modules \(C_i\); \(i = 1, \ldots, n\). Hence by the dual method to the proof of Proposition 1.3 we have

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Proposition 5.4. Let $A$ be an artinian ring. Then the following conditions are equivalent.

1. $A$ is of left colocal type.
2. For any colocal-simple sets $(C_i, S_i); i = 1, 2,$ and any isomorphism $\theta: S_1 \to S_2,$ $\theta$ is $(C_1, C_2)$-coextendible or $\theta^{-1}$ is $(C_2, C_1)$-coextendible.

Proposition 5.5. If $A$ is of left colocal type, then it holds $[D_i(U): D_2(U)], \leq 2$ for any uniserial module $U$ with $|U| \geq 3.$

Proof. Assume there exists a uniserial module $U$ with $|U| \geq 3$ and $[D_i(U): D_2(U)], \geq 3.$ Then by [6. Lemma 5.3] we can construct a colocal module represented by the following diagram (which is a dual one to (3) in Remark 3).

By the dual method to the proof (in the case $i = 3$) of Lemma 1.4, there exists a colocal-simple sets $(C_i, S_i) (i = 1, 2)$ and an isomorphism $\theta: S_1 \to S_2$ which does not satisfy (2) in Proposition 5.4. Therefore $A$ is not of left colocal type, which is a contradiction. Thus the proof is complete.

Examples. Let $D$ and $E$ (resp. $F$ and $G$) be division rings such that $E$ is a subring of $D$ and $[D: E], \geq 2, \infty \geq [D: E], \geq 3$ (resp. $G$ is a subring of $F$ and $\infty \geq [F: G], \geq 3,$ $[F: G], = 2$) (see [5] for the existence of these division rings). Let

$$A_n = \begin{pmatrix} E & E \\ & EE \\ & \vdots \\ & EEE...E \\ & DDD...DD \end{pmatrix}, \quad B_n = \begin{pmatrix} G & G \\ & GG \\ & \vdots \\ & GGG...G \\ & FFF...FF \end{pmatrix}$$

be subrings of $M_n(D)$ and $M_n(F),$ where $M_n(D)$ and $M_n(F)$ denote the full-matrix rings over $D$ and $F$ with degree $n \geq 2,$ respectively. Generally if a ring $A$ is of right local type (or left colocal type), then so is any factor ring of $A.$ On the other hand, for integers $m \leq n \leq 2,$ $A_n$ (resp. $B_n$) is clearly a factor ring of $A_m$ (resp. $B_m$). For any $n \geq 2,$ $A_n$ (resp. $B_n$) satisfies (b) in Theorem 4.6 (resp. (d) in Theorem 4.7) (see [6, Example 3]). In [6,
Examples 3 and 4] we showed that $B_1$ and $B_2$ do not satisfy (c) and (a), respectively (i.e. $B_1$ is not of left colocal type and $B_2$ is not of right local type). But it is immediate from Propositions 5.2 (or 5.1), 5.5 and Theorems 4.6, 4.7 that $A_1$ and $B_2$ are not of right local type and $A_2$ and $B_3$ are not of left colocal type.

**References**


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