Configuration Spaces with Partially Summable Labels and Homology Theories

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Abstract

It is shown that any subset of a topological abelian monoid gives rise to a generalized homology theory that is closely related to the notion of labeled configuration space. Applications of the main theorem include generalizations of the classical Dold-Thom and the Barratt-Priddy-Quillen-Segal theorems.
CONFIGURATION SPACES WITH PARTIALLY SUMMABLE LABELS AND HOMOLOGY THEORIES

Dedicated to the memory of Professor Katsuo Kawakubo

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Abstract. It is shown that any subset of a topological abelian monoid gives rise to a generalized homology theory that is closely related to the notion of labeled configuration space. Applications of the main theorem include generalizations of the classical Dold-Thom and the Barratt-Priddy-Quillen-Segal theorems.

Introduction

Let \( M \) be a topological abelian partial monoid, e.g. any subset of a topological abelian monoid. Following [2], we associate with any pointed space \( X \) the space \( \tilde{C}^M(X) \) of configurations of distinct points in \( X \) with labels in \( M \). If \( M \) is taken to be the abelian monoid of non-negative integers \( \mathbb{N} \) then we get a topological abelian monoid \( \text{SP}(X, \ast) \) generated by the points of \( X \), with its basepoint \( \ast \) identified with 0. On the other hand, if \( M = \{1\} \subset \mathbb{N} \) then \( \tilde{C}^{(1)}(X) \) is the space of finite subsets of \( X \), modulo the relation \( \ast = \text{null} \). Thus the notion of labeled configuration spaces generalizes both infinite symmetric products [1] and classical configuration spaces [7].

In general, \( \tilde{C}^M(X) \) is not a functor of \( X \), nor does it preserve homotopy equivalences. Nevertheless we can show that if we replace \( \tilde{C}^M(X) \) by the limit of the inclusions \( \tilde{C}^M(\mathbb{R}^n \times X) \subset \tilde{C}^M(\mathbb{R}^{n+1} \times X) \), where \( \mathbb{R}^n \times X = \mathbb{R}^n \times X/\mathbb{R}^n \times \ast \), then the correspondence

\[
X \mapsto \tilde{C}^M(\mathbb{R}^\infty \times X) = \bigcup_n \tilde{C}^M(\mathbb{R}^n \times X)
\]

defines a generalized homology theory.

More precisely, we have

Main Theorem. Given a topological abelian partial monoid \( M \) there exists a functor \( E^M \), of the category of pointed spaces into itself, together with a natural transformation \( E^M(X) \to \Omega E^M(\Sigma X) \) enjoying the following properties:

(P1) \( E^M(\ast) = \ast \).
(P2) For any pointed space \( X \), the map \( E^M(X) \to \Omega E^M(\Sigma X) \) is a homotopy equivalence.

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For any pair of pointed spaces \((X, A)\), the sequence
\[
E^M(A) \to E^M(X) \to E^M(X \cup CA)
\]
induced by the cofibration sequence \(A \to X \to X \cup CA\) is a homotopy fibration sequence.

If \(X\) is a pointed finite CW-complex then \(E^M(X)\) has the weak homotopy type of a group completion of \(\widetilde{CM}(\mathbb{R}^\infty \times X)\).

Notice that \(\widetilde{CM}(\mathbb{R}^\infty \times X)\) is an admissible \(H\)-space with respect to the multiplication induced by a linear isometry \(\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty\).

In short, the theorem says that there is a generalized homology theory
\[
\tilde{h}_* = \pi_*E^M
\]
such that for a pointed finite CW-complex \(X\),
\[
\tilde{h}_*(X) \cong \pi_*\widetilde{CM}(\mathbb{R}^\infty \times X)
\]
holds if \(\widetilde{CM}(\mathbb{R}^\infty \times X)\) is grouplike. (This is always the case if \(X\) is connected).

Among the applications of the main theorem we have the following:

1. Let \(M\) be an arbitrary topological abelian monoid. Then \(\widetilde{CM}(X)\) is a continuous functor of \(X\) and there is a sequence of natural isomorphisms of homology theories
\[
\pi_*E^M(X) \cong \pi_*\Omega\widetilde{CM}(\Sigma X) \cong \tilde{h}_*(X; \pi_*G)
\]
defined on the category of pointed CW-complexes; here \(G\) is the group completion of \(M\) and \(\tilde{h}_*(X; \pi_*G)\) is the ordinary homology of \(X\) with coefficients in the graded group \(\pi_*G\). Accordingly there holds a generalized Dold-Thom isomorphism
\[
\pi_n\widetilde{CM}(X) \cong \tilde{h}_n(X; \pi_*G) = \sum_r \tilde{h}_{n-r}(X; \pi_rG), \ n \geq 0,
\]
if \(M\) is grouplike or if \(X\) has the homotopy type of a connected CW-complex.

2. Any pointed space \(M\) can be viewed as the subset of generators for the topological abelian monoid \(\text{SP}(M, \ast)\), and hence we can form \(\widetilde{CM}(X)\) and \(E^M(X)\). There is a chain of natural weak equivalences
\[
E^M(X) \to \Omega^\infty E^M(\Sigma^\infty X) \leftarrow \Omega^\infty \Sigma^\infty (X \wedge M)
\]
inducing a natural isomorphism of homology theories
\[
\pi_*E^M(X) \cong \pi_*^S(X \wedge M).
\]
This can be regarded as a generalization of the Barratt-Priddy-Quillen theorem (the case \(M = X = S^0\)). It follows by (P4) that \(\widetilde{CM}(\mathbb{R}^n \times X)\) approximates the homotopy type of \(\Omega^\infty \Sigma^\infty (X \wedge M)\) as \(n \to \infty\) if \(X\) is connected.
The paper is organized as follows. Section 1 introduces the notion of labeled configuration spaces. Section 2 provides a construction of $E^M(X)$ and proves the first three properties in the statement of the main theorem. Section 3 provides several applications of our theory, including certain generalizations of the classical Dold-Thom and the Barratt-Priddy-Quillen theorems. Section 4 gives the main part of the proof of the last property (P4). Finally, Section 5 completes the proof of (P4) by proving the key proposition, Proposition 4.7.

Throughout the paper we shall work in the category of compactly generated spaces. Basepoints are generically denoted by $\ast$ and are assumed to be non-degenerate.

1. Labeled configuration spaces

By a topological abelian partial monoid we shall mean a topological space $M$ equipped with subsets $D_n \subseteq M^n$, $n \geq 2$, and operations $D_n \to M$, $(a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$, enjoying the following property:

If $a_1, \ldots, a_n \in M$ and $I_1, \ldots, I_r, J_1, \ldots, J_s$ are subsets of $\{1, 2, \ldots, n\}$ such that $I_1 \times \cdots \times I_r = J_1 \times \cdots \times J_s = \{1, 2, \ldots, n\}$ holds then we have

$$\sum_{i \in I_1} a_i + \cdots + \sum_{i \in I_r} a_i = \sum_{j \in J_1} a_j + \cdots + \sum_{j \in J_s} a_j$$

whenever both sides of the equation are defined.

We always assume that $M$ is embeddable in a topological abelian partial monoid with unit element 0 and we write $\overline{M} = M \cup \{0\}$. Observe that any subset of a topological abelian monoid (with unit) is a abelian partial monoid. In particular, any topological space $Y$ can be viewed as an abelian partial monoid with $D_n = \emptyset$, for $Y$ is the set of generators for the topological abelian monoid $\text{SP}(Y)$.

To each space (without basepoint) $X$ we assign a topological category $Q^M(X)$ defined as follows:

(1) Objects of $Q^M(X)$ are the points of $\prod_{p \geq 0} X^p \times \overline{M}^p$.

(2) A morphism from $(x_1, \ldots, x_p, a_1, \ldots, a_p)$ to $(y_1, \ldots, y_q, b_1, \ldots, b_q)$ is a map of finite sets $\theta: \{1, \ldots, p\} \to \{1, \ldots, q\}$ such that

(a) $x_i = y_{\theta(i)}$ for $1 \leq i \leq p$, and

(b) $b_j = \sum_{i \in \theta^{-1}(j)} a_i$ for $1 \leq j \leq q$ (meaning that $b_j = 0$ if $\theta^{-1}(j) = \emptyset$).

Let us denote this morphism by $(\theta^* y, a) \xrightarrow{\theta} (y, \theta_* a)$, where $\theta^*: X^q \to X^p$ and $\theta_*: A^p \to A^q$ are given by the formulas

$$\theta^* y = (y_{\theta(1)}, \ldots, y_{\theta(p)}), \quad \theta_* a = \left(\sum_{i \in \theta^{-1}(1)} a_i, \ldots, \sum_{i \in \theta^{-1}(q)} a_i\right).$$
Then \( \text{Mor } \mathcal{Q}^M(X) \) can be identified with the subset of 
\[
\prod_{p,q \geq 0} X^q \times \text{Map}(\{1, \ldots, p\}, \{1, \ldots, q\}) \times \overline{M}^p
\]
consisting of those triples \((y, \theta, a)\) such that every component of \(\theta_*a\) is in \(\overline{M}\).

(3) The composition of two morphisms \((\theta^*y, a) \xrightarrow{\theta} (y, \theta_*a)\) and \((\psi^*z, b) \xrightarrow{\psi} (z, \psi_*b)\), where \(y = \psi^*z\) and \(b = \theta_*a\), is defined to be the morphism
\[
(\theta^*y, a) = ((\psi\theta)^*z, a) \xrightarrow{\psi\theta} (z, (\psi\theta)_*a) = (z, \psi_*b).
\]

In other words, we have \((z, \psi, b)(y, \theta, a) = (z, \psi, a)\) whenever \(y = \psi^*z\) and \(b = \theta_*a\). Note that \((\psi\theta)_* = \psi_*\theta_*\) holds, for \(A\) is abelian.

**Definition 1.1.** Given a space \(X\), let \(B^M(X)\) denote the identification space of \(\text{Ob } \mathcal{Q}^M(X)\) with respect to the least equivalence relation such that two objects are equivalent if there is a morphism between them. We denote by \(C^M(X)\) the subspace of \(B^M(X)\) consisting of those classes \([x, a]\) such that the components of \(x\), say \(x_1, \ldots, x_p\), are mutually distinct in \(X\).

Each point of \(C^M(X)\) has a representative of the form \((x_1, \ldots, x_p, a_1, \ldots, a_p)\) such that \(a_i \neq 0\) and \(x_i \neq x_j\) if \(i \neq j\). Such a representative will be called reduced. Clearly reduced representatives are isomorphic with each other through permutations of indices, and hence determine a unique finite subset \(\{(x_1, a_1), \ldots, (x_p, a_p)\}\) of \(X \times M\) such that \(a_i \neq 0\) and \(x_i \neq x_j\) if \(i \neq j\). Thus we may regard \(C^M(X)\) as the configuration space of finite distinct points in \(X\) with labels in \(M\), modulo the relation \((x, 0) = \text{null}\).

It enjoys the property that \(\{(x_1, a_1), \ldots, (x_p, a_p)\}\) converges to \(\{(x, a)\}\) as \(x_i \to x\) if, and only if, \(a_1 + \cdots + a_p\) converges to \(a \in \overline{M}\). (Compare [2, §3] as well as [9]).

Note that we have \(C^M(X) = \overline{C}^M(X)\). Note also that \(C^M(X)\) is not necessarily a functor of \(X\).

**Example 1.2.** (1) Let \(M\) be the subset \(\{1\}\) of the abelian monoid \(\mathbb{N}\). Then
\[
C^{\{1\}}(X) = \prod_{n \geq 0} C_n(X),
\]
where \(C_n(X)\) is the configuration spaces of \(n\) distinct points in \(X\).

(2) For \(M = \mathbb{N}\), we have
\[
C^\mathbb{N}(X) = \prod_{n \geq 0} \text{SP}^n(X)
\]
where \(\text{SP}^n(X)\) is the \(n\)-fold symmetric products of \(X\). In other words, \(C^\mathbb{N}(X)\) is the topological free abelian monoid generated by the points of \(X\).

If \(X\) is the complex line \(\mathbb{C}\) then under the correspondence
\[
\{(z_1, n_1), \ldots, (z_p, n_p)\} \mapsto (z - z_1)^{n_1} \cdots (z - z_p)^{n_p},
\]
$C^N(C)$ can be identified with the space of monic polynomials over $C$.

(3) Let $M = N \times 0 \cup 0 \times N \subset N^2$. Then $C^M(C)$ is identified with the space of pairs $(p, q)$ of coprime monic polynomials. Clearly we have $C^M(C) = \coprod_{m,n \geq 0} C^M_{m,n}(C)$, where $C^M_{m,n}(C)$ is the component consisting of those $(p, q)$ such that $\deg p = m$ and $\deg q = n$. In particular, $C^M_{n,n}(C)$ is identified with the space of rational algebraic maps of degree $n$ of the Riemann sphere $S^2 = C \cup \infty$ into itself. (See \[10\]).

(4) The example above can be generalized in several ways. If we take $M = \{(n_1, \ldots, n_m) \mid n_1 \cdots n_m = 0\}$ then $C^M(C)$ is the space of $m$-tuples of polynomials with no common root; its component consisting of those triples $(p_1, \ldots, p_m)$ with $\deg p_1 = \cdots = \deg p_m = n$ is identified with the space of rational algebraic maps of degree $n$ of $S^2$ into the $m$-dimensional complex projective space $P^m$. (See \[10\]). On the other hand, if $M = \bigcup_{k=1}^m \{0\}^{k-1} \times N \times \{0\}^{m-k}$, then $C^M(C)$ is the space of $m$-tuples $(p_1, \ldots, p_m)$ of mutually coprime monic polynomials. (See \[3\]).

(5) Let $M = Z$. Then $C^Z(X) = AG(X)$ is the topological free abelian group generated by the points of $X$. More generally, if $R$ is a topological ring then $C^R(X)$ is the topological free $R$-module generated by the points of $X$.

We next introduce a pointed version of the notion above. For $p \geq 0$, let $p$ denote the pointed finite set $\{0, 1, \ldots, p\}$ based at 0. If $X$ is a pointed space, we have a topological category $\tilde{Q}^M(X)$ defined as follows:

1. Objects of $\tilde{Q}^M(X)$ are the points of $\coprod_{p \geq 0} X^p \times \overline{M}^p$.
2. A morphism from $(x_1, \ldots, x_p, a_1, \ldots, a_p)$ to $(y_1, \ldots, y_q, b_1, \ldots, b_q)$ is a basepoint preserving map $\theta : p \rightarrow q$ such that
   - $x_i = y_{\theta(i)}$ for $1 \leq i \leq p$ (meaning that $x_i = *$ if $\theta(i) = 0$), and
   - $b_j = \sum_{i \in \theta^{-1}(j)} a_i$ for $1 \leq j \leq q$.

Again, we denote this morphism by $(\theta^* y, a) \xrightarrow{\theta} (y, \theta_* a)$, where $\theta^* : X^q \rightarrow X^p$ and $\theta_* : A^p \rightarrow A^q$ are given by the formula (a) and (b) above. Then $\text{Mor} \tilde{Q}^M(X)$ can be identified with the subset of

$$\coprod_{p,q \geq 0} X^q \times \text{Map}_0(p, q) \times \overline{M}^p$$

consisting of those triples $(y, \theta, a)$ such that every component of $\theta_* a$ is in $\overline{M}$.

(3) The composition of morphisms is given by $(z, \psi, b)(y, \theta, a) = (z, \psi \theta, a)$, where $y = \psi^* z$ and $b = \theta_* a$.

One easily verifies that if $X^\times = X \coprod^* \times$ is a space with disjoint basepoint $*$ then the inclusion $Q^M(X) \xrightarrow{I} \tilde{Q}^M(X^\times)$ has a natural left adjoint.
\( \tilde{Q}^M(X_+) \xrightarrow{R} Q^M(X) \) such that \( RI = \text{Id} \). Thus the functor \( \tilde{Q}^M \) can be regarded as a natural extension of \( Q^M \).

**Definition 1.3.** Given a pointed space \( X \), let \( \tilde{B}^M(X) \) denote the identification space of \( \text{Ob} \tilde{Q}^M(X) \) with respect to the least equivalence relation such that two objects are equivalent if there is a morphism between them. We define \( \tilde{C}^M(X) \) to be the subspace of \( \tilde{B}^M(X) \) consisting of those classes \([x,a]\) such that the components \( x_1, \ldots, x_p \) of \( x \) are mutually distinct in \( X \). \( \tilde{C}^M(X) \) has the basepoint \( \emptyset \) represented by \( X^0 \times \overline{M}^0 \).

Each point of \( \tilde{C}^M(X) \) can be identified with a finite subset \( \{(x_1, a_1), \ldots, (x_p, a_p)\} \) of \( X \times M \) such that \( a_i \neq 0, x_i \neq *, \) and \( x_i \neq x_j \) if \( i \neq j \). In addition to the property similar to that of \( C^M(X) \), \( \tilde{C}^M(X) \) enjoys the property that \( \{(x_i, a_i)\} \) converges to \( \emptyset \) as \( x_i \to * \). (That is to say, the basepoint \( * \) is a ‘vanishing point’ of \( \tilde{C}^M(X) \)).

The following is immediate from the definitions.

**Proposition 1.4.** For any space (without basepoint) \( X \), the natural map \( C^M(X) \xrightarrow{\omega} \tilde{C}^M(X_+) \) induced by the inclusion \( X \subset X \coprod * = X_+ \) is a homeomorphism.

**Example 1.5.** (1) Any pointed space \( M \) can be viewed as the set of generators for the topological abelian monoid \( \text{SP}(M, \ast) \) generated by the points of \( M \), where \( \ast \) is identified with 0. The corresponding \( \tilde{C}^M(X) \) is the space of finite subsets of \( X \) “labeled by \( M \)” in the sense of [7].

(2) For any pointed space \( X \), \( \tilde{C}^N(X) \) is identified with \( \text{SP}(X, \ast) \). In particular, if \( X \) is the Riemann sphere \( S^2 = \mathbb{C} \cup \infty \) then \( \text{SP}(S^2, \infty) \) can be identified with the infinite dimensional complex projective space \( \mathbb{P}^\infty \) formed from the vector space \( \mathbb{C}[z] \) of polynomials by mapping \( \{(z_i, n_i)\} \), where \( z_i = [u_i, v_i] \in \mathbb{P}^1 \), to the class of the polynomial \( \prod (u_i - v_i z) \).

(3) Let \( M = \mathbb{N} \times 0 \cup 0 \times \mathbb{N} \subset \mathbb{N}^2 \). Then there is a homotopy equivalence

\[
\mathbb{P}^\infty \vee \mathbb{P}^\infty = \tilde{C}^N(S^2) \vee \tilde{C}^N(S^2) \xrightarrow{\sim} \tilde{C}^M(S^2)
\]

induced by the inclusions \( \mathbb{N} \to \mathbb{N} \times 0 \) and \( \mathbb{N} \to 0 \times \mathbb{N} \). Proposition (1.5) of [10] says that there is a homotopy equivalence \( \tilde{Q} \xrightarrow{\sim} \Omega^2 \tilde{C}^M(S^2) \), where \( \tilde{Q} \) is a ‘group completion’ of the \( H \)-space \( C^M(\mathbb{C}) \cong \tilde{C}^M(\mathbb{R}^2 \times S^0) \). This equivalence is a prototype of our main theorem. \( \square \)

**Remark 1.6.** To any subset \( M \) of a (not necessarily abelian) topological monoid \( A \) we can assign a variant of \( \tilde{C}^M(X) \) constructed from the subcategory of \( \tilde{Q}^M(X) \) whose morphisms are those \( (\theta^* y, a) \xrightarrow{\theta} (y, \theta_* a) \) such that \( \theta \) is order-preserving. Here order-preserving means that \( 0 < \theta(i) < \theta(j) \) implies
i < j. For example, if $M = \mathbb{N}$ then we get the reduced product space $X_\infty$ in the sense of James [4].

2. Homology theories associated with labeled configuration spaces

In this section we construct a generalized homology theory starting from the classifying space of the category $\overline{Q}^M(X)$. Recall that if $\mathcal{X}$ is a topological category, its classifying space $|\mathcal{X}|$ is defined to be the realization of the simplicial space $N\mathcal{X} = \{N_n\mathcal{X}\}$ whose $n$-simplices are chains of arrows $x_0 \to x_1 \to \cdots \to x_n$.

Given a pointed space $X$, let $\overline{Q}^M(X) = |\overline{Q}^M(X)|$ with the basepoint $\emptyset = X^0 \times \overline{M}^0$, and put

$$F^M(X) = \overline{Q}^M(X)/\overline{Q}^M(*), \quad E^M(X) = \Omega F^M(\Sigma X).$$

Here we write $\Sigma Z = S^1 \wedge Z$ and $\Omega Z = \text{Map}_0(S^1, Z)$ for a pointed space $Z$.

As $F^M(*) = *$, the continuous map

$$\text{Map}_0(X,Y) \to \text{Map}_0(F^M(X), F^M(Y)), \quad f \mapsto F^M(f)$$

preserves basepoints. Hence there are natural maps

$$\mu_{Y,X}: Y \wedge F^M(X) \to F^M(Y \wedge X), \quad \mu_{Y,X}: Y \wedge E^M(X) \to E^M(Y \wedge X)$$

induced, by adjunction, from the composite maps

$$Y \xrightarrow{l} \text{Map}_0(X,Y \wedge X) \xrightarrow{T} \text{Map}_0(T(X),T(Y \wedge X)), \quad T = F^M, E^M,$$

where $l$ is given by $l(y)(x) = y \wedge x$. It follows that

**Proposition 2.1.** Both $F^M$ and $E^M$ preserve homotopies. In particular, if $X \simeq Y$ then we have $F^M(X) \simeq F^M(Y)$ and $E^M(X) \simeq E^M(Y)$.

We now state the main result of this section. Recall from [5] that a (homology) group completion $g: Y \to Z$ is an $H$-map between admissible (e.g. homotopy associative and homotopy commutative) $H$-spaces such that $\pi_0 Z$ is the universal group associated with $\pi_0 Y$ and $H_\bullet(Z; k)$ is the localization of the Pontrjagin ring $H_\bullet(Y; k)$ at its submonoid $\pi_0 Y$ for every ring of coefficients $k$. For any $X$, $\overline{Q}^M(X)$ is an admissible $H$-space since $\overline{Q}^M(X)$ is a permutative category with respect to the operation

$$\overline{Q}^M(X) \times \overline{Q}^M(X) \xrightarrow{\oplus} \overline{Q}^M(X), \quad (x,a) \oplus (y,b) = (x,y,a,b).$$

**Theorem 2.2.** Let $X$ be a pointed space. Then

1. the composite $\nu: \overline{Q}^M(X) \to E^M(X)$ of the natural map $\overline{Q}^M(X) \to F^M(X)$ with the map $F^M(X) \to \Omega F^M(\Sigma X) = E^M(X)$ adjoint to $\mu_{S^1,X}$ is a group completion map,
(2) the map \( E^M(X) \to \Omega E^M(\Sigma X) \) adjoint to \( \mu_{S^1,X} \) is a homotopy equivalence, and

(3) for every pointed subspace \( A \) of \( X \), \( E^M(A) \to E^M(X) \to E^M(X \cup CA) \) is a homotopy fibration sequence, that is, the natural map from \( E^M(A) \) to the homotopy fiber of \( E^M(X) \to E^M(X \cup CA) \) is a homotopy equivalence.

This implies

**Corollary 2.3.** For every pointed space \( X \), \( E^M(X) \) has a structure of an infinite loop space.

and also,

**Corollary 2.4.** The correspondence \( X \mapsto \pi \cdot E^M(X) \), together with the natural equivalence \( \pi \cdot E^M(X) \cong \pi_{+1}E^M(\Sigma X) \) induced by \( E^M(X) \cong \Omega E^M(\Sigma X) \), defines a generalized homology theory on the category of pointed spaces.

Observe that, under the natural transformation

\[
\lim_k \pi_{n+k}(X \wedge E^M(S^k)) \to \lim_k \pi_{n+k}E^M(X \wedge S^k) \cong \pi_n E^M(X)
\]

induced by the maps \( X \wedge E^M(S^k) \to E^M(X \wedge S^k) \), \( \pi \cdot E^M \) is equivalent to the generalized homology theory associated with the \( \Omega \)-spectrum \( \{E^M(S^n)\} \). Clearly \( \pi \cdot E^M \) satisfies the wedge axiom.

To prove Theorem 2.2 we need several lemmas similar to those used in the proof of Theorem 1.7 of [12].

**Lemma 2.5.** If \((X, A)\) is an NDR pair then so is \((\tilde{Q}^M(X), \tilde{Q}^M(A))\).

**Proof.** By definition, \((X, A)\) is an NDR pair if there are a map \( u: X \to I \) such that \( A = u^{-1}(0) \) and a homotopy \( h: I \times X \to X \) such that \( h(0, x) = x \) for \( x \in X \), \( h(t, a) = a \) for \( (t, a) \in I \times A \), and \( h(1, x) \in A \) for \( x \in u^{-1}[0, 1] \).

Let \( \Delta^n \) be the standard \( n \)-simplex with vertices \( e_0, e_1, \ldots, e_n \). Given a sequence of real numbers \( \nu = (\nu_0, \ldots, \nu_n) \) such that \( 0 \leq \nu_i \leq 1 \), let \( w_\nu \) be the map \( \Delta^n \to I \) which sends the barycenter \( b_\sigma \) of a face \( \sigma = \{e_{i_0} \cdots e_{i_k}\} \) to \( \max\{\nu_{i_0}, \ldots, \nu_{i_k}\} \) and is linear on each face \( \{b_{\sigma_0} \cdots b_{\sigma_k}\} \) of the barycentric subdivision \( Sd \Delta^n \) of \( \Delta^n \); that is,

\[
w_\nu(s_0 b_{\sigma_0} + \cdots + s_k b_{\sigma_k}) = s_0 w_\nu(b_{\sigma_0}) + \cdots + s_k w_\nu(b_{\sigma_k}) \text{ if } s_0 + \cdots + s_k = 1.
\]

Let \( \delta_i \) denote the inclusion of \( \Delta^{n-1} \) as the \( i \)-th face of \( \Delta^n \) and \( \sigma_i \) the \( i \)-th projection \( \Delta^{n+1} \to \Delta^n \). Then we have

\[
w_\nu \delta_i = w_{d_i \nu}, \quad w_\nu \sigma_i = w_{s_i \nu},
\]
Lemma 2.8. For any $X$, $Y$ the natural map $\tilde{Q}^M(X \vee Y) \to \tilde{Q}^M(X) \times \tilde{Q}^M(Y)$ induced by the evident projections is a homotopy equivalence.
Proof. Let \( \pi \) denote the composite
\[
\tilde{Q}^M(X) \times \tilde{Q}^M(Y) \xrightarrow{I_X \times I_Y} \tilde{Q}^M(X \vee Y)^2 \oplus \tilde{Q}^M(X \vee Y),
\]
where \( I_X \) and \( I_Y \) are induced by the inclusions \( X \to X \vee Y \) and \( Y \to X \vee Y \) respectively. Then \( \pi \) is an adjoint inverse to \( \tilde{Q}^M(X \vee Y) \to \tilde{Q}^M(X) \times \tilde{Q}^M(Y) \); hence its realization gives a homotopy inverse to \( \tilde{Q}^M(X \vee Y) \to \tilde{Q}^M(X) \times \tilde{Q}^M(Y) \).

Now suppose \( X_\bullet \) is a pointed simplicial space. Then the maps \( X_n \times \Delta^n \to |X_\bullet|, (x, s) \mapsto [x, s] \), factor through \( X_n \wedge \Delta^n_+ = X_n \times \Delta^n / * \times \Delta^n \) and we obtain
\[
\tilde{Q}^M(X_n) \times \Delta^n \to \tilde{Q}^M(X_n) \wedge \Delta^n_+ \to \tilde{Q}^M(X_n \wedge \Delta^n_+) \to \tilde{Q}^M(|X_\bullet|)
\]
which, in turn, induce a pointed map \( |	ilde{Q}^M(X_\bullet)| \to \tilde{Q}^M(|X_\bullet|) \).

Lemma 2.9. Given a pointed simplicial space \( X_\bullet \), the map \( |	ilde{Q}^M(X_\bullet)| \to \tilde{Q}^M(|X_\bullet|) \) is a homeomorphism.

Proof. As is well known, \( |	ilde{Q}^M(X_\bullet)| = \left[ [m] \mapsto |\tilde{Q}^M(X_m)| \right] \) is homeomorphic to the successive realization
\[
\left[ [n] \mapsto |N_n \tilde{Q}^M(X_\bullet)| \right] = \left[ [n] \mapsto |[m] \mapsto N_n \tilde{Q}^M(X_m)| \right]
\]
of the bisimplicial space \( ([m], [n]) \mapsto N_n \tilde{Q}^M(X_m) \). To prove the lemma, it suffices to show that the natural map \( |N_n \tilde{Q}^M(X_\bullet)| \to N_n \tilde{Q}^M(|X_\bullet|) \) is a homeomorphism for \( n \geq 0 \). But this follows from the fact that \( |X_\bullet^p| \to |X_\bullet|^p \) is a homeomorphism for \( p \geq 0 \).

We are ready to prove Theorem 2.2. In fact, we can prove a more general result. A functor \( T \) of the category of pointed spaces into itself is called continuous if the function
\[
\text{Map}_0(X, Y) \to \text{Map}_0(T(X), T(Y)), \quad f \mapsto T(f)
\]
is continuous for any \( X \) and \( Y \). If \( T \) is continuous and \( T(*) = * \) then there are pairings
\[
\mu_{Y,X} : Y \wedge T(X) \to T(Y \wedge X), \quad \mu'_{Y,X} : T(Y) \wedge X \to T(Y \wedge X)
\]
obtained, by adjunction, from the composite
\[
Y \xrightarrow{L} \text{Map}_0(X, Y \wedge X) \xrightarrow{T} \text{Map}_0(T(X), T(Y \wedge X)),
\]
where \( L \) is given by \( l(y)(x) = y \wedge x \), and the similar one
\[
X \to \text{Map}_0(T(Y), T(Y \wedge X)).
\]
It follows, in particular, that any homotopy \( h: I_+ \wedge X \to Y \) induces one \( I_+ \wedge T(X) \to T(Y) \) between \( T(h_0) \) and \( T(h_1) \); hence \( T \) preserves homotopy.

Given a continuous functor \( T \) of the category of pointed spaces into itself, put \( F(X) = T(X)/T(*) \) and \( E(X) = \Omega F(SX) \). (When \( T(X) = \hat{Q}^M(X) \), \( F(X) = F^M(X) \) and \( E(X) = E^M(X) \)). As \( F(*) = * \) and \( E(*) = * \), there are natural maps

\[
F(X) \xrightarrow{\varepsilon} \Omega F(SX) = E(X), \quad E(X) \xrightarrow{\varepsilon} \Omega E(SX)
\]

adjoint to \( S^1 \wedge F(X) \to F(S^1 \wedge X) \) and \( S^1 \wedge E(X) \to E(S^1 \wedge X) \) respectively.

**Theorem 2.10.** Suppose \( T \) satisfies the following conditions:

(C1) If \((X, A)\) is an NDR pair then so is \((T(X), T(A))\).
(C2) \( T(*) \) is contractible.
(C3) For any \( X, Y \) the natural map \( T(X \vee Y) \to T(X) \times T(Y) \) induced by the evident projections is a homotopy equivalence.
(C4) For any pointed simplicial space \( X_\bullet \) the natural map \( |T(X_\bullet)| \to T(|X_\bullet|) \) is a homotopy equivalence.

Let us regard \( T(X) \) as an \( H \)-space whose composition law is given by the composite

\[
T(X) \times T(X) \xrightarrow{p^{-1}} T(X \vee X) \xrightarrow{m} T(X),
\]

where \( p^{-1} \) is a homotopy inverse to the equivalence \( T(X \vee X) \to T(X) \times T(X) \) and \( m \) is induced by the folding map \( X \vee X \to X \). Then we have

1. the composite \( \nu: T(X) \to E(X) \) of the natural map \( T(X) \to F(X) \)
   with \( F(X) \xrightarrow{\varepsilon} E(X) \) is a group completion,
2. the map \( E(X) \xrightarrow{\varepsilon} \Omega E(SX) \) is a homotopy equivalence, and
3. for every pointed subspace \( A \) of \( X \), \( E(A) \to E(X) \to E(X \cup CA) \) is a homotopy fibration sequence.

**Proof.** Given a pair of pointed maps \( f: A \to X \), \( g: A \to Y \), let \( Z(f, g)_\bullet \) denote the pointed simplicial space defined as follows: \( Z(f, g)_n = X \vee A^{\vee n} \vee Y \), where \( A^{\vee n} \) is the \( n \)-fold wedge sum of \( A \), and for every \( \theta: [m] \to [n] \), \( \theta^*: Z(f, g)_n \to Z(f, g)_m \) is given by \( \theta^*(z) = z \) if \( z \in X \vee Y \), and

\[
\theta^*(\iota_j(a)) = \begin{cases} f(a) & \text{if } j \leq \theta(0), \\ \iota_k(a) & \text{if } \theta(k-1) < j \leq \theta(k), \ 1 \leq k \leq m, \\ g(a) & \text{if } \theta(m) < j. \end{cases}
\]

Here \( \iota_j \) denotes the inclusion of \( A \) into the \( j \)-th summand of \( A^{\vee n} \). As every point of \( X \vee A^{\vee n} \vee Y \) is the image of a unique point of \( X \vee A \vee Y \) under some degeneracy map, the realization of \( Z(f, g)_\bullet \) is the quotient of \( (X \vee Y) \coprod I \times (X \vee A \vee Y) \) by the identifications, \((t, z) \sim z, (0, a) \sim f(a), (1, a) \sim g(a), \)

\[
(0, a) \sim (1, a) \sim (0, a) \sim (1, a)
\]

\[
\theta^*(z) \sim \theta^*(z) \sim \theta^*(z) \sim \theta^*(z)
\]

\[
(0, a) \sim (1, a) \sim (0, a) \sim (1, a)
\]
where \( t \in I, z \in X \vee Y \) and \( a \in A \). Thus \( |Z(f, g)_\bullet| \) is homeomorphic to the reduced double mapping cylinder
\[
Z(f, g) = X \cup_f (I \times A) \cup_g Y/I \times *.
\]

By (C4), there is a natural equivalence
\[
|T(Z(f, g)_\bullet)| \to T(Z(f, g)).
\]
Moreover, by (C1), \( T(Z(f, g)_\bullet) \) is a proper simplicial space (or good simplicial space in the sense of [8]). Hence the natural map \( ||T(Z(f, g)_\bullet)|| \to |T(Z(f, g)_\bullet)| \) is an equivalence. Here \( ||T(Z(f, g)_\bullet)|| \) denotes the fat realization of \( T(Z(f, g)_\bullet) \).

To see that \( \nu: T(X) \to E(X) \) is a group completion, let \( f \) and \( g \) be the constant map \( X \to * \) and write \( Z(f, g)_\bullet = X^{\vee \bullet} \). Then the simplicial space \( T(X^{\vee \bullet}) \) is the composite of the \( \Gamma \)-space \( n \mapsto \Lambda(n) = T(X \land n) = T(X^{\vee n}) \) with the contravariant functor \( \Delta \to \Gamma, [n] \mapsto n \).

But the properties (C2) and (C3) ensure that \( \Lambda \) satisfies the conditions (i) and (ii) of [8, Definition 1.2]. Thus, by the main theorems of [8] (or [6, §3]), the map
\[
\Lambda(1) \to \Omega B \Lambda(1) = \Omega(|\Lambda|/\Lambda(0))
\]
induced by the natural map \( \Delta^1 \times \Lambda(1) \to |\Lambda| \) is a group completion.

As \( \nu \) can be written as the composite
\[
T(X) = \Lambda(1) \to \Omega(|\Lambda|/\Lambda(0)) \xrightarrow{\psi} E(X),
\]
where \( \psi \) is induced by the equivalence \( |\Lambda| = |T(X^{\vee \bullet})| \to T(\Sigma X) \), we conclude that \( \nu \) is a group completion.

The second statement is proved by the similar argument with \( X \) replaced by \( \Sigma X \). Since \( \Lambda(1) = T(\Sigma X) \) has a continuous homotopy inverse induced by the map
\[
S^1 \land X \to S^1 \land X, \quad [t, x] \mapsto [1 - t, x],
\]
\( \Lambda(1) \to \Omega B \Lambda(1) \) is a homotopy equivalence by Proposition 1.5 of [8]. This implies that \( F(\Sigma X) \to E(\Sigma X) \) is an equivalence; hence so is \( E(X) \to \Omega E(\Sigma X) \).

To prove the last statement, we need only show that
\[
(2.1) \quad T(A) \to T(X) \to T(X \cup CA)
\]
is a homotopy fibration sequence if \( T(A) \) has a homotopy inverse. But this is equivalent to say that
\[
(2.2) \quad T(A) \xrightarrow{T(j)} T(X \cup ZA) \xrightarrow{T(p)} T(X \cup CA),
\]
is a homotopy fibration sequence. Here \( X \cup ZA = Z(i, 1_A) \) is the reduced mapping cylinder of the inclusion \( i: A \to X \), \( j \) is the inclusion of \( A \) into...
{1} \times A \subset X \cup ZA and \( p: X \cup ZA \to X \cup CA \) is the map which collapses \{1\} \times A to the basepoint.

Let us write \( Z(i, 1_A)_* = X \vee A^\vee \vee A \) and \( Z(i, \ast)_* = X \vee A^\vee \), where \( \ast \) is the constant map \( X \to \ast \). Then \( j \) is induced by the inclusion \( j_\ast \) of \( A \) as the last summand of \( X \vee A^\vee \vee A \) and \( p \) is induced by the simplicial map

\[ p_\ast: X \vee A^\vee \vee A \to X \vee A^\vee \]

which collapses the last summand \( A \) to the basepoint. There is a commutative diagram

\[
\begin{array}{ccc}
T(A) & \longrightarrow & T(X \vee A) \\
\downarrow & & \downarrow \\
T(A) & \longrightarrow & T(X \vee A^\vee \vee A) \\
\end{array}
\]

(2.3)

\[
\begin{array}{ccc}
T(A) & \longrightarrow & T(X \vee A^\vee \vee A) \\
\downarrow & & \downarrow \\
T(X \vee A^\vee \vee A) & \longrightarrow & T(X \vee A^\vee) \\
\end{array}
\]

in which the vertical arrows are the natural maps sending 0-simplexes into realizations.

As \( T(p_\ast) \) is a map of proper simplicial spaces, we can apply the argument of the proof of Proposition 1.5 of [8], with the standard realization instead of the fat one. Thus the right hand square of (2.3) is homotopy-cartesian if \( T(A) \) has a homotopy inverse. As \( T(A) \to T(X \vee A) \to T(X) \) is a homotopy fibration sequence, we conclude that (2.2) is also a homotopy fibration sequence. This completes the proof of Theorem 2.2.

**Remark 2.11.** If we take the non-abelian version of \( \tilde{Q}^M(X) \) discussed in Remark 1.6 then the corresponding \( T(X) \) no longer satisfies (C3). Therefore the commutativity is essential in the construction of \( E^M(X) \).

The following variation of Theorem 2.10 is also useful. Again, let \( T \) be a continuous functor of the category of pointed spaces into itself.

**Theorem 2.12.** Suppose \( T \) satisfies the following conditions:

- (C2)' \( T(\ast) = \ast \).
- (C3) For any \( X, Y \), the natural map \( T(X \vee Y) \to T(X) \times T(Y) \) induced by the projections is a homotopy equivalence.
- (C4)' For any pointed simplicial space \( X_\ast \), the natural map \( \|T(X_\ast)\| \to T(\|X_\ast\|) \) is a homotopy equivalence.

Let \( E(X) = \Omega T(\Sigma X) \) for any pointed space \( X \). Then we have

1. the map \( T(X) \xrightarrow{\xi} E(X) \), adjoint to the natural map \( S^1 \wedge T(X) \to T(S^1 \wedge X) \), is a group completion,
2. \( E(X) = \Omega T(\Sigma X) \xrightarrow{\Omega \xi} \Omega E(\Sigma X) \) is a homotopy equivalence, and
3. for every pointed subspace \( A \) of \( X \), \( E(A) \to E(X) \to E(X \cup CA) \) is a homotopy fibration sequence.
Proof. By arguing as in the proof of Theorem 2.10, but replacing the standard realization with the fat one, we see that the natural map \( T(X) \to \Omega \| T(X^{\bullet}) \| \) is a group completion for any \( X \), and is an equivalence if \( T(X) \) has a continuous homotopy inverse. As the composite

\[
\Omega \| T(X^{\bullet}) \| \to \Omega T(\| X^{\bullet} \|) \to \Omega T(\| X^{\bullet} \|) = E(X)
\]

is an equivalence, we see that the first and the second properties hold. The last property is proved similarly. \( \square \)

3. Classical examples

3.1. The stable homotopy theory. In [8] Segal showed that for any space (without basepoint) \( X \) there exists a \( \Gamma \)-space \( B\Sigma X \) such that the associated spectrum \( B(B\Sigma X) \) is equivalent to the suspension spectrum \( S(X_+) \).

By definition, \( B\Sigma X \) is the \( \Gamma \)-space associated with the symmetric monoidal category equivalent to \( Q^{(1)}(X) \). Hence there is a natural homotopy equivalence

\[
B\Sigma X(1) = |Q^{(1)}(X)| \simeq \tilde{Q}^{(1)}(X_+)
\]

induced by the inclusion \( Q^{(1)}(X) \to \tilde{Q}^{(1)}(X_+) \). It follows by Proposition 3.6 of [8] that \( E^{S^0}(X_+) = E^{(1)}(X_+) \) has the weak homotopy type of \( \Omega^{\infty}\Sigma^{\infty}(X_+) \). (The case \( X_+ = S^0 \) is the Barratt-Priddy-Quillen theorem).

More generally, we can prove

**Theorem 3.1.** Let \( M \) be a pointed space having the homotopy type of a \( CW \)-complex. If we regard \( M \) as the set of generators for \( SP(M, \ast) \) then there is a natural isomorphism of generalized homology theories

\[
\pi_{\bullet}E^M(X) \cong \pi_{\bullet}S^M(X \land M)
\]

defined on the category of pointed spaces having the homotopy type of a \( CW \)-complex.

**Proof.** Let \( T(X) = \tilde{Q}^M(X)/\tilde{Q}^{(0)}(X) \) and \( \tilde{E}^M(X) = \Omega(T(\Sigma X)/T(\ast)) \). As \( \tilde{Q}^{(0)}(X) \) is contractible, the natural map \( \tilde{Q}^M(X) \to T(X) \) is an equivalence; hence so is \( E^M(X) \to \tilde{E}^M(X) \). Let \( \psi: X \land M \to \tilde{E}^M(X) \) be the natural map induced by the composite

\[
X \times M \xymatrix{ \ar[r]^\psi & \text{Ob} \tilde{Q}^M(X) \ar[r] & |\tilde{Q}^M(X)| = \tilde{Q}^M(X) \ar[r] & T(X) \ar[r] & \tilde{E}^M(X)
}\]

Then there is a diagram of natural maps

\[
\Omega^{\infty}\Sigma^{\infty}(X \land M) \xymatrix{ \ar[r]^{\Omega^{\infty}\psi} & \Omega^{\infty}\tilde{E}^M(\Sigma^{\infty}X) \ar[l]|{\varepsilon^{\infty}} & \tilde{E}^M(X) \ar[l]|{\varepsilon^{\infty}} \ar[l]|{\cong} & E^M(X)
}\]

in which \( \Omega^{\infty}\psi \) is the limit of

\[
\Omega^\ast: \Omega^\ast\Sigma^\ast(X \land M) = \Omega(\Sigma^\ast X \land M) \to \Omega^\ast \tilde{E}^M(\Sigma^\ast X)
\]
and \( \varepsilon^\infty \) is the natural map
\[
\hat{E}^M(X) \to \lim_n \Omega^n \hat{E}^M(\Sigma^n X) = \Omega^\infty \hat{E}^M(\Sigma^\infty X).
\]
As \( \varepsilon^\infty \) is a weak equivalence, we obtain a natural transformation
\[
\tau: \pi_\bullet^\tau(X \wedge M) = \pi_\bullet \Omega^\infty \Sigma^\infty (X \wedge M) \to \pi_\bullet E^M(X).
\]
Observe that \( \tilde{Q}^M(X) \) satisfies the properties (C1)–(C4) with respect to \( M \) as well as \( X \). Hence \( \pi_\bullet E^M(X) \cong \pi_\bullet \hat{E}^M(X) \) defines a homology theory in either variable \( X \) or \( M \). Therefore, to prove that \( \tau \) is an isomorphism we need only show the case \( X = M = S^0 \). But this follows from the classical Barratt-Priddy-Quillen theorem. (See Proposition 3.5 of [8]).

Note that if \( X \) is pathwise connected then so is \( \tilde{C}^M(\mathbb{R}^\infty \times X) \). Therefore \( \tilde{C}^M(\mathbb{R}^\infty \times X) \) has the weak homotopy type of \( E^M(X) \cong \Omega^\infty \Sigma^\infty (X \wedge M) \) if \( X \) has the homotopy type of a connected finite \( CW \)-complex. But this also follows from the next theorem together with the fact that the inclusion \( C^{X \wedge M}(\mathbb{R}^\infty) \to \tilde{C}^M(\mathbb{R}^\infty \times X) \) is a homotopy equivalence if \( X \) admits an embedding into \( \mathbb{R}^\infty \). (Compare Proposition 4.4).

**Theorem 3.2.** The natural map \( C^X(\mathbb{R}^\infty) \to \Omega C^{\Sigma X}(\mathbb{R}^\infty) \) is a group completion and there is a natural isomorphism of homology theories
\[
\pi_\bullet \Omega C^{\Sigma X}(\mathbb{R}^\infty) \cong \pi_\bullet^\tau X.
\]

**Proof.** Let \( T(X) = C^X(\mathbb{R}^\infty) \). We first show that \( T \) satisfies the conditions (C2)', (C3) and (C4)' of Theorem 2.12. It is obvious that (C2)' holds. To see that (C3) holds, choose a linear isometry \( l: \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty \). Then there is a composite
\[
C^X(\mathbb{R}^\infty) \times C^Y(\mathbb{R}^\infty) \xrightarrow{(i_*, j_*)} C^{X \vee Y}(\mathbb{R}^\infty) \overset{\tau}{\to} C^{X \vee Y}(\mathbb{R}^\infty)
\]
where \( i_* \) and \( j_* \) are induced by the inclusions \( X \) and \( Y \) into \( X \vee Y \), and \( \tau \) takes \( \{(v_i, p_i)\}, \{(w_j, q_j)\} \) to \( \{(l(0, v_i), p_i)\} \cup \{(l(e, w_j), q_j)\} \) for some \( e \neq 0 \). This gives a homotopy inverse to \( T(X \vee Y) \to T(X) \times T(Y) \), for the space of linear isometries of \( \mathbb{R}^\infty \) is contractible. Finally, (C4)' is equivalent to the condition that the natural map
\[
|\tau C^{X_\bullet}(\mathbb{R}^\infty)| \to |C^{X_\bullet}(\mathbb{R}^\infty)| \cong C^{X_\bullet}(\mathbb{R}^\infty)
\]
is a homotopy equivalence, where \( \tau X_\bullet \) denotes the proper simplicial space associated with \( X_\bullet \). (See [8, Appendix A]). But for every \( n \geq 0 \), the composite
\[
C^{X_n}(\mathbb{R}^\infty) \xrightarrow{\tilde{\tau}} \tau_n C^{X_\bullet}(\mathbb{R}^\infty) \rightarrow C^{\tau_n X_\bullet}(\mathbb{R}^\infty)
\]
is an equivalence induced by \( X_n \xrightarrow{\tilde{\tau}} \tau_n X_\bullet \). Hence \( |\tau C^{X_\bullet}(\mathbb{R}^\infty)| \to |C^{X_\bullet}(\mathbb{R}^\infty)| \) is a homotopy equivalence by Theorem A.4 of [5].
It follows by Theorem 2.12 that the natural map $C^X(\mathbb{R}^\infty) \to \Omega C^{\Sigma X}(\mathbb{R}^\infty)$ is a group completion, and the correspondence $X \mapsto \pi_* \Omega C^{\Sigma X}(\mathbb{R}^\infty)$ defines a generalized homology theory. Moreover, there is a natural transformation of homology theories $\nu: \pi_* \Omega C^{\Sigma X}(\mathbb{R}^\infty) \to \pi_* \Omega \Sigma X$, taken by the argument similar to that of Theorem 3.2.

3.2. The ordinary homology theory. Now suppose $M$ is a topological abelian monoid. Then $\tilde{C}^M(X) = \tilde{B}^M(X)$ is a functor of $X$, and there is a natural map

$$E^M(X) \to \Omega \tilde{C}^M(\Sigma X)$$

induced by the functor $\tilde{Q}^M(X) \to \tilde{C}^M(X)$ which takes a morphism $(\theta^* y, a) \to (y, \theta_* a)$ to the class $[\theta^* y, a] = [y, \theta_* a]$.

Given a graded group $G_\bullet$, let $\tilde{H}_\bullet(X; G_\bullet)$ denote the ordinary homology of $X$ with coefficients in $G_\bullet$.

The following generalizes the results of Dold-Thom [1].

**Theorem 3.3.** Let $M$ be an arbitrary topological abelian monoid. Then the natural map $\tilde{C}^M(X) \to \Omega \tilde{C}^M(\Sigma X)$ is a group completion, and there are natural isomorphisms of homology theories

$$\pi_* E^M(X) \cong \pi_* \Omega \tilde{C}^M(\Sigma X) \cong \tilde{H}_\bullet(X; \pi_* G),$$

where $G$ is the group completion of $M$, i.e. $G = \Omega BM$. In particular, we have

$$\pi_* \tilde{C}^M(X) \cong \tilde{H}_\bullet(X; \pi_* G) = \sum_r \tilde{H}_{n-r}(X; \pi_r G)$$

if $M$ is grouplike or if $X$ is connected.

**Proof.** We first show that the functor $\tilde{C}^M$ satisfies the conditions (C2)', (C3) and (C4)' of Theorem 2.12. It is obvious that (C2)' holds. (C3) follows from the fact that the composite

$$\tilde{C}^M(X) \times \tilde{C}^M(Y) \xrightarrow{(i_*j_*)} \tilde{C}^M(X \vee Y) \xrightarrow{\oplus} \tilde{C}^M(X \vee Y)$$

is the inverse to $\tilde{C}^M(X \vee Y) \to \tilde{C}^M(X) \times \tilde{C}^M(Y)$. Here $i_*$ and $j_*$ are induced by the inclusions $X \to X \vee Y$ and $Y \to X \vee Y$ respectively, and $\oplus$ is the multiplication given by $[x, a] \oplus [y, b] = [x, y, a, b]$. (C4)' is equivalent to say that

$$|\tau \tilde{C}^M(X_\bullet)| \to |\tilde{C}^M(\tau X_\bullet)| \cong \tilde{C}^M(|\tau X_\bullet|)$$

is a homotopy equivalence, and is proved by the argument similar to that of Theorem 3.2.
By Theorem 2.12, \( \tilde{C}^M(X) \to \Omega \tilde{C}^M(\Sigma X) \) is a group completion and the correspondence \( X \mapsto \pi_* \Omega \tilde{C}^M(\Sigma X) \) defines a generalized homology theory which is represented by the \( \Omega \)-spectrum \( \{ \Omega \tilde{C}^M(S^{n+1}) \} \). However, as \( \tilde{C}^M(S^{n+1}) \) are topological abelian monoids, \( \{ \Omega \tilde{C}^M(S^{n+1}) \} \) is equivalent to the generalized Eilenberg-MacLane spectrum \( \prod_n K(\pi_n G, n) \). (Cf. Satz 7.1 of [1]). Thus there is a natural isomorphism of homology theories

\[
\pi_* \Omega \tilde{C}^M(\Sigma X) \cong \tilde{H}_*(X; \pi_* G).
\]

On the other hand, as \( \tilde{Q}^M(S^0) \to \tilde{C}^M(S^0) = M \) has a right adjoint \( M \to \tilde{Q}^M(S^0) \), \( a \mapsto (1, a) \), the induced map \( \tilde{Q}^M(S^0) \to \tilde{C}^M(S^0) \) is a homotopy equivalence. Therefore \( E^M(S^0) \to \Omega \tilde{C}^M(\Sigma S^0) \) is a weak equivalence, and so is \( E^M(X) \to \Omega \tilde{C}^M(\Sigma X) \) for any pointed \( CW \)-complex \( X \). This completes the proof of the theorem. \( \square \)

### 3.3. Theories associated with subsets of \( N^m \).

As we stated in Examples 1.2 and 1.5, the labeled configuration space associated with a subset of \( N^m \) is interesting in that it is related to the topology of the space of rational algebraic maps. The following proposition together with (P4) implies that if \( M \) is sufficiently large, \( \pi_* \tilde{C}^M(R^n \times X) \) converges to \( \tilde{H}_*(X; \mathbb{Z}^m) \) as \( n \to \infty \).

**Proposition 3.4.** Let \( M \) be a subset of \( N^m \). Suppose \( M \) contains all the subsets of the form \( \{0\}^{k-1} \times N \times \{0\}^{m-k} \), \( 1 \leq k \leq m \). Then there is a natural isomorphism of homology theories

\[
\pi_* E^M(X) \cong \tilde{H}_*(X; \mathbb{Z}^m).
\]

**Proof.** Let \( M' = \bigcup_{k=1}^m \{0\}^{k-1} \times N \times \{0\}^{m-k} \). We shall show that the \( H \)-map

\[
\tilde{Q}^{M'}(S^0) \to \tilde{Q}^M(S^0)
\]

induced by the inclusion \( M' \subset M \) is a homotopy equivalence. This implies, in particular, \( E^{M'}(S^0) \simeq E^{N^m}(S^0) \). But \( E^{N^m}(S^0) \simeq \mathbb{Z}^m \) as we have already seen. Hence \( E^M(S^0) \simeq \mathbb{Z}^m \) for any \( M \) containing \( M' \).

As \( \tilde{Q}^M(S^0) \) is homotopy equivalent to \( Q^M(\{1\}) = |Q^M(\{1\})| \), we need only show that the inclusion \( I: Q^{M'}(\{1\}) \to Q^M(\{1\}) \) has a homotopy inverse. For brevity, let us write \( Q^M(\{1\}) = Q^M \). The elements of \( Q^M \) are identified with finite sequences of the elements of \( M \). Let \( \eta \) be the functor \( Q^M \to Q^{M'} \) given by

\[
\eta(a_1, \ldots, a_p) = (a_1^1 e_1, \ldots, a_1^m e_m, \ldots, a_p^1 e_1, \ldots, a_p^m e_m) \in (M')^{mp},
\]

where \( a_j = (a_1^j, a_m^j), 1 \leq j \leq p \), and \( e_k = (0, \ldots, 1, \ldots, 0) \in \{0\}^{k-1} \times N \times \{0\}^{m-k} \). Then its realization \( J: Q^M(\{1\}) \to Q^{M'}(\{1\}) \) gives a homotopy inverse to \( I \). In fact, if \( Q^M_0 \) denotes the full subcategory of \( Q^{M'} \) consisting of those objects \((a_1, \ldots, a_p)\) such that \( a_k \neq 0 \) for \( 1 \leq k \leq p \) then the
inclusion $Q_0^{M'} \to Q^{M'}$ has the evident retraction $\rho: Q^{M'} \to Q_0^{M'}$; hence $|Q_0^{M'}| \simeq |Q^{M'}| = Q^{M'}(\{1\})$. But it is easily verified that $\rho\eta$ gives a left adjoint to the composite $Q_0^{M'} \to Q^{M'} \to Q^M$. Therefore $JI \simeq \text{Id}$. Similarly we can show that $IJ \simeq \text{Id}$. □

4. The relation between $\tilde{C}^M$ and $E^M$

If $X$ is a pointed space, we write $\mathbb{R}^n \times X = \mathbb{R}^n \times X/\mathbb{R}^n \times \{*\}$. Let $L(n)$ be the space of linear isometries $(\mathbb{R}^\infty)^n \to \mathbb{R}^\infty$. For each $l \in L(n)$, consider the composite

$$\widetilde{B}^M((\mathbb{R}^\infty)^n \times X)^n \xrightarrow{(l \times 1)^n} \widetilde{B}^M(\mathbb{R}^\infty \times X)^n \xrightarrow{(l \times 1)^n} \widetilde{B}^M((\mathbb{R}^\infty)^n \times X)^n,$$

where $i_j$ is the inclusion of $\mathbb{R}^\infty$ into the $j$-th component of $(\mathbb{R}^\infty)^n$ and $\oplus$ is the juxtaposition map. Clearly this restricts to a continuous map

$$\tilde{C}^M(\mathbb{R}^\infty \times X)^n \to \tilde{C}^M(\mathbb{R}^\infty \times X).$$

Thus we obtain a linear isometries operad action on $\tilde{C}^M(\mathbb{R}^\infty \times X)$, hence a group completion of $\tilde{C}^M(\mathbb{R}^\infty \times X)$ into an infinite loop space.

The following theorem is nothing but (P4) of the main theorem.

**Theorem 4.1.** If $X$ is a pointed finite CW-complex then $E^M(X)$ has the weak homotopy type of a group completion of $\tilde{C}^M(\mathbb{R}^\infty \times X)$.

From now on we assume, by replacing $M$ by $\overline{M}$ if necessary, that $M$ contains 0.

To establish a relationship between $E^M(X)$ and $\tilde{C}^M(\mathbb{R}^\infty \times X)$, we introduce a pointed map

$$\Phi: \tilde{Q}^{[S^*M]}([S_*X]) \to [S_*\tilde{C}^M(\mathbb{R}^\infty \times X)],$$

where $S_*X = \{S_nX\}$ is the total singular complex of $X$ and $\tilde{Q}^{[S^*M]}([S_*X])$ is the realization of the diagonal of the bisimplicial set

$$([m], [n]) \mapsto N_nS^{mM}(S_mX) = S_m(N_n\tilde{Q}^M(X)).$$

As the evaluation maps $|S_*((N_n\tilde{Q}^M(X))| \to N_n\tilde{Q}^M(X)$ are weak equivalences, the induced map $\tilde{Q}^{[S^*M]}([S_*X]) \to \tilde{Q}^M(X)$ is a homology equivalence. (Cf. Theorem A.4 of [5]). But an $H$-map between grouplike admissible $H$-spaces is a weak homotopy equivalence if and only if it is a homology equivalence. (Cf. [5, Remark 1.5]). Hence

**Lemma 4.2.** The natural map $\rho: E^{[S^*M]}([S_*X]) \to E^M(X)$ induced by the evaluation maps $|S_*M| \to M$ and $|S_*X| \to X$ is a weak homotopy equivalence.
Let $\Delta^n$ be the standard $n$-simplex with vertices $e_0, e_1, \ldots, e_n$. Let $\delta_j$ be the inclusion of $\Delta^{n-1}$ as the $j$-th face of $\Delta^n$ and $\sigma_j$ the $j$-th projection $\Delta^{n+1} \to \Delta^n$. Write $\partial_j \Delta^n = \delta_j(\Delta^{n-1})$ and $\partial \Delta^n = \bigcup_{j=0}^n \partial_j \Delta^n$.

Lemma 4.3. There is a system of continuous maps $R(\theta, i) : \Delta^n \to \mathbf{R}^\infty$, defined for every sequence of pointed maps $\theta = (p_0 \xrightarrow{\theta_1} p_1 \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_n} p_n)$ and every $i \in p_0$, enjoying the following properties:

1. $R(\theta, 0) \equiv 0$,
2. for any $s \in \Delta^n - \partial_0 \Delta^n$, $R(\theta, i)(s) \neq R(\theta, j)(s)$ if $i \neq j$,
3. $R(\theta, i)\sigma_j = R(s_j \theta, i)$, $R(\theta, i)\delta_0 = R(d_0 \theta, \theta_1(i))$ and $R(\theta, i)\delta_j = R(d_j \theta, i)$ if $j > 0$.

Here we denote $s_j \theta = (\ldots, \theta_j, 1, \theta_{j+1}, \ldots)$ for $0 \leq j \leq n$, and

$$d_j \theta = \begin{cases} 
(\theta_2, \ldots, \theta_n) & \text{if } j = 0 \\
(\ldots, \theta_{j-1}, \theta_{j+1}, \theta_j, \theta_{j+2}, \ldots) & \text{if } 1 \leq j \leq n-1 \\
(\theta_1, \ldots, \theta_{n-1}) & \text{if } j = n.
\end{cases}$$

Proof. For every sequence $\theta = (p_0 \to p_1 \to \cdots \to p_n)$ let $|\theta|$ be a copy of $p_0$ and let $S = \bigcup_{|\theta|} |\theta|$ be the union of all $|\theta|$. We may identify $\mathbf{R}^\infty$ with the linear space $\mathbf{R}(S)$ spanned by $S$.

The construction of $R(\theta, i)$ proceeds by induction on the length $n$ of $\theta$. For $n = 0$ we define $R(\theta, i)$ by $R(\theta, i)(e_0) = i \in |\theta| \subset \mathbf{R}(S)$. Assume that we have assigned $R(\theta', i')$ for every sequence $\theta'$ of length $< n$. Assume further that the image of such $R(\theta', i')$ is contained in $\mathbf{R}(S_{n-1})$, where $S_{n-1}$ is the subset of $S$ consisting of all $|\theta|$ such that the length of $\theta$ is less than $n$.

Let $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ be a sequence of length $n$. Let $b_n$ be the barycenter of $\Delta^n$, so that every element of $\Delta^n = b_n * \partial \Delta^n$ can be written in the form $(1-t)b_n + t\delta_j x$ for some $j$ and $x \in \Delta^{n-1}$. Then we define $R(\theta, i)$ as follows:

1. if $\theta = s_j \theta'$ for some $\theta'$ of length $n-1$ then

$$R(\theta, i)(x) = R(\theta', i)(\delta_j(x))$$

for any $x \in \Delta^n$,

2. if $\theta \neq s_j \theta'$ for all $\theta'$ then

$$R(\theta, i)((1-t)b_n + t\delta_j x) = (1-t)i_\theta + tR(d_j \theta, i')(x),$$

where $i_\theta \in |\theta|$ corresponds to $i$ under the bijection $|\theta| \cong p_0$ and $i' = \theta_1(i)$ if $j = 0$, $i' = i$ otherwise.

As $i_\theta \in S_n - S_{n-1}$, one easily verifies that $R(\theta, i)$ enjoys the required properties and that the image of $R(\theta, i)$ is contained in $\mathbf{R}(S_n)$. Thus the proof proceeds by induction on $n$. \qed
Let $D(X)_\bullet$ be the diagonal of the bisimplicial set 

$$([m],[n]) \mapsto N_n \tilde{Q}^{S_m,M}(S_nX).$$

Then each element $f$ of $D(X)_n$ is of the form 

$$(x^0,a^0) \overset{\theta_1}{\to} (x^1,a^1) \overset{\theta_2}{\to} \cdots \overset{\theta_n}{\to} (x^n,a^n),$$

where $(x^i,a^i) \in S_nX^{p_i} \times S_nM^{p_i}$, $x^{i-1} = \theta_i^* x^i$ and $a^i = \theta_{i+1} a^{i-1}$. Let us write 

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n), \quad x^0 = (x_1, \ldots, x_{p_0}), \quad a^0 = (a_1, \ldots, a_{p_0})$$

and define $\Phi_n(f) : \Delta^n \to \tilde{B}^M(\mathbb{R}^\infty \times X)$ by 

$$\Phi_n(f)(s) = [(R(\theta,1)(s), x_1(s)), \ldots, (R(\theta,p_0)(s), x_{p_0}(s)), a_1(s), \ldots, a_{p_0}(s)].$$

Let us write $\Phi : D(X)_\bullet \to S_\bullet \tilde{C}^M(\mathbb{R}^\infty \times X)$.

Proposition 4.4. If $X$ admits an embedding into $\mathbb{R}^\infty$ then the inclusion 

$$\tilde{C}^M(\mathbb{R}^\infty \times X)' \to \tilde{C}^M(\mathbb{R}^\infty \times X)$$

is a homotopy equivalence.

Proof. Let $i$ be an embedding of $X$ into $\mathbb{R}^\infty$ and define a homotopy $h : I \times X \to \mathbb{R}^\infty$ by $h(t,x) = (1-t)i(x)$. If we write $h_0(x) = h(t,x)$ then $h_0 = i$ and $h_1$ is the constant map with value 0. Let $l$ be a linear isometry $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$. Then there is a homotopy 

$$H : I_+ \wedge \tilde{B}^M(\mathbb{R}^\infty \times X) \to \tilde{B}^M(\mathbb{R}^\infty \times X)$$

such that $H_t$ is induced by the composite 

$$\mathbb{R}^\infty \times X \xrightarrow{1 \times \text{diag}} \mathbb{R}^\infty \times X \xrightarrow{1 \times h_t \times 1} \mathbb{R}^\infty \times \mathbb{R}^\infty \times X \xrightarrow{l \times 1} \mathbb{R}^\infty \times X.$$ 

One easily verifies that

(1) $H$ restricts to $I_+ \wedge \tilde{C}^M(\mathbb{R}^\infty \times X) \to \tilde{C}^M(\mathbb{R}^\infty \times X)$ and also to $I_+ \wedge \tilde{C}^M(\mathbb{R}^\infty \times X)' \to \tilde{C}^M(\mathbb{R}^\infty \times X)'$,

(2) the image of $f_0 = H_0|\tilde{C}^M(\mathbb{R}^\infty \times X)$ is in $\tilde{C}^M(\mathbb{R}^\infty \times X)'$, and

(3) $f_1 = H_1|\tilde{C}^M(\mathbb{R}^\infty \times X)$ is induced by $l' \times 1 : \mathbb{R}^\infty \times X \to \mathbb{R}^\infty \times X$, where $l'$ is the composite $\mathbb{R}^\infty \cong \mathbb{R}^\infty \times \{0\} \subset \mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{l} \mathbb{R}^\infty$.  

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As the space of linear isometries of $\mathbb{R}^\infty$ is contractible, $l'$ is isotopic to the identity through linear isometries. Thus we have $f_0 \simeq f_1 \simeq 1$, showing that $f_0$ is a homotopy inverse to the inclusion $\tilde{C}^M (\mathbb{R}^\infty \times X)' \to \tilde{C}^M (\mathbb{R}^\infty \times X)$. □

Clearly the correspondence $X \mapsto \tilde{C}^M (\mathbb{R}^\infty \times X)'$ is a continuous functor of $X$. Let $T(X) = \tilde{C}^M (\mathbb{R}^\infty \times X)'$ and $E(X) = \Omega T(\Sigma X)$. Then

**Proposition 4.5.** The natural map $\nu: T(X) \to E(X)$ is a group completion and the correspondence $X \mapsto \pi_* E(X)$ defines a generalized homology theory on the category of pointed spaces.

**Proof.** It suffices to show that $T$ satisfies the conditions (C2)$'$, (C3) and (C4)$'$ of Theorem 2.12. But this follows from the argument similar to that used in the proof of Theorem 3.2. □

It follows, by Propositions 4.4 and 4.5, that $E(X)$ has the weak homotopy type of a group completion of $\tilde{C}^M (\mathbb{R}^\infty \times X)$. Moreover, there is a diagram

$$E^M(X) \xrightarrow{\rho} E^{[S^*M]}([S_*X]) \xrightarrow{\psi} E(X),$$

in which $\psi$ is induced by the composite

$$\tilde{Q}^{[S^*M]}([S_*X]) \xrightarrow{\Phi'} [S_* \tilde{C}^M (\mathbb{R}^\infty \times X)'] \to \tilde{C}^M (\mathbb{R}^\infty \times X)' = T(X).$$

Now Theorem 4.1 follows from

**Proposition 4.6.** If $X$ is a pointed finite CW-complex then

$$\psi: E^{[S^*M]}([S_*X]) \to E(X)$$

is a weak equivalence.

Beware that $E^{[S^*M]}([S_*X])$ is not continuous in $X$. Nevertheless, the correspondence $X \mapsto \pi_* E^{[S^*M]}([S_*X])$ defines a generalized homology theory on pointed finite CW-complexes, if its suspension isomorphism is defined to be the composite

$$\pi_* E^{[S^*M]}([S_*X]) \xrightarrow{\varepsilon_*} \pi_* \Map_0([S_*S^1], E^{[S^*M]}([S_*\Sigma X]))$$

$$\xrightarrow{\iota_*^{-1}} \pi_* \Map_0(S^1, E^{[S^*M]}([S_*\Sigma X])) = \pi_{n+1} E^{[S^*M]}([S_*\Sigma X]).$$

Here $\varepsilon$ is the adjoint to $[S_*S^1] \wedge E^{[S^*M]}([S_*X]) \xrightarrow{\rho} E^{[S^*M]}([S_*S^1 \wedge X])$ and $\iota$ is induced by the equivalence $[S_*S^1] \to S^1$. Clearly

$$\psi_*: \pi_* E^{[S^*M]}([S_*X]) \to \pi_* E(X)$$

is a natural transformation of homology theories.

Thus, to prove Proposition 4.6, we need only show the case $X = S^0$. But this is a consequence of
Proposition 4.7. The map 
\[ \Phi': Q^{|S^\bullet M|(S^0)} \to |S^\bullet C^M(\mathbb{R}^\infty \times S^0)'| = |S^\bullet C^M(\mathbb{R}^\infty)| \]
is a homology equivalence.

We give a proof of this proposition in the next section.

5. Proof of Proposition 4.7

We shall prove the case \( M \) is discrete. The general case follows from this, for \( \Phi' \) is obtained as the realization of the map of proper simplicial spaces 
\[ Q^n(0) \to |S^\bullet C^n(\mathbb{R}^\infty)|, \quad n \geq 0. \]
As in the preceding section, we assume \( M \) contains 0.

Definition 5.1. A continuous map \( g \) of a simplex \( \sigma = [p_0 p_1 \ldots p_n] \) into \( C^M(\mathbb{R}^\infty) \) is called proper if \( n = 0 \) or if there are maps \( v_1, \ldots, v_p \) of \( \sigma \) into \( \mathbb{R}^\infty \) and elements \( a_1, \ldots, a_p \) of \( M \) satisfying the following conditions:

1. \( g(s) = [v_1(s), \ldots, v_p(s), a_1, \ldots, a_p] \in C^M(\mathbb{R}^\infty) \) for all \( s \in \sigma \),
2. if \( v_i(s) = v_j(s) \) holds for some \( s \in \sigma - \partial_0 \sigma \), where \( \partial_0 \sigma = [p_1 p_2 \ldots p_n] \), then \( v_i = v_j \) all over \( \sigma \),
3. the restriction of \( g \) to \( \partial_0 \sigma \) is proper with respect to the restrictions of the maps \( v_1, \ldots, v_p \) to \( \partial_0 \sigma \) and labels \( a_1, \ldots, a_p \).

Observe that if \( g: \sigma \to C^M(\mathbb{R}^\infty) \) is proper then so is its restriction to each face 
\[ \partial_i \sigma = [p_0 \cdot p_{i-1} p_{i+1} \cdot p_n], \quad 0 \leq i \leq n. \]
Moreover, its restriction to each simplex of \( Sd \sigma \) is also proper. Here \( Sd \sigma \) denotes the barycentric subdivision of \( \sigma \); a simplex of of which is of the form 
\[ [b_{\sigma_0} b_{\sigma_1} \ldots b_{\sigma_r}], \quad \text{where} \sigma_0, \sigma_1, \ldots, \sigma_r \text{ are faces of } \sigma \text{ such that } \sigma_0 > \sigma_1 > \cdots > \sigma_r \text{ and } b_{\sigma_i} \text{ is the barycenter of } \sigma_i, \quad 0 \leq i \leq r. \]

For each \( n \geq 0 \), let \( S^n C^M(\mathbb{R}^\infty)' \) denote the set of proper maps from \( \Delta^n \to C^M(\mathbb{R}^\infty) \). Then \( S^\bullet C^M(\mathbb{R}^\infty)' = \{ S^n C^M(\mathbb{R}^\infty)' \} \) is a subcomplex of \( S^\bullet C^M(\mathbb{R}^\infty) \) and \( \Phi' \) factors as 
\[ \tilde{Q}^M(S^0) \xrightarrow{\Phi'} |S^\bullet C^M(\mathbb{R}^\infty)'| \xrightarrow{I} |S^\bullet C^M(\mathbb{R}^\infty)|, \]
where \( I \) is the inclusion map. Let \( \Phi'' \) denote the composite 
\[ Q^M(\{1\}) \cong \tilde{Q}^M(S^0) \xrightarrow{\Phi'} |S^\bullet C^M(\mathbb{R}^\infty)'|, \]
where \( Q^M(\{1\}) = |Q^M(\{1\})| \). Then Proposition 4.7 is a consequence of the following two lemmas.
Lemma 5.2. $\Phi'': Q^M(\{1\}) \to |S_*C^M(\mathbb{R}^\infty)|''$ is a homotopy equivalence.

Lemma 5.3. $I: |S_*C^M(\mathbb{R}^\infty)|'' \to |S_*C^M(\mathbb{R}^\infty)|$ is a homology equivalence.

Proof of Lemma 5.2. To construct a homotopy inverse to $\Phi''$, we introduce a simplicial subdivision of $\Delta^n$. For every $n \geq 0$, let $[[n]]$ denote the partially ordered set $\{(i,j) \mid 0 \leq i \leq j \leq n\}$, where $(i,j) \leq (i',j')$ if and only if both $i \geq i'$ and $j \leq j'$ hold. (Thus we have $(i,j) \leq (0,n)$ for any $(i,j) \in [[n]]$). Let us regard $[[n]]$ as a category in which a morphism from $(i,j)$ to $(i',j')$ is the relation $(i,j) \leq (i',j')$. Then there is a piecewise-linear homeomorphism $\phi_n$ from the realization of $[[n]]$ to $\Delta^n$ such that $\phi_n(j,j) = e_j$ and $\phi_n(i,j) = \frac{1}{2}(e_i + e_j)$ (i.e. the middle point of $[e_i, e_j]$) if $i < j$. Given a linearly ordered subset $J = \{(i_0,j_0) < \cdots < (i_k,j_k)\}$ of $[[n]]$, let $\Delta_J$ denote the simplex spanned by the vertices $\phi_n(i_0,j_0), \ldots, \phi_n(i_k,j_k)$. Then the collection $\{\Delta_J\}$ defines a simplicial subdivision of $\Delta^n$ having $2^n$ $n$-simplices corresponding to those $J = \{(i_0,j_0) < \cdots < (i_n,j_n)\}$ such that $(i_0,j_0) = (i,i)$ for some $i$ and $(i_n,j_n) = (0,n)$. This subdivision coincides with the edgewise subdivision defined in [7, Appendix 1].

Let $g: \Delta^n \to C^M(\mathbb{R}^\infty)$ be a proper simplex. Then there are maps $v_1, \ldots, v_p$ of $\Delta^n$ into $\mathbb{R}^\infty$ and elements $a_1, \ldots, a_p$ of $M$ such that the following holds:

$$g(s) = [v_1(s), \ldots, v_p(s), a_1, \ldots, a_p], \quad s \in \Delta^n$$

For each $j$, let $(x^{i,j}_{p(j,j)}, a^{i,j}_{p(j,j)}, \ldots, a^{j,j}_{p(j,j)})$ be the unique reduced representative of $g(e_j)$ such that $x^{i,j}_{p(j,j)} < \cdots < x^{j,j}_{p(j,j)}$ with respect to the lexicographical order in $\mathbb{R}^\infty$. Then we have

$$a^{i,j}_k = \sum_{m \in \Lambda(i,j;k)} a^{i,j}_m,$$

where $\Lambda(i,j;k)$ denotes the set of those integers $m$ such that $v_m(e_j) = x^{i,j}_{k}$. More generally, for any pair $(i,j)$ such that $i \leq j$, let

$$\bigcup_m \{v_m(e_j)\} = \{x^{i,j}_1, \ldots, x^{i,j}_{p(i,j)}\}, \quad x^{i,j}_1 < \cdots < x^{i,j}_{p(i,j)},$$

where $m$ runs through the integers such that $v_m(e_i) = x^{i,i}_{k}$ for some $k$, $1 \leq k \leq p(i,i)$. Obviously $\{x^{i,j}_1, \ldots, x^{i,j}_{p(i,j)}\}$ is a subset of $\{x^{i',j}_{i'}, \ldots, x^{i,j}_{p(i,j)}\}$ if $i' < i \leq j$. Let

$$a^{i,j}_k = \sum_{m \in \Lambda(i,j;k)} a^{i,j}_m,$$

where $\Lambda(i,j;k) = \{m \mid v_m(e_j) = x^{i,j}_{k}\}$. Again, $(x^{i,j}_{1}, \ldots, x^{i,j}_{p(i,j)}; a^{i,j}_{1}, \ldots, a^{i,j}_{p(i,j)})$ is a (possibly non-reduced) representative of $g(e_j)$.

For any triple $(i,i',j)$ with $i' < i \leq j$, there is an injection

$$\iota_{i > i'j}: \{1, \ldots, p(i,j)\} \to \{1, \ldots, p(i',j)\}$$
such that \( t_i > i, j \star (x_1^{i,j}, \ldots, x_p^{i,j}) = (x_1^{i,j}, \ldots, x_p^{i,j}) \) holds. On the other hand, if \( i \leq j < j' \), there is a surjection

\[
\pi_{i,j < j'}: \{1, \ldots, p(i, j)\} \to \{1, \ldots, p(i, j')\}
\]

defined by choosing any \( m \) such that \( v_m(e_j) = x_1^{i,j} \) and putting \( \pi_{i,j < j'}(k) = l \) if \( v_m(e_{j'}) = x_l^{i,j'} \). Clearly we have

\[
\pi_{i,j < j'} \pi_{i,j < j''} = \pi_{i,j < j''}
\]

\[
t_i > i', j \pi_{i,j < j'} = t_i > i', j
\]

\[
\pi_{i', j < j'} \pi_{i,j < j'} = t_i > i', j' \pi_{i,j < j'}
\]

Moreover, if we write \( a^{i,j} = (a_1^{i,j}, \ldots, a_p^{i,j}) \) then we have

\[
\pi_{i,j < j'} a^{i,j} = a^{i,j'}, \quad t_i > i', j a^{i,j} = a^{i,j'}.
\]

Hence there is a functor \( \tilde{g}: [[n]] \to Q^M(\text{point}) \) such that \( \tilde{g}(i, j) = a^{i,j} \) and

\[
\tilde{g}((i, j) < (i', j')) = (a^{i,j} \pi_{i,j < j'}, a^{i,j'}) = (a^{i,j} t_i > i', j a^{i,j'}).
\]

Now we define \( \Theta: |S_n C^M(\mathbb{R}^\infty)^n| \to |Q^M(\text{point})| \) as follows. Suppose \( [g, s] \) is a point of \( |S_n C^M(\mathbb{R}^\infty)^n| \), where \( g \in S_n C^M(\mathbb{R}^\infty)^n \) and \( s \in \Delta^n \). Choose a linearly ordered subset \( J \) of \([n]\) such that \( s \in \text{Int} \Delta_J \). Let \( \xi_J \) be the affine homeomorphism \( \Delta_J \cong \Delta_J^{\text{aff}} \) which preserves the order of the vertices, and let \( \tilde{g}_J \) be the composite

\[
[j] \xrightarrow{\pi} J \xrightarrow{\tilde{g}_J} Q^M(\text{point}).
\]

Then \( \Theta([g, s]) \) is defined to be the class of \( (\tilde{g}_J, \xi_J(s)) \) \( \in N_J Q^M(\text{point}) \times \Delta_J^{\text{aff}} \).

We will show that \( \Theta \) is a homotopy inverse to \( \Phi'^n \). Given

\[
f = (a_0 \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_n} a^n) \in N_n Q^M(\text{point}),
\]

let \( \tilde{f} \) denote the functor \( [[n]] \to Q^M(\text{point}) \) which takes \( (i, j) \) to \( a^j \) and such that

\[
\tilde{f}((i, j) < (i', j')) = \theta_j \cdots \theta_{j+1}: a^j \to a^{j'}, \quad \tilde{f}((i, j) < (i', j)) = \text{Id}: a^j \to a^j.
\]

Let \( F \) be the self-map of \(|Q^M(\text{point})|\) given by

\[
F([f, s]) = [\tilde{f}_J, \xi_J(s)] \quad \text{if } s \in \text{Int} \Delta_J.
\]

Then there is a homotopy \( \Theta \Phi' \cong F \) induced by the evident natural transformation \( \tilde{F}_n(f) \to \tilde{f} \). There is also a homotopy \( F \cong \text{Id} \) given by \( h_t([f, s]) = F([f, h'_t(s)]) \), where \( h'_t \) is a self-map of \( \Delta^n \) such that

\[
h'_t \left( \frac{1}{2} (e_i + e_j) \right) = \frac{1 - t}{2} e_i + \frac{1 + t}{2} e_j
\]
and is linear on each $\Delta_j$. Then $h'_1$ maps $\Delta^n$ onto $\Delta_{j_0}$, where $J_0 = \{(0, 0) < (0, 1) < \cdots < (0, n)\}$, and $\xi_{j_0}h'_1$ is the identity of $\Delta^n$. As $\tilde{f}_{j_0} = f$, $h_1$ is the identity of $|Q^M(\text{point})|$. Therefore $\Theta\Phi'' \simeq \text{Id}$.

On the other hand, we can inductively construct a homotopy $\Phi''\Theta \simeq \text{Id}$ by using the general position argument as in the proof of Lemma 4.3. The verification will be left to the reader. \hfill \square

To prove Lemma 5.3, we need the following lemma.

**Lemma 5.4.** Let $X = |K|$ and $Y = |T|$ be polyhedra and let $f: X \to W$ be a continuous map of $X$ into an open subset $W$ of $Y$. Suppose that $K$ is a finite complex and there is a subcomplex $L$ of $K$ such that the restriction of $f$ to $A = |L|$ is simplicial with respect to $L$ and $T$. Suppose further that $W$ is the union of the open stars $O(p, T)$, where $p$ runs through the vertices contained in $W$. Then there exists a subdivision $K'$ of $K$, leaving $L$ unaltered, and a simplicial map $g: X \to W$, with respect to $K'$ and $T$, such that $g(x) = f(x)$ for $x \in A$ and

$$f \simeq g: X \to W \text{ rel } A.$$

**Proof.** By taking the barycentric subdivision of $K$ relative to $L$, we may assume that $L$ is a full subcomplex of $K$. Let $\bar{L} = \{\sigma \in K \mid \sigma \cap |L| = \emptyset\}$ and let $K_1 = \text{Sd}_{\bar{L}} K$ be the barycentric subdivision of $K$ relative to $L \cup \bar{L}$. Then there is a simplicial map $h: X \to X$, with respect to $K_1$, such that

1. if $v \in L \cup \bar{L}$ then $h(v) = v$,
2. if $v \not\in L \cup \bar{L}$, so that $v$ is the barycenter of some simplex $\sigma$ such that $\sigma \not\in L$ and $\sigma \cap |L| \neq \emptyset$, then $h(v)$ is a vertex of $\sigma$ that belongs to $L$.

As we have $O(v, K_1) \subset O(h(v), K_1)$ for every $v \in K_1$, $h$ is a simplicial approximation of the identity and

$$1 \simeq h: X \to X \text{ rel } A. \tag{5.1}$$

Now let $K_2 = \text{Sd}_L K_1$. Then $h(N(v, K_2)) \subset h(O(v, K_1)) = O(v, L)$ holds for $v \in L$. Here $N(v, K_2)$ denotes the (closed) star of $v$ in $K_2$. As $f|A$ is simplicial with respect to $L$ and $T$, we have $fh(N(v, K_2)) \subset f(O(v, L)) \subset O(f(v), T)$ or, equivalently,

$$N(v, K_2) \subset (fh)^{-1}(O(f(v), T)). \tag{5.2}$$

Let $U$ be a covering of $X$ consisting of those open sets

$$U(p) = (fh)^{-1}(O(p, T)),$$

where $p$ runs through the vertices contained in $W$. Then, by the auxiliary theorem of [11, p. 318], there is a subdivision $K'$ of $K_2$, leaving $L$ unaltered,
enjoying the property that for every vertex \( v \) of \( K' \) there is a vertex \( g(v) \) contained in \( W \) such that

\[
N(v, K') \subset U(g(v)) = (fh)^{-1}(O(g(v), T)).
\]

Here we may assume, by (5.2), that \( g(v) = f(v) \) if \( v \in L \).

Evidently the correspondence \( v \mapsto g(v) \) determines a simplicial approximation \( g: X \to W \) of \( fh \) with respect to \( K' \) and \( T \). Moreover, as \( g|A = f|A = fh|A \), we have

\[
fh \simeq g: X \to W \text{ rel } A.
\]

It follows that \( f \simeq g: X \to W \text{ rel } A \), where \( g \) is simplicial with respect to a subdivision \( K' \), leaving \( L \) unaltered, and \( T \). \( \Box \)

We are now ready to prove Lemma 5.3.

**Proof of Lemma 5.3.** For \( n \geq 0 \), let \( S_nC^M(\mathbb{R}^{\infty})'' \) be the abelian group generated by all proper maps \( \Delta^n \to C^M(\mathbb{R}^{\infty}) \). Then \( S\cdot C^M(\mathbb{R}^{\infty})'' = \{ S_nC^M(\mathbb{R}^{\infty})'' \} \) is a subcomplex of the singular chain complex \( S\cdot C^M(\mathbb{R}^{\infty}) \) of \( C^M(\mathbb{R}^{\infty}) \). We will show that the inclusion

\[
I\cdot : S\cdot C^M(\mathbb{R}^{\infty})'' \to S\cdot C^M(\mathbb{R}^{\infty})
\]

is a chain homotopy equivalence. This, of course, implies the lemma.

For this purpose, we introduce a triangulation of \( B^M(\mathbb{R}^{\infty}) \). Let \( X\cdot \) be the ordered simplicial set associated with the standard triangulation of \( \mathbb{R} \), with the integers as vertices. For \( N \geq 0 \), let \( X^N\cdot \) be the \( N \)-fold cartesian product of \( X\cdot \). Then \( X^N\cdot \) is isomorphic to the ordered simplicial set associated with the product triangulation of \( \mathbb{R}^N \); hence \( |X^N\cdot| \cong \mathbb{R}^N \). Moreover, if \( \theta \) is a map of finite sets then \( \theta^*: (\mathbb{R}^N)^q \to (\mathbb{R}^N)^p \) is induced by a simplicial map \( (X^N\cdot)^q \to (X^N\cdot)^p \). Clearly the equivalence relation on \( \coprod_{p \geq 0} (X^N\cdot)^p \times M^p \) generated by the relations \( (\theta^* x, a) \sim (x, \theta a) \) is compatible with the simplicial structure. Thus we obtain a simplicial set \( P\cdot \) such that \( |P\cdot| \cong B^M(\mathbb{R}^N) \) holds.

As \( P\cdot \) is a regular simplicial set, its realization is a regular \( CW \)-complex. Hence we can triangulate \( B^M(\mathbb{R}^N) = |P\cdot| \) by successively taking the elementary subdivision \( b_{\sigma} \ast (\partial \sigma)' \) of every closed cell \( \sigma \) of positive dimension. Here \( b_{\sigma} \) is the barycenter of \( \sigma \) and \( (\partial \sigma)' \) is the (already constructed) subdivision of \( \partial \sigma \). This triangulation is compatible with the natural map \( B^M(\mathbb{R}^N) \to B^M(\mathbb{R}^{N+1}) \) induced by the inclusion \( \mathbb{R}^N \subset \mathbb{R}^{N+1} \). Thus we obtain a triangulation \( T \) of \( B^M(\mathbb{R}^{\infty}) \). It is readily seen that \( C^M(\mathbb{R}^{\infty}) \) is the union of open stars \( O(p, T) \), where \( p \) runs through the vertices contained in \( C^M(\mathbb{R}^{\infty}) \).
To each $f : \Delta^n \to C^M(\mathbb{R}^\infty)$ we assign a homotopy $\tilde{f} : I \times \Delta^n \to C^M(\mathbb{R}^\infty)$, with $\tilde{f}_0 = f$, and simplicial subdivisions $K_f$ of $\Delta^n$ and $L_f$ of $I \times \Delta^n$ enjoying the following properties:

1. $(d_i(\tilde{f}))_t = d_i(\tilde{f}_t)$ for $0 \leq i \leq n$ and $t \in I$, where $d_i f = f\delta_i : \Delta^{n-1} \to C^M(\mathbb{R}^\infty)$,
2. $K_f|\partial_i \Delta^n = \{\sigma \in K_f | \sigma \subset \partial_i \Delta^n\} = K_{d_i f}$,
3. $L_f|\partial(I \times \Delta^n) = \{0\} \times \Delta^n \cup \{1\} \times K_f \cup \bigcup_{0 \leq i \leq n}(1 \times \delta_i)L_{d_i f}$,
4. $\tilde{f}_1$ is simplicial with respect to $K_f$ and $T$, and its restriction to $\sigma$ is proper for every $\sigma \in \text{Sd} \ K_f$,
5. if $f \in S_n C^M(\mathbb{R}^\infty)^n$ then the restriction of $\tilde{f}$ to $|\tau|$ is proper for every $\tau \in \text{Sd}_{\{0\} \times \Delta^n} L_f$.

The construction is by induction on $n$. For $n = 0$, we take as $\tilde{f}$ a path in $C^M(\mathbb{R}^\infty)$ joining $f$ to a vertex $p$ of $T$ such that $f \in O(p,T)$. Suppose that we have constructed $\tilde{f}'$, $K_{f'}$ and $L_{f'}$ for every $f' : \Delta^k \to C^M(\mathbb{R}^\infty)$ with $k < n$. Then for each $f : \Delta^n \to C^M(\mathbb{R}^\infty)$ and each $i$ with $0 \leq i \leq n$ there are a homotopy $\tilde{d}_i f$ and subdivisions $K_{d_i f}$ of $\partial_i \Delta^n$ and $L_{d_i f}$ of $I \times \Delta^{n-1}$ enjoying the properties mentioned above. Let $\tilde{d} f : I \times \partial \Delta^n \to C^M(\mathbb{R}^\infty)$ be the homotopy determined by the homotopies $d_i f$. Then, by the homotopy extension property, there is a homotopy $F : I \times \Delta^n \to C^M(\mathbb{R}^\infty)$ such that $F_0 = f$ and $F|I \times \partial \Delta^n = \tilde{d} f$.

Let $L$ be the subdivision of $\partial \Delta^n$ determined by the subdivisions $K_{d_i f}$ and let $K = b_n * L$ be the elementary subdivision of $\Delta^n$ arising from $L$. Then $L$ is a subcomplex of $K$ and $F_1|\partial \Delta^n = \tilde{d} f_1$ is simplicial with respect to $L$ and $T$. Therefore we obtain, by Lemma 5.4, a subdivision $K_f$ of $K$, leaving $L$ unaltered, and a simplicial map $g : \Delta^n \to C^M(\mathbb{R}^\infty)$ with respect to $K_f$ and $T$ such that the following holds:

$$g|\partial \Delta^n = F_1|\partial \Delta^n = \tilde{d} f_1, \quad F_1 \simeq g \text{ rel } \partial \Delta^n.$$

Combining these two homotopies $f = F_0 \simeq F_1$ and $F_1 \simeq g \text{ rel } \partial \Delta^n$, we obtain a homotopy $\tilde{f}$ with $\tilde{f}_0 = f$ and $\tilde{f}_1 = g$.

We will show that the restriction of $g$ to $\sigma$ is proper for every $\sigma \in \text{Sd} \ K_f$. By definition, $\sigma$ is of the form $[b_{\sigma_0}b_{\sigma_1} \cdots b_{\sigma_r}]$, where $\sigma_1, \ldots, \sigma_r$ are simplexes of $K_f$ such that $\sigma_0 > \sigma_1 > \cdots > \sigma_r$. As $g$ is simplicial with respect to $K_f$ and $T$, its restriction to $\sigma$ is an affine map of $\sigma$ onto the (possibly degenerate) simplex $\tau = [b_{\tau_0}b_{\tau_1} \cdots b_{\tau_r}]$, where $\tau_i = g(\sigma_i) \in T$. Such a map is certainly proper by the construction of $T$.

We also define $L_f$ to be the elementary subdivision $b * L'$, where $b = (\frac{1}{2}, b_n) \in I \times \Delta^n$ and

$$L' = \{0\} \times \Delta^n \cup \{1\} \times K_f \cup \bigcup_i (1 \times \delta_i)L_{d_i f}.$$
If \( f \in S_n C^M(\mathbb{R}^\infty)'' \) then, by using the general position argument as in the proof of Lemma 4.3, we can choose such an \( \tilde{f} \) that its restriction to each simplex of \( Sd_{(0)} \times \Delta^n L_f \) is proper. Thus \( \tilde{f} \), together with \( K_f \) and \( L_f \), enjoys all the required properties.

Now let \( J_n \) be a homomorphism \( S_n C^M(\mathbb{R}^\infty) \rightarrow S_n C^M(\mathbb{R}^\infty)'' \) which takes each generator \( f : \Delta^n \rightarrow C^M(\mathbb{R}^\infty) \) to the chain

\[
[\tilde{f}]_1 = \sum_{\sigma \in Sd} \varepsilon(\sigma, \Delta^n)(\tilde{f}_1 \circ i_\sigma) \in S_n C^M(\mathbb{R}^\infty)'',
\]

where \( \sigma \) runs through the \( n \)-dimensional simplexes of \( Sd K_f \) is the affine homeomorphism \( \Delta^n \cong \sigma \) which preserves the order of vertices, and \( \varepsilon(\sigma, \Delta^n) \) is either 1 or \(-1\) according as the orientation of \( \sigma \) induced by \( i_\sigma \) is compatible with the standard orientation of \( \Delta^n \) or not. By the properties of \( \tilde{f}_1 \) and \( K_f \), \( \{J_n\} \) defines a chain map

\[
J_* : S_* C^M(\mathbb{R}^\infty) \rightarrow S_* C^M(\mathbb{R}^\infty)''.
\]

Similarly let \( h_n \) be a homomorphism \( S_n C^M(\mathbb{R}^\infty) \rightarrow S_{n+1} C^M(\mathbb{R}^\infty) \) which takes a generator \( f \) to

\[
[\tilde{f}^1] = \sum_{\tau \in Sd_{(0)} \times \Delta^n} L_{f, n+1} \varepsilon(\tau, I \times \Delta^n)(\tilde{f}^1 \circ i_\tau) \in S_{n+1} C^M(\mathbb{R}^\infty),
\]

where \( \tau \) runs through the \((n+1)\)-dimensional simplexes of \( Sd_{(0)} \times \Delta^n L_f \) is the affine homeomorphism \( \Delta^{n+1} \cong |\tau| \) which preserves the order of vertices, and \( \varepsilon(\tau, I \times \Delta^n) \) is either 1 or \(-1\) according as the orientation of \( |\tau| \) induced by \( i_\tau \) is compatible with the orientation of \( I \times \Delta^n \) or not. Here \( I \times \Delta^n \) is so oriented that we have

\[
\partial(I \times \Delta^n) = -(\{0\} \times \Delta^n) + (\{1\} \times \Delta^n) - \sum_{i=0}^{n} (-1)^i (I \times \partial_i \Delta^n).
\]

Then we have

\[
\partial_{n+1} h_n(f) = \partial_{n+1} \left( \sum_{\tau \in Sd_{(0)} \times \Delta^n} L_{f, n+1} \varepsilon(\tau, I \times \Delta^n)(\tilde{f} \circ i_\tau) \right)
= \sum_{\tau \in Sd_{(0)} \times \Delta^n} L_{f, n+1} \sum_{i=0}^{n+1} (-1)^i \varepsilon(\tau, I \times \Delta^n)(\tilde{f} \circ i_\tau \circ \delta_i)
= \sum_{\tau \in A_{f, n+1}} \varepsilon(\tau, I \times \Delta^n)(\tilde{f} \circ i_{\delta_0 \tau}),
\]

where \( A_{f, n+1} \) denotes the set of those \( \tau \in Sd_{(0)} \times \Delta^n L_{f, n+1} \) such that \( \partial_0 \tau \) is contained in \( \partial(I \times \Delta^n) \) and \( i_{\delta_0 \tau} = i_\tau \circ \delta_0 \). (By the construction of \( Sd_{(0)} \times \Delta^n L_f \), \( \partial_i \tau \) cannot be contained in \( \partial(I \times \Delta^n) \) unless \( i = 0 \).)
But we have
\[ \varepsilon(\tau, I \times \Delta^n)(\tilde{f} \circ i_{\partial_0 \tau}) = \begin{cases} 
-f & \text{if } \partial_0 \tau = \{0\} \times \Delta^n \\
\varepsilon(\sigma, \Delta^n)(\tilde{f}_1 \circ i_\sigma) & \text{if } \partial_0 \tau = \{1\} \times \sigma, \\
-(-1)^i \varepsilon(\tau', I \times \Delta^{n-1})(d_i f \circ i_{\tau'}) & \sigma \in \text{Sd} K^n_f \\
& \text{if } \partial_0 \tau = (1 \times \delta_i) \tau', \\
& \tau' \in \text{Sd}_{\{0\} \times \Delta^n} L_{d_i f}.
\end{cases} \]

Therefore

\[ \partial_{n+1} h_n(f) + h_{n-1} \partial_n(f) = J_n(f) - f \]

holds for any generator \( f \) of \( S_n C^M(\mathbb{R}^\infty) \). This, of course, implies that \( h = \{h_n\} \) defines a chain homotopy \( I_* J_* \simeq \text{Id} \).

On the other hand, if \( f \) belongs to \( S_n C^M(\mathbb{R}^\infty)'' \) then \( [\tilde{f}] \in S_{n+1} C^M(\mathbb{R}^\infty)'' \). Hence there are homomorphisms \( k_n \colon S_n C^M(\mathbb{R}^\infty)'' \to S_{n+1} C^M(\mathbb{R}^\infty)'' \) defining a chain homotopy \( J_* I_* \simeq \text{Id} \).

This completes the proof of Lemma 5.3. \( \Box \)

**References**


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