Some periodicity conditions for rings

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SOME PERIODICITY CONDITIONS FOR RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

HOWARD E. BELL* and ADIL YAQUB

1. Introduction. Let \( Z_+ \) denote the set of positive elements of the ring \( \mathbb{Z} \) of integers. Let \( R \) be an arbitrary associative ring, with center denoted by \( C \). Call an element \( a \in R \) potent if there exists an integer \( n > 1 \) for which \( a^n = a \); call \( R \) a \( J \)-ring if every element is potent. Define \( R \) to be periodic if for each \( x \in R \), there exist distinct \( m, n \in \mathbb{Z}_+ \) such that \( x^m = x^n \).

It is well known that if \( R \) is periodic, then \( R = P + N \), where \( P \) and \( N \) denote respectively the sets of potent and nilpotent elements of \( R \). Whether \( R = P + N \) implies that \( R \) is periodic is apparently not known, except in the presence of additional hypotheses. A recent result in this area, due to Bell and Tominaga [3], is the following:

**Theorem B-T.** If \( R \) is a ring in which every element has a unique representation as a sum of a potent element and a nilpotent element, then \( R \) is a direct sum of a \( J \)-ring and a nil ring. In particular, \( R \) is periodic.

The major purpose of this paper is to study periodicity of rings in which the zero divisors satisfy conditions of Bell-Tominaga type. Specifically, letting \( D \) and \( E \) denote respectively the sets of right zero divisors and idempotents of \( R \), we consider the following conditions:

(†) Each \( x \in D \) is uniquely representable in the form \( x = a + u \), where \( a \in E \) and \( u \in N \).

(††) Each \( x \in D \) is uniquely representable in the form \( x = a + u \), where \( a \in P \) and \( u \in N \).

Condition (†) was introduced by Abu-Khuzam and Yaqub in [1], and (††) is a natural analogue. The condition of the final theorem of [1] suggests another condition we shall explore briefly, namely \( D \subseteq P \cup N \).

An indispensable tool in the study of periodicity is a result due to Chacron [4] (see also [2]): specifically, if \( R \) is a ring such that for each \( x \in R \), there exist \( m \in \mathbb{Z}_+ \) and \( p(x) \in \mathbb{Z}[X] \) for which \( x^m = x^{m+1}p(x) \), then \( R \) is periodic. It follows at once that if \( R = P + N \) and \( N \) is an ideal, then \( R \) must be periodic.

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2. Rings with (††). Before stating the first theorem, we establish two lemmas, the first of which applies to rings satisfying either (†) or (††).

Lemma 1. Let \( R \) be any ring with the property that each \( x \in D \) has at most one representation as a sum of an idempotent and a nilpotent element. Let \( e \) be an arbitrary idempotent of \( R \).

(i) If \( e \notin D \), then \( x = xe \) for each \( x \in R \); if \( e \in D \), then \( ex = xe \) for each \( x \in R \). In particular, \( eR = eRe \).

(ii) If \( \bar{R} = R/A_R(R) \), where \( A_R(R) \) is the right annihilator of \( R \), then every idempotent of \( \bar{R} \) is central.

Proof. (i) For arbitrary \( x \in R \), consider the idempotents \( f_x = e - (ex - exe) \) and \( g_x = e - (xe - exe) \). In case \( e \notin D \), \( (x - xe)e = 0 \) implies \( x = xe \). In case \( e \in D \), by hypothesis, \( e = f_x + (ex - exe) = g_x + (xe - exe) \) implies \( ex - xe = xe - exe \), i.e., \( ex = xe \).

(ii) Suppose, to the contrary, that there exists a non-central idempotent \( f_x = f + A_R(R) \) in \( \bar{R} \). Then \( R(f^2 - f) = 0 \); in particular, \( f^2 = f^3 \), so that \( e = f^2 \) is a non-central idempotent of \( R \) with \( f = \bar{e} \). By (i), \( e \in D \) and \( Re = R \), and furthermore \( R(ex - x) = 0 \) for each \( x \in R \), whence we see that \( \bar{e}x = \bar{x} = \bar{exe} \). This is a contradiction.

Lemma 2. Let \( R \) be any ring which satisfies (††) and has the property that \( N^2 \subseteq N \). If \( a \in P \) and \( u \in N \), then \( au \in N \).

Proof. Suppose that \( u^k = 0 \) and \( a^n = a \), \( n > 1 \); and consider \( e = a^{n - 1} \). By Lemma 1 (i), either \( e \in C \) or \( e \) is a right identity element for \( R \).

Note that \( (a^iua^{n - i - 1})^k = a^i(ue)^{k - 1}ua^{n - i - 1} \) \((i = 1, 2, \ldots, n - 2)\), which is equal to either \( a^iue^ka^{n - i - 1} = 0 \) or \( a^iue^k = 0 \), depending on the nature of \( e \). In either event, \( a^iua^{n - i - 1} \in N \). Since \( N^2 \subseteq N \), we now get \( (au)^{n - 1} = (aua^{n - 2})(a^2ua^{n - 3}) \cdots (a^{n - 2}ua)u \in N \). Thus \( au \in N \).

Theorem 1. Let \( R \) be a ring satisfying (††). If \( N^2 \subseteq N \), then either \( N = D \) or \( R \) is a direct sum of a \( J \)-ring and a nil ring.

Proof. Assume \( N \neq D \), and choose an element \( x \) of \( D \setminus N \): \( yx = 0 \), \( y \neq 0 \). We write \( x = a + u \), where \( a^n = a \neq 0 \) and \( u \in N \). We assume without loss \( n \geq 3 \). Then \( e = a^{n - 1} \) is a non-zero idempotent with \( ea = ae = a \). Left-multiplying the above by \( a^{n - 2} \) gives \( a^{n - 2}x = e + a^{n - 2}u \), where \( a^{n - 2}u \in N \cap eR \) by Lemma 2. Hence \( a^{n - 2}x \) is invertible in \( eR = eRe \) (Lemma 1 (i)): \( a^{n - 2}xw = e \) with some \( w \in eR \). Our first task is to show that \( e \) is in \( D \), and
consequently in \( C \) (Lemma 1 (i)). Suppose, to the contrary, that \( e \in D \). Then \( e \) is a right identity element of \( R \), and so \( 0 = y \varepsilon x a^{n-1} = y a \cdot a^{n-2} x a^{n-2} = y a \cdot e a^{n-2} = y e = y \), a contradiction.

Noting that \( eR \subseteq D \) and \( e \) is central, we can easily see that \( eR \) satisfies the hypothesis of Theorem B-T, hence is a \( J \)-ring. It follows that \( eN = 0 \). Now consider the direct decomposition \( R = eR \oplus A(e) \), where \( A(e) \) is the annihilator of \( e \). Since \( A(e) \subseteq D \) and \( eN = 0 \), we see that \( A(e) \) satisfies the hypothesis of Theorem B-T, hence is a direct sum of a \( J \)-ring and a nil ring; consequently so is \( R \) itself.

**Corollary 1.** Let \( R \) satisfy (††), and suppose that \( D \neq N \). If \( N \) is commutative, then \( R \) is commutative.

**Proof.** Since \( N \) commutative implies \( N^2 \subseteq N \), the result follows from Theorem 1 and the well-known fact that \( J \)-rings are commutative.

In what follows, we shall frequently find it convenient to compute in \( \overline{R} = R/A_+(R) \). For \( x \in R \), denote by \( \bar{x} \) the element \( x + A_+(R) \) of \( \overline{R} \); and let \( \overline{N} \) be the set of nilpotent elements of \( \overline{R} \). Note that \( x \in N \) if and only if \( \bar{x} \in \overline{N} \), hence \( N \) is an ideal of \( R \) if and only if \( \overline{N} \) is an ideal of \( \overline{R} \).

**Theorem 2.** Let \( R \) satisfy (††), and suppose that \( N^2 \subseteq N \). If \( R = P + N \), then \( R \) is periodic.

**Proof.** If \( N \neq D \), the conclusion is immediate from Theorem 1. Suppose, then, that \( N = D \). Since nil rings are obviously periodic, we may assume that \( R \neq N \), in which case the hypothesis \( R = P + N \) guarantees the existence of a non-zero idempotent \( e \). The proof of Lemma 1 (ii) shows that for each such \( e \), \( \bar{e} \) is a multiplicative identity element for \( \overline{R} \). It follows that for any non-zero \( a \in P \), \( \bar{a} \) is invertible in \( \overline{R} \).

Now, consider \( x \in R \setminus N \). Since \( x \notin N \), there exists a non-zero \( a \in P \) and \( u \in N \) such that \( x = a + u \); and we may assume \( a^n = a \) with \( n > 2 \). Then, as in the proof of Theorem 1, we can see that \( a^{n-2} \bar{x} \) is invertible in \( \overline{R} \). Thus, as is well known, \( \overline{R} \) is a local ring with radical \( \overline{N} \). Therefore, \( N \) is an ideal of \( R \), and Chacron's result implies that \( R \) is periodic.

3. **Rings with (†).** While (†) does not seem to yield nice direct-sum decompositions, we can establish periodicity theorems which parallel Theorem 1 and Theorem 2. The following lemma will be needed.
Lemma 3. Let $R$ be any ring satisfying (†).

(i) If $x \in D$, then there exists a positive integer $m$ and central idempotent $f \in \langle x \rangle$ such that $x^m = x^m f$. If, furthermore, $x \in yR$ (resp. $x \in R y$), then $x^m \in y^2 R$ (resp. $x^m \in R y^2$).

(ii) If $u \in N$ and $r \in R$, then $ur \in N$ and $ru \in N$.

Proof. (i) Writing $x = e + u$, where $e \in E$ and $u \in N$, and noting that $\bar{e}$ is central (Lemma 1 (i)), we see that $\bar{x} - \bar{x}^2 = \bar{e} + \bar{u} - (\bar{e} + \bar{u})^2 = \bar{u} - \bar{u}^2 - 2\bar{e}\bar{u} \in \bar{N}$. Thus $(x-x^2)^m = 0$ with some $m \in Z$. By standard computation, we obtain $x^m = x^m x'$ with some $x' \in \langle x \rangle \cap E$ and $x^m = x^m f$; and $f$ is necessarily central by Lemma 1 (i). Now, assume further that $x = y r$ ($r \in R$). Since $f$ is central, we see that $(y r)^m = (y r)^m f = y r f (y r)^{-1} \in y^2 R$.

(ii) Suppose the assertion is false. Then choose $u \in N$, of minimal index of nilpotency, such that $u R \nsubseteq N$. Then $u r \in N$ for some $r \in R$, and $u^2 R \subseteq N$. Since $u r \in D$, there exists a positive integer $m$ such that $(u r)^m \in u^2 R \subseteq N$, by (i). This is a contradiction.

Theorem 3. Let $R$ be any ring satisfying (†) and having $N \neq D$. Then $R$ is periodic. Moreover, if $N$ is commutative, then $R$ is commutative.

Proof. Since $D \supseteq N$, Lemma 3 (i) guarantees the existence of a non-zero central idempotent $e$ in $D$. Then for any $x \in R$, $ex \in D$; hence by the proof of Lemma 3 (i), $e(x-x^2)^m = (ex-(ex)^2)^m = 0$ for some $m \in Z$. Therefore $(x-x^2)^m \in D$. Again by the proof of Lemma 3 (i), $(x-x^2)^m - (x-x^2)^m \in N$. It is now clear that Chacron's condition is satisfied for all $x \in R$, so that $R$ is periodic.

Suppose now that $N$ is commutative. Continuing with the same central idempotent $e$, we have $R = R_1 \oplus R_2$, where $R_1 = eR$ and $R_2 = A(e)$. Clearly $R_1 \subseteq D$. Moreover, if $x \in R_1$ is represented as $f + v$ with $f \in E$ and $v \in N$, it is easy to show that $f \in D$, hence $f \in C$(Lemma 1 (i)). It is now clear that $R$ is commutative.

Theorem 4. If $R = P + N$ and $R$ satisfies (†), then $R$ is periodic.

Proof. If $N \neq D$, Theorem 3 yields the desired conclusion. If $N = D$, the proof of Theorem 2 works, the only change being the replacement of Lemma 2 by Lemma 3 (ii).

4. Rings with $D \subseteq P \cup N$. Our final theorem provides another appli-
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Theorem 5. Let $R$ be a ring with $N$ commutative. If each element of $D$ is either potent or nilpotent, then $N$ is an ideal. Moreover, if $D \neq N$, then $R$ is periodic.

Proof. Of course, $(N, +)$ is a subgroup of $(R, +)$. Assume $N$ is not an ideal, and choose $u \in N$, of minimal index of nilpotency, for which $uR \nsubseteq N$. Then $u^2R \subseteq N$. Let $r$ be an arbitrary element of $R$. Since $ur \in D$, $ur$ is either potent or nilpotent, hence there exists $k \in \mathbb{Z}_+$ such that $e = (ur)^k$ is an idempotent, possibly 0. Since $re - ere \in N$ and $ue \in u^2R$, we have

$$(ur)^k e^2 = u[re - ere, u]r + u^2(re - ere) r + ureuer \in u^2R \subseteq N,$$

contrary to the original supposition. Thus, $N$ is an ideal.

Assume now that $N \neq D$. Let $d$ be a fixed element of $D \setminus N$, so that $d$ is potent and some power of $d$ is a nonzero idempotent $e$ in $D$. Let $r$ be an arbitrary element of $R$, and note that both $er$ and $r - er$ are in $D$.

If $er \in N$, then $e(r^m - r) \in N$ for all $k \in \mathbb{Z}_+$. On the other hand, if $er \in P$, there exists $m > 1$ such that $(er)^m = er$. Now $ere \equiv er \mod N$, so $(er)^j \equiv er^j$ for all $j$. In particular, $(er)^m = er \equiv er^m \mod N$ and hence $e(r^m - r) \in N$.

Repeat the argument with $r - er$ instead of $er$. Noting that if $er$ and $r - er$ are both potent, we can find a single $m$ which works for both. Thus in all cases, there exists $m > 1$ such that $e(r^m - r) \in N$ and $r^m - r - e(r^m - r) \in N$. Consequently $r^m - r \in N$, and Chacron's condition is satisfied.

References


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