Closed ideals in non-unital matrix rings

Shoji Kyuno* Mi-Soo B. Smith† Nobuo Nobusawa‡

*Tohoku Gakuin University
†Chaminade University Of Honolulu
‡University Of Hawaii

Copyright ©1987 by the authors. Mathematical Journal of Okayama University is produced by The Berkeley Electronic Press (bepress). http://escholarship.lib.okayama-u.ac.jp/mjou
CLOSED IDEALS IN NON-UNITAL MATRIX RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

SHOJI KYUNO, MI-SOO B. SMITH and NOBU NOBUSAWA

1. Introduction. Closed ideals in a non-unital ring were first introduced in [3] to study the correspondence of ideals in Morita equivalent rings. Let $I$ be an ideal of a ring $R$. We say that $I$ is lower closed if $RIR = I$. Every irreducible ideal of $R$ is lower closed. On the other hand, we say that $I$ is upper closed if $R^{-1}IR^{-1} = I$, where $R^{-1}IR^{-1} = \{x \in R \mid RxR \subseteq I\}$. Every prime ideal is upper closed. Some properties which are usually satisfied by ideals of a unital ring fail for general ideals of a non-unital ring. But, they are satisfied by closed ideals in the above sense. For example, it is well known that there is a one-to-one correspondence between ideals of two Morita equivalent unital rings. This is not true in case of non-unital rings. However, if we restrict to closed ideals, the same property holds. (See Theorem 3 and Theorem 5.) In this paper, we consider two types of matrix rings. One is the total matrix ring over a ring, and the other is a Morita context which is considered as a subring of a matrix ring of $2 \times 2$ over a ring. Let $R$ be a non-unital ring, and $R_n$ the total matrix ring of $n \times n$ over $R$. It is known that an arbitrary ideal of $R_n$ is not necessarily the total matrix ring $A_n$ over an ideal $A$ of $R$ (contrary to the unital ring case). In 2, we show that every closed (lower or upper) ideal of $R_n$ is expressed as $A_n$ with some ideal $A$ of $R$. Moreover, it will be shown that the ideal of $R_n$ is lower (or upper) closed if and only if $A$ is lower (or upper) closed. In 3, we deal with a Morita context ring $C = C_{11} \oplus C_{22} \oplus C_{12} \oplus C_{21}$, where $C_{ij}$ are submodules satisfying that $C_{ij}C_{jk} \subseteq C_{ik}$ and $C_{ij}C_{km} = 0$ if $j \neq k$. It is easy to see that $C$ is considered as a subring of a total matrix ring of $2 \times 2$ over a ring. Here, we do not assume that $C$ has the identity. However, we have to assume that $C_{12}C_{21} = C_{11}$ and $C_{21}C_{12} = C_{22}$. First, we show that every closed ideal $I$ of $C$ is a Morita context ring $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$, where $I_{ij} = I \cap C_{ij}$ are $C_{ij}$-$C_{jj}$-submodules of $C_{ij}$. We can define upper and lower closed submodules, and we will show that $I$ is lower (or upper closed if and only if all $I_{ij}$ are lower (or upper closed. When there exists an ideal $I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21}$, we say that submodules $I_{11}$, $I_{22}$, $I_{12}$ and $I_{21}$ correspond to each other via the ideal $I$. We can show that the correspondence is
one-to-one among all upper (or lower) closed submodules. Especially, $I_{11}$ and $I_{22}$ are ideals of $C_{11}$ and $C_{22}$, and the correspondence between closed ideals is one-to-one. $C_{11}$ and $C_{22}$ are Morita equivalent in a general sense, and the above result is a generalization of the result in case of a unital ring. See [1].

2. Closed ideals in a total matrix ring. Let $R_n$ be the total matrix ring of $n \times n$ over a non-unital ring $R$. Let $e_{ij}$ be the matrix units. Note that $e_{ij}$ do not exist in $R_n$. However, the formal multiplication by $e_{ij}$ will always make sense in the following context.

**Proposition 1.** Let $I$ be an ideal of $R_n$. There exist ideals $A$ and $B$ of $R$ such that

$$R_nIR_n \subseteq B_n \subseteq I \subseteq A_n \subseteq R_n^{-1}IR_n^{-1}.$$  

**Proof.** Let $I(i, j) = \{ r \in R \mid r \text{ appears in the } (i, j)\text{-entry of some element of } I \}$. It is clear that $I(i, j)$ is an ideal of $R$. We have

$$(RI(i, j)R)e_{km} \subseteq I,$$

because $(RI(i, j)R)e_{km} = (Re_{km})R \subseteq RnIRn \subseteq I$. Let $A = \sum I(i, j)$. $A$ is an ideal of $R$, and $I \subseteq A_n$. On the other hand, $R_nA_nR_n \subseteq I$, because $e_{kk}(R_nA_nR_n)e_{mm} = (RAR)e_{km} \subseteq I$ by (1). Therefore, $A_n \subseteq R_n^{-1}IR_n^{-1}$. We obtained $I \subseteq A_n \subseteq R_n^{-1}IR_n^{-1}$. Let $B = RAR$. Then, $B_n = R_nA_nR_n$. Since $I \subseteq A_n \subseteq R_n^{-1}IR_n^{-1}$, we have $R_nIR_n \subseteq B_n \subseteq I$. The proof of Proposition 1 is completed.

From Proposition 1, we can conclude that if $I$ is lower (or upper) closed ideal of $R_n$, it is the total matrix ring over an ideal of $R$.

**Theorem 1.** Let $I$ be an ideal of $R_n$. $I$ is upper closed if and only if $I = A_n$ with an upper closed ideal $A$ of $R$. $I$ is lower closed if and only if $I = B_n$ with a lower closed ideal $B$ of $R$.

**Proof.** First, suppose that $I$ is upper closed. Then, $I = A_n$ with an ideal $A$ as noted above. It is clear that $R_n(R^{-1}AR^{-1})_nR_n \subseteq A_n$. Since $A_n$ is upper closed, we have $(R^{-1}AR^{-1})_n \subseteq A_n$ and $R^{-1}AR^{-1} \subseteq A$. Therefore, $R^{-1}AR^{-1} = A$ and $A$ is upper closed. Conversely, let $A$ be an upper closed ideal of $R$. We show that $A_n$ is upper closed. Let $x$ be an element of $R_n$ such that $R_nxR_n \subseteq A_n$. Let $r_{ij}$ be an element of $R$ such that $e_{ii}xe_{jj} = r_{ij}e_{ij}$. 

http://escholarship.lib.okayama-u.ac.jp/mjou/vol29/iss1/11
Then, \((R_{r_{ij}}R)e_{km} = (Re_{ki})x(Re_{jm}) \subseteq A_n\). So, \(R_{r_{ij}}R \subseteq A\). Since \(A\) is upper closed, \(r_{ij} \subseteq A\). Hence, \(x \in A_n\) and \(A_n\) is upper closed. Secondly, suppose that \(I\) is lower closed. Then, \(I = B_n\) with an ideal \(B\). \((RBR)_n = R_nB_nR_n = R_nIR_n = I = B_n\). So, \(RBR = B\) and \(B\) is lower closed. Conversely, let \(B\) be a lower closed ideal of \(R\). Then, \(R_nB_nR_n = (RBR)_n = B_n\). Hence, \(B_n\) is lower closed. The proof of Theorem 1 is completed.

**Corollary.**

(i) If \(I\) is a prime ideal of \(R_n\), then \(I = P_n\) with a prime ideal \(P\) of \(R\).

(ii) If \(x \in R_nxR_n\) for every element \(x\) of an ideal \(I\) of \(R_n\), then \(I = A_n\) with an ideal \(A\) of \(R\).

**Proof.** (i) A prime ideal of \(R_n\) is upper closed. For, let \(I\) be a prime ideal of \(R_n\). Then, \(R_n(R_n^{-1}IR_n^{-1})R_n \subseteq I\) implies \(R_n^{-1}IR_n^{-1} \subseteq I\). Thus, \(I\) is upper closed. Then, \(I = P_n\) with an ideal \(P\) of \(R\). We want to show that \(P\) is prime. Let \(CD \subseteq P\) for ideals \(C\) and \(D\) of \(R\). \(C_nD_n = (CD)_n \subseteq P_n = I\) implies \(C_n \subseteq I\) or \(D_n \subseteq I\). If \(C_n \subseteq I = P_n\), then \(C \subseteq P\). If \(D_n \subseteq I\), then \(D \subseteq P\). \(P\) is a prime ideal.

(ii) Suppose that the condition of (ii) of Corollary is satisfied. Then, \(R_nIR_n = I\), and \(I\) is lower closed. So, \(I = A_n\) with an ideal \(A\) of \(R\).

(i) of Corollary is obtained by Sands [4]. (ii) of Corollary is obtained by Luh [2].

3. Closed ideals in a Morita context ring. A subring \(S\) of \(R_2\) is called a Morita context ring (or a M. c. ring) if \(S = S_{11} \oplus S_{22} \oplus S_{12} \oplus S_{21}\), where \(S_{ij} = e_{ii}Se_{jj}\). Thus, \(S\) is a M. c. ring if and only if \(S\) contains all \(S_{ij}\). Note that \(S_{ij}S_{jk}\) is contained in \(S_{ik}\) but is not necessarily equal to \(S_{ik}\). \(S_{ii}\) are rings, and \(S_{ij}\) are \(S_{ii}\)-\(S_{jj}\)-bimodules. In the following, we fix a M. c. ring \(C\), which satisfies the conditions \(C_{12}C_{21} = C_{11}\) and \(C_{21}C_{12} = C_{22}\). Under this basic assumption, we have \(C_{11}C_{12} = C_{12}C_{22}\) and \(C_{21}C_{22} = C_{22}C_{21}\). Let \(I\) be an ideal of \(C\). \(I\) is not necessarily a M. c. ring, and in this direction we have Proposition 2 which is an analogue of Proposition 1.

**Proposition 2.** Let \(I\) be an ideal of \(C\). Then, there exist ideals \(A\) and \(B\) which are M. c. rings such that \(CIC \subseteq B \subseteq I \subseteq A \subseteq C^{-1}IC^{-1}\).

**Proof.** Let \(A = I_{11} \oplus I_{12} \oplus I_{22} \oplus I_{21}\). \(A\) is an ideal of \(C\) as well as a M. c. ring. Clearly, \(I \subseteq A\). We have \(C_{ik}I_{km}C_{mj} \subseteq C_{ik}IC_{mj} \subseteq I\). Hence.
CAC \subseteq I \text{ and hence } A \subseteq C^{-1}IC^{-1}. \text{ Next, let } B = CAC. \text{ } B_{ij} = e_{ii}CACe_{jj} \subseteq CAC = B. \text{ So, } B \text{ is a M. c. ring. Clearly, } B \text{ is an ideal of } C. \text{ It is also clear that } CIC \subseteq B \subseteq I.

Proposition 2 implies that a lower (or upper) closed ideal of } C \text{ is a M. c. ring.}

**Lemma.** \( C_{ij}C_{jk} \) is either \( C_{ii} \) or \( C_{ii}C_{ik} \). Similarly, \( C_{ij}C_{jk} \) is either \( C_{kk} \) or \( C_{ik}C_{kk} \).

**Proof.** If \( i \neq j \neq k \), then \( i = k \) and \( C_{ij}C_{jk} = C_{ij}C_{ji} = C_{ii} \). Otherwise, \( C_{ij}C_{jk} = C_{ij}C_{ik} \) due to the fact \( C_{12}C_{22} = C_{11}C_{12} \) and \( C_{22}C_{11} = C_{22}C_{22} \). The second part is similarly proven.

Let \( M_{ij} \) stand for a \( C_{ii}C_{jj} \)-submodule of \( C_{ij} \) in general. We say that \( M_{ij} \) is lower closed if \( C_{ii}M_{ij}C_{jj} = M_{ij} \).

**Theorem 2.** An ideal \( I \) of \( C \) is lower closed if and only if \( I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21} \) with lower closed \( I_{ij} \).

**Proof.** Suppose that \( I \) is lower closed. Then, \( I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21} \) as above. We want to show that \( I_{ij} \) is lower closed. Since \( I = CIC = C(CIC)C \), we have \( I_{ij} = \sum_{s,k,m,l} C_{is}C_{sk}I_{km}C_{ml}C_{uj} \). Now, by Lemma, \( C_{is}C_{sk} \cdot I_{km}C_{mj}C_{jj} \subseteq C_{ii}I_{ij}C_{jj} \). So, \( I_{ij} \subseteq C_{ii}I_{ij}C_{jj} \), or \( I_{ij} \) is lower closed. Conversely, suppose that \( I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21} \) with lower closed \( I_{ij} \). Then, \( CIC \supseteq C_{11}I_{11}C_{11} \oplus C_{22}I_{22}C_{22} \oplus C_{11}I_{12}C_{22} \oplus C_{22}I_{21}C_{11} = I \). Thus, \( CIC = I \) and \( I \) is lower closed.

Let \( \pi_{ij} \) be the mapping of the set of ideals of \( C \) to the set of \( C_{ii}C_{jj} \)-submodules of \( C_{ij} \) such that \( \pi_{ij}(I) = e_{ii}Ie_{jj} \).

**Theorem 3.** \( \pi_{ij} \) induces a bijection of the set of lower closed ideals of \( C \) to the set of lower closed \( C_{ii}C_{jj} \)-submodules of \( C_{ij} \).

**Proof.** Let \( I = I_{11} \oplus I_{22} \oplus I_{12} \oplus I_{21} \) be a lower closed ideal of \( C \), where \( I_{ij} \) are all lower closed. We show that \( I_{km} = C_{ki}I_{ij}C_{jm} \) for any \( i, j, k \) and \( m \). For example, suppose that \( i = k \) and \( j \neq m \). Then, \( I_{km} = I_{im} \supseteq C_{ii}I_{ij}C_{jm} \supseteq C_{ii}(I_{im}C_{mj})C_{jm} = C_{ii}I_{im}C_{nm} = I_{im} = I_{km} \). Therefore, \( I_{km} = C_{ii}I_{ij}C_{jm} = C_{ki}I_{ij}C_{jm} \) as required. All the other cases are similarly proven. Now, \( I_{km} = C_{ki}I_{ij}C_{jm} \) implies that \( I_{km} \) is uniquely determined by \( I_{ij} \) for any
$k$ and $m$. Therefore, $I$ is uniquely determined by $I_{ij}$. Conversely, let $M_{ij}$ be a lower closed $C_{ii'}C_{jj'}$-submodule of $C_{ij}$. Let $M_{km} = C_{kk'}M_{ij}C_{jm}$. It is easily verified that $M_{km}$ is lower closed. Let $I = I_{i1} \oplus I_{i2} \oplus I_{12} \oplus I_{11}$. We can show that $I$ is a lower closed ideal. This completes the proof of Theorem 3.

In order to discuss the upper closed case, we define the operators $C_{ij}^{-1}$ as follows. Define $C_{ik}^{-1}M_{ij} = \{ x \in C_{ik} | C_{ik}x \subseteq M_{ij} \}$ and $M_{ij}C_{kj}^{-1} = \{ x \in C_{ik} | xC_{kj} \subseteq M_{ij} \}$. Now, we say that $M_{ij}$ is upper closed if $C_{ik}^{-1}M_{ij}C_{kj}^{-1} = M_{ij}$.

**Theorem 4.** Let $I$ be an ideal of $C$. $I$ is upper closed if and only if $I = I_{i1} \oplus I_{i2} \oplus I_{12} \oplus I_{11}$ with upper closed $I_{ij}$.

**Proof.** First, suppose that $I$ is an upper closed ideal of $C$. Then, $I = I_{i1} \oplus I_{i2} \oplus I_{12} \oplus I_{11}$ as above. We want to show that $I_{ij}$ is upper closed. For it, observe that $CC(C_{ii}^{-1}I_{ij}C_{jj}^{-1})CC \subseteq I$ which follows due to Lemma. So, $C_{ii}^{-1}I_{ij}C_{jj}^{-1} \subseteq C^{-1}(C^{-1}IC^{-1})C^{-1} = I$, or $C_{ii}^{-1}I_{ij}C_{jj}^{-1} \subseteq I \cap C_{ij} = I_{ij}$. Thus, $I_{ij}$ is upper closed. Conversely, suppose that $I = I_{i1} \oplus I_{i2} \oplus I_{12} \oplus I_{11}$ with upper closed $I_{ij}$. Let $x$ be an element of $C$ such that $CxC \subseteq I$. Express $x = x_{i1} + x_{i2} + x_{12} + x_{11}$ with $x_{ij} \in C_{ij}$. Then, $C_{ii}x_{ij}C_{jj} = C_{ii}x_{ij}C_{jj} \subseteq I \cap C_{ij} = I_{ij}$. Since $I_{ij}$ is upper closed, $x_{ij} \in I_{ij}$, which implies that $x \in I$, or $I$ is upper closed.

**Theorem 5.** $\pi_{ij}$ induces a bijection of the set of upper closed ideals of $C$ to the set of upper closed $C_{ii'}C_{jj'}$-submodules of $C_{ij}$.

**Proof.** Let $I = I_{i1} \oplus I_{i2} \oplus I_{12} \oplus I_{11}$ be an upper closed ideal of $C$, where $I_{ij}$ are all upper closed. We show that $M_{km} = C_{ik}^{-1}I_{ij}C_{jm}$ for any $i$, $j$, $k$ and $m$. For example, suppose that $i = k$ and $j \neq m$. Then, $M_{km} = I_{km} \subseteq C_{ii}^{-1}I_{ij}C_{jm} \subseteq C_{ii}^{-1}(I_{km}C_{jm})C_{mj}^{-1} = C_{ii}^{-1}I_{km}C_{jm}^{-1}$ (due to the fact $C_{mj}C_{jm} = C_{mm}$). So, $M_{km} = C_{ii}^{-1}I_{ij}C_{jm}^{-1} = C_{ik}^{-1}I_{ij}C_{jm}^{-1}$ as required. All the other cases are similarly proven. Thus, all $I_{km}$ and hence $I$ are uniquely determined by $I_{ij}$. Conversely, if $M_{ij}$ is an upper closed $C_{ii'}C_{jj'}$-submodule of $C_{ij}$, we let $M_{km} = C_{ik}^{-1}M_{ij}C_{jm}$ and let $I = I_{i1} \oplus I_{i2} \oplus I_{12} \oplus I_{11}$. $I$ is an upper closed ideal and its projection to the $(i,j)$-component is $M_{ij}$.
S. KYUNO, M.-S. B. SMITH and N. NOBUSAWA

REFERENCES


SHOJI KYUNO
DEPARTMENT OF MATHEMATICS
TOMIOKU GAKUIN UNIVERSITY
TAGAJO, MIYAGI 985, JAPAN

MI-SOO B. SMITH
COLLEGE OF ARTS AND SCIENCES
CHAMINADE UNIVERSITY OF HONOLULU
HONOLULU, HAWAII 96816, U. S. A.

NOBU NOBUSAWA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HAWAII
HONOLULU, HAWAII 96822, U. S. A.

(Received May 9, 1986)