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Tomas Edson Barros*  Alcibiades Rigas†
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1. Introduction

The term non-cancellation phenomenon is used here in the following sense:

Differentiable manifolds $M_1$, $M_2$ and $N$ are given to satisfy

(i) $M_1 \times N$ is diffeomorphic to $M_2 \times N$;

(ii) $M_1 \not\cong M_2$, where $\cong$ means same homotopy type.

Charlap in 1965, furnished one of the early examples of such phenomenon in [Ch], where $M_1$, $M_2$ are flat riemannian manifolds and $N = S^1$. Such an example is obtained as a consequence of his classification for $\mathbb{Z}p$-manifolds (riemannian manifolds with holonomy group equal to $\mathbb{Z}p$) with $p$ prime.

In 1969, Hilton and Roitberg [HR2] considered the case where $M_1$ and $M_2$ are total spaces of principal bundles and $N$ the corresponding structural group, more precisely, they consider principal $S^3$-bundles over spheres $S^n$. The principal $S^3$-bundles over $S^n$ are classified by $\pi_n(BS^3) \cong \pi_{n-1}(S^3)$ (cf. [St]). So, we have for each $\alpha \in \pi_{n-1}(S^3)$ the corresponding $S^3$-bundle $S^3 \cdots E_\alpha p_\alpha \rightarrow S^n$ classified by the adjoint $\alpha_0 \in \pi_n(BS^3)$ of $\alpha$. Given $\alpha, \beta \in \pi_{n-1}(S^3)$ let $E_{\alpha,\beta}$ be the principal $S^3$-bundle over $E_{\alpha}$ induced from the bundle $E_\beta$ by the projection $p_\alpha : E_\alpha \rightarrow S^n$. We have in this way the following commutative diagram:

$$
\begin{array}{ccc}
S^3 & \rightarrow & S^3 \\
\vdots & & \vdots \\
E_{\alpha,\beta} & \longrightarrow & E_\beta \\
\downarrow & & \downarrow^{p_\beta} \\
E_{\alpha} & \rightarrow & S^n \rightarrow_{\beta_0} BS^3.
\end{array}
$$

Diagram 1
Theorem 1 (Hilton-Roitberg [HR2]).

i) \(E_\alpha \simeq E_\beta \iff \alpha = \pm \beta\);

ii) Let \(\alpha \in \pi_{n-1}(S^3)\) be an element of order \(k\) and \(\beta = l\alpha, l \in \mathbb{Z}\). If there exists \(l', l' \equiv l \mod k\), such that

\[
\frac{l'(l' - 1)}{2} \omega \circ \Sigma^3 \alpha = 0 \in \pi_{n+2}(S^3),
\]

where \(\omega \in \pi_6(S^3)\) is the generator, \(\Sigma^3\) is the 3-fold suspension of \(\alpha\), then the bundle

\[S^3 \cdots E_\alpha \beta \rightarrow E_\alpha\]

is trivial.

We observe that by the construction of the induced bundle it follows easily that \(E_{\alpha \beta} = E_{\beta \alpha}\). This observation together with the above theorem give the following example of non-cancellation phenomena:

Consider \(S^3\)-bundles over \(S^7\), \(M_1 = E_\alpha\), where \(\alpha = \omega \in \pi_6(S^3) \cong \mathbb{Z}_{12}\) is a generator, and \(M_2 = E_\beta\) with \(\beta = 7\alpha\). Since \(\frac{7(7-1)}{2} \omega \circ \Sigma^3 \alpha = 21\omega \circ \Sigma^3 \alpha = 0\) by \(\pi_9(S^3) \cong \mathbb{Z}_3\), similarly as \(\alpha = 7\beta\) we also have \(\frac{7(7-1)}{2} \omega \circ \Sigma^3 \beta = 0\). As \(E_\omega\) is the canonical \(S^3\)-bundle \(Sp(2)\) over \(S^7\) we have now

\[(1) \quad Sp(2) \times S^3 = E_{7\omega} \times S^3 \quad \text{and} \quad Sp(2) \not\cong E_{7\omega}.\]

This example is also relevant in a different context:

\(Sp(2)\) and \(E_{7\omega}\) are the only total spaces of principal \(S^3\)-bundles over \(S^7\), up to orientation, that admit a loop-space structure (cf. [HMR2], [CM] or [Z]). This, together with the second part of (1), tells us that \(Sp(2)\) and \(E_{7\omega}\) are \(H\)-spaces with distinct \(H\)-structures.

In the examples above \(M_1, M_2\) and \(N\) are at most 2-connected. In 1972 Hilton, Mislin and Roitberg [HMR1] provided examples of \(M_1, M_2\) and \(N\) all arbitrarily highly connected.

These examples of non-cancellation phenomena are obtained by indirect ways, that is, the diffeomorphism (1) is not explicitated.

Hilton and Roitberg ([HR1], [HR2]) consider the cellular decomposition of the spaces \(E_\alpha (\alpha \in \pi_{n-1}(S^3))\):

\[E_\alpha = (S^3 \cup_\alpha e^n) \cup e^{n+3} = C_\alpha \cup e^{n+3}.\]

It is shown that under the same conditions in which we have \(E_\alpha \times S^3 = E_\beta \times S^3\) we also have \(C_\alpha \vee S^3 \simeq C_\beta \vee S^3\) (where \(\vee\) denotes the one point union) although, in general \(E_\alpha \not\cong E_\beta\) and \(C_\alpha \not\cong C_\beta\). They suggest then a more careful analysis of the diffeomorphism between \(E_\alpha \times S^3\) and \(E_\beta \times S^3\).
The subject of this paper is to analyse the example of Hilton-Roitberg above, trying to give an idea of the complexity of the diffeomorphism between $Sp(2) \times S^3$ and $E_{7\omega} \times S^3$.

To do this, we worked with the models for principal $S^3$-bundles over $S^7$ denoted in [R] by $\tilde{P}_n$. Such bundles are represented by 10-dimensional submanifolds of $Sp(n)$ and have transition functions $g_{VU}^n : U \cap V \rightarrow S^3$ relative to an open covering of $S^7$ by just two sets $U = \{(a_b^a) \in S^7; a \neq 0\}$ and $V = \{(a_b^b) \in S^7; b \neq 0\}$, given by $g_{VU}^n(a_b^a) = \frac{\rho_{n-1}(\tilde{a})^{n-1}a^{n-1}}{(|a||b|)^2(n-1)}$ and the method used in [R] shows that

$$g_{VU}^n \stackrel{H}{\approx} 1 \implies$$

The bundle $\tilde{P}_n$ is trivial and a global section can be constructed explicitly by means of $H$.

There exists a diffeomorphism $\beta : S^3 \times S^3 \times (0, \frac{\pi}{2}) \rightarrow U \cap V$ such that $g_{VU}^n \circ \beta(A, B, \theta) = B^{n-1}(A\tilde{B})^{n-1} \tilde{A}^{n-1}$. Thus, working in the group of homotopy classes of maps $[S^3 \times S^3, S^3]$ we obtained that $B^8(A\tilde{B})^8 \tilde{A}^8 \simeq 1$ and the implication (2) can be realized for $n = 9$, which coincides with the classification of the bundles $\tilde{P}_n$ given in [B2]. We use the same idea to construct a diffeomorphism between $Sp(2) \times S^3$ and $E_{7\omega} \times S^3$. In this case the bundles $E_{\alpha\beta}$ are modeled through the $\tilde{P}_n$'s via the pull-back construction providing the principal $S^3$-bundles $\tilde{P}_{n,m}$ over $\tilde{P}_n$, in such a way that the homotopy classes of transition functions of these bundles are in one to one correspondence with $[S^3 \times S^3 \times S^3, S^3]$, so the method applied above works here but the calculations are much more complicated.

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2. THE BUNDLES $\tilde{P}_n$ AND $\tilde{P}_{n,m}$

Let $M_n = M_n(a, b, x_1, x_2, \ldots, x_n) \in Sp(n) \ (n \geq 3)$ be given by

$$M_3 = \begin{pmatrix} a & -b|b|^2 & x_1 \\ b & b\bar{a}b & x_2 \\ 0 & a\sqrt{1 + |b|^2} & x_3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} a & -b|b|^2L(4)^{-1} & 0 & x_1 \\ b & b\bar{a}bL(4)^{-1} & 0 & x_2 \\ 0 & a|a|^2L(4)^{-1} & -b & x_3 \\ 0 & a\bar{a}bL(4)^{-1} & a & x_4 \end{pmatrix},$$

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where $L(4) = \sqrt{|a|^4 + |b|^4}$, and for $n \geq 5$

$$M_n = \begin{pmatrix}
  a & \frac{-b|b|^2}{L(n)} & 0 & \ldots & 0 & 0 & 0 & 0 & x_1 \\
  b & \frac{b|b|^2}{L(n)} & 0 & \ldots & 0 & 0 & 0 & 0 & x_2 \\
  0 & \frac{(ab)f_{n-5}}{L(n)} & \frac{f_{n-4}}{L(n)} & \ldots & 0 & 0 & 0 & 0 & x_3 \\
  0 & \frac{(ab)f_{n-6}}{L(n)} & \frac{f_{n-4}}{L(n)} & \ldots & 0 & 0 & 0 & 0 & x_4 \\
  0 & \frac{(ab)^2f_{n-7}}{L(n)} & \frac{f_{n-4}}{L(n)} & \ldots & 0 & 0 & 0 & 0 & x_5 \\
  0 & \frac{(ab)^3f_{n-8}}{L(n)} & \frac{f_{n-4}}{L(n)} & \ldots & 0 & 0 & 0 & 0 & x_6 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & \frac{(ab)^{n-7}f_2}{L(n)} & \frac{(ab)^{n-8}f_2}{L(n)} & \ldots & -b & 0 & 0 & 0 & x_{n-4} \\
  0 & \frac{(ab)^{n-6}f_1}{L(n)} & \frac{(ab)^{n-7}f_1}{L(n)} & \ldots & \frac{a}{L_4} & \frac{f_1}{L_4} & -b & 0 & 0 & x_{n-3} \\
  0 & \frac{(ab)^{n-5}f_0}{L(n)} & \frac{(ab)^{n-6}f_0}{L(n)} & \ldots & \frac{(ab)a}{L_2} & \frac{f_0}{L_2} & \frac{a}{L_1} & -b & 0 & x_{n-2} \\
  0 & \frac{(ab)^{n-4}a}{L(n)} & \frac{(ab)^{n-5}a}{L(n)} & \ldots & \frac{(ab)^2a}{L_3} & \frac{(ab)^3a}{L_3} & \frac{(ab)^2a}{L_2} & \frac{a}{L_1} & a & x_n \\
\end{pmatrix},$$

where

$$\begin{align*}
L_1^2 &= |a|^4 + |b|^2, \\
L_k^2 &= |a|^{2(k+1)}(L_1L_2L_3 \ldots L_{k-1})^2 + |b|^2, \quad k = 2, 3, 4, \ldots, n - 4, \\
L(n)^2 &= |a|^{2(n-2)}(L_1L_2L_3 \ldots L_{n-4})^2 + |b|^4, \\
f_0 &= |a|^4, \\
f_k &= |a|^{2(k+2)}(L_1L_2L_3 \ldots L_{k-1})^2, \quad k = 2, 3, 4, \ldots, n - 4.
\end{align*}$$

The bundles $S^3 \rightarrow \tilde{P}_n \rightarrow S^7$ are such that $\tilde{P}_2$ is the canonical $S^3$-bundle $S^2$ over $S^7$ and for $n > 2$, $\tilde{P}_n = \{ M \in Sp(n); M = M_n(a, b, x_1, \ldots, x_n), \}$ for certain $a, b, x$, in $\mathbb{H}$, $\tilde{p}_n(M_n(a, b, x_1, \ldots, x_n)) = (a^b)$ is $S^7$ and $S^3$ acts on $\tilde{P}_n$ by multiplication on the right over the last column. We have the classification theorem.

**Theorem 2 ([B2]).** If $\omega \in \pi_6(S^3)$ is the preferred generator then

$$\tilde{P}_n \cong E_{\varphi(n-1)} \omega,$$

where $\varphi(n) = \binom{n+1}{2} = \frac{(n+1)n}{2}$.

We calculate here the transition functions of the bundles $\tilde{P}_n$ over $S^7$ and of the bundles $E_{\alpha \beta}$ over $E_{\alpha}$ with respect to a certain open covering of $S^7$ and $E_{\alpha}$ respectively containing just 2 open sets.
Let $S^7 = \{ (a, b) \in \mathbb{H}^2; a\bar{a} + b\bar{b} = 1 \}$ be the seven sphere and $U, V$ the open subsets defined by $U = \{ (a, b) \in S^7; a \neq 0 \}$, $V = \{ (a, b) \in S^7; b \neq 0 \}.$ Thus for $n \geq 5$, $\tilde{p}_n^{-1}(U) = \{ M_n(a, b, y_1, y_2, \ldots, y_n) \in Sp(n); a \neq 0 \}$ and $\tilde{p}_n^{-1}(V) = \{ M_n(a, b, z_1, z_2, \ldots, z_n) \in Sp(n); b \neq 0 \}$, where

$$y_1 = -(a\bar{b})|a|^{-2}y_2,$$

$$|y_2| = |y_2(a, b)| = |a|^{n-1}(L_1L_2 \cdots L_{n-4})L(n)^{-1},$$

$$y_k = -(a\bar{b})^{k-1}|a|^{-2(k-1)}(L_{n-(k+1)}L_{n-k} \cdots L_{n-4})^{-2}y_2, \quad 3 \leq k \leq n - 2,$$

$$y_{n-1} = -(a\bar{b})^{n-2}|a|^{-2(n-2)}(L_1L_2 \cdots L_{n-4})^{-2}y_2,$$

$$y_n = -(a\bar{b})^{n-1}|a|^{-2(n-1)}(L_1L_2 \cdots L_{n-4})^{-2}y_2,$$

$$z_1 = (b\bar{a})^{n-2}|b|^{-2(n-2)}(L_1L_2 \cdots L_{n-4})^2z_n,$$

$$z_2 = -(b\bar{a})^{n-1}|b|^{-2(n-1)}(L_1L_2 \cdots L_{n-4})^2z_n,$$

$$z_k = (b\bar{a})^{n-k}|b|^{-2(n-k)}(L_1L_2 \cdots L_{n-(k+2)})^2x_n, \quad 3 \leq k \leq n - 3,$$

$$z_{n-2} = (b\bar{a})^2|b|^{-4}z_n,$$

$$z_{n-1} = (b\bar{a})|b|^{-2}z_n,$$

$$|z_n| = |z_n(a, b)| = |b|^{n-1}(L_1L_2 \cdots L_{n-4}L(n))^{-1}$$

are obtained solving the equations $(\text{col } n) \cdot (\text{col } i) = 0 \quad (i = 1, 2, \ldots, n - 1)$ and using the fact that $a \neq 0$ and $b \neq 0$ in $U$ and $V$ respectively (cf. [R]).

We note that an element of $\tilde{p}_n^{-1}(U)$ depends only on the values of $a, b$ and $y_2$, thus we can write $M_n(a, b, y_1, y_2, \ldots, y_n) = M_n(a, b, y_2) \in \tilde{p}_n^{-1}(U)$. Similarly, $\tilde{p}_n^{-1}(V)$ depends only on the values of $a, b$ and $z_n$, then $M_n(a, b, z_1, z_2, \ldots, z_n) = M_n(a, b, z_n) \in \tilde{p}_n^{-1}(V)$.

We define the partial sections $S^m_U : U \rightarrow \tilde{p}_n^{-1}(U)$ over $U$ and $S^m_V : V \rightarrow \tilde{p}_n^{-1}(V)$ over $V$ by

$$S^m_U\left(\begin{array}{c} a \\ b \end{array}\right) = M_n(a, b, y_2), \quad y_2 = a^{n-1}(L_1L_2 \cdots L_{n-4})L(n)^{-1},$$

$$S^m_V\left(\begin{array}{c} a \\ b \end{array}\right) = M_n(a, b, z_n), \quad z_n = -\bar{b}^{n-1}(L_1L_2 \cdots L_{n-4}L(n))^{-1}.$$

Note that $y_2$ and $z_n$ are restricted only by their modules, i.e., they belong to certain $S^3$’s. Their values were chosen so that the transition function $g^n_{VU}$ can be factored through the $S^3 \wedge S^3 = S^6$ (cf. §3).

If $g^n_{VU} : U \cap V \rightarrow S^3$ is a transition function of the bundle $\tilde{P}_n$ with respect to the open sets $U$ and $V$, then

$$S^n_V\left(\begin{array}{c} a \\ b \end{array}\right) g^n_{VU}\left(\begin{array}{c} a \\ b \end{array}\right) = S^n_U\left(\begin{array}{c} a \\ b \end{array}\right), \quad \forall \left(\begin{array}{c} a \\ b \end{array}\right) \in U \cap V.$$
As the action of \( S^3 \) on \( \tilde{P}_n \) is by multiplication from the right in the last column, we have

\[
(4) \quad z_i g_{iU}^n \left( \begin{array}{c} a \\ b \end{array} \right) = y_i, \quad i = 1, 2, \ldots, n,
\]

so, for example

\[
z_n g_{nU}^n \left( \begin{array}{c} a \\ b \end{array} \right) = y_n = -(\alpha \bar{b})^{n-1}|a|^{-2(n-1)}(L_1 L_2 \ldots L_{n-4})^{-2} \bar{a}^{n-1}(L_1 L_2 \ldots L_{n-4})L(n)^{-1},
\]

hence

\[
-\bar{b}^{n-1}(L_1 L_2 \ldots L_{n-4}L(n))^{-1} g_{nU}^n \left( \begin{array}{c} a \\ b \end{array} \right) = -\frac{(\alpha \bar{b})^{n-1} \bar{a}^{n-1}}{(L_1 L_2 \ldots L_{n-4})|a|^{2(n-1)}L(n)},
\]

thus

\[
-\bar{b}^{n-1}(-\bar{b}^{n-1}) g_{nU}^n \left( \begin{array}{c} a \\ b \end{array} \right) = \frac{b^{n-1}(\alpha \bar{b})^{n-1} \bar{a}^{n-1}}{|a|^{2(n-1)}},
\]

therefore

\[
(5) \quad g_{nU}^n \left( \begin{array}{c} a \\ b \end{array} \right) = \frac{b^{n-1}(\alpha \bar{b})^{n-1} \bar{a}^{n-1}}{(|a||b|)^{2(n-1)}}.
\]

Analogously we can define sections \( S^k_U : U \rightarrow \tilde{P}_k^{-1}(U) \) and \( S^k_V : V \rightarrow \tilde{P}_k^{-1}(V) \) \((k = 2, 3, 4)\) in such a way that \( g_{VU}^k \left( \begin{array}{c} a \\ b \end{array} \right) = \frac{b^{k-1}(\alpha \bar{b})^{k-1} \bar{a}^{k-1}}{(|a||b|)^{2(k-1)}} \) is a transition function of \( \tilde{P}_k \) with respect to the open sets \( U \) and \( V \). Thus we have

**Lemma 1.** If \( S^7 = \{ \left( \begin{array}{c} a \\ b \end{array} \right) \in H; a \bar{a} + \bar{b}^2 = 1 \} \) is the 7-sphere and \( U, V \) are the open subsets \( U = \{ \left( \begin{array}{c} a \\ b \end{array} \right) \in S^7; a \neq 0 \} \) and \( V = \{ \left( \begin{array}{c} a \\ b \end{array} \right) \in S^7; b \neq 0 \} \) respectively, then a transition function of the bundle \( S^3 \rightarrow \tilde{P}_n \rightarrow S^7 \) with respect to the open sets \( U \) and \( V \) is the function \( g_{VU}^n : U \cap V \rightarrow S^3 \) given by (5) above.

We now apply an analogous procedure to obtain transition functions of \( E_{a\bar{b}} \).

Let \( \tilde{P}_{n,m} \) be the principal \( S^3 \)-bundle over \( \tilde{P}_n \) induced from \( \tilde{P}_m \) by the projection \( \tilde{p}_n : \tilde{P}_n \rightarrow S^7 \). So, we have the commutative diagram:

\[
\begin{array}{ccc}
S^3 & S^3 & S^3 \\
\vdots & \vdots & \vdots \\
\tilde{P}_{n,m} & \tilde{P}_m & Sp(2) \\
\downarrow & \downarrow \tilde{p}_m & \\
\tilde{P}_n & \tilde{p}_n & S^7 \\
\downarrow & \varphi(m^{-1}) & \\
\tilde{P}_n & S^7 & S^7.
\end{array}
\]

Diagram 2.
By the definition of the induced bundle we have $\tilde{P}_{n,m} = \{(M_n, M_m) \in \tilde{P}_n \times \tilde{P}_m; \tilde{p}_n(M_n) = \tilde{p}_m(M_m)\}$. This provides a model for $E_{\varphi(n-1)\omega, \varphi(m-1)\omega}$ where $\omega \in \pi_6(S^3)$ is the preferred generator.

Let us consider the open sets $\tilde{U}_n = \tilde{p}_n^{-1}(U) = \{M_n(a, b, y_2) \in \tilde{P}_n; a \neq 0\}$ and $\tilde{V}_n = \tilde{p}_n^{-1}(V) = \{M_n(a, b, z_n) \in \tilde{P}_n; b \neq 0\}$. For each $k, t \in \mathbb{Z}$ such that $\frac{m-tkn+tk-1}{t}$ is an integer, let us define partial sections $s_{U_n}^{kt}(M) : \tilde{U}_n \rightarrow \tilde{P}_{n,m}$ over $\tilde{U}_n$ and $s_{V_n}^{kt}(M) : \tilde{V}_n \rightarrow \tilde{P}_{n,m}$ over $\tilde{V}_n$ by

$$s_{U_n}^{kt}(M_n(a, b, y_2)) = (M_n(a, b, y_2), M_m(a, b, Y_2(k, t))),$$

$$s_{V_n}^{kt}(M_n(a, b, z_n)) = (M_n(a, b, z_n), M_m(a, b, Z_m(k, t))),$$

where

$$Y_2(k, t) = (y_2^k a^{m-tnk+tk-1})^t \frac{L_1 L_2 \ldots L_{m-4} L(n)^{tk}}{(L_1 L_2 \ldots L_{m-4})^{tk} L(m)},$$

$$Z_m(k, t) = (-1)^{tk+1} \frac{L_1 L_2 \ldots L_{m-4} L(n)^{kt}}{(L_1 L_2 \ldots L_{m-4})^{tk} L(m)}.$$

By using the expression of $y_n$ on page 77, if $M_n(a, b, y_2) = M_n(a, b, z_n)$ over $\tilde{U}_n \cap \tilde{V}_n$ then

$$z_n = \frac{-(ab)^n y_2}{a^{2(n-1)}(L_1 L_2 \ldots L_{m-4})^2} \Rightarrow z_{tk}^n = \frac{(-1)^k ((ab)^n y_2)^k}{a^{2k(n-1)}(L_1 L_2 \ldots L_{m-4})^{2k}}.$$

A transition function $g_{n,m,k,t} : \tilde{U}_n \cap \tilde{V}_n \rightarrow S^3$ of $\tilde{P}_{n,m}$ with respect to the open sets $\tilde{U}_n$ and $\tilde{V}_n$ can be given solving the equation

$$s_{V_n}^{kt}(M). g_{n,m,k,t}(M) = s_{U_n}^{kt}(M), \quad \forall M \in \tilde{U}_n \cap \tilde{V}_n.$$

It follows from the expressions on page 77 that

$$Y_m(k, t) = \frac{-(ab)^{m-1} (y_2^k a^{m-tnk+tk-1})^t L(n)^{tk}}{|a|^{2(m-1)}(L_1 L_2 \ldots L_{m-4})(L_1 L_2 \ldots L_{m-4})^{tk} L(m)}.$$

Setting $M_n = M_n(a, b, y_2)$ it follows from (7) that

$$Z_m(k, t) g_{n,m,k,t}(M_n) = Y_m(k, t),$$

hence

$$(-1)^{tk+1} \frac{(z_n^k b^{m-tnk+tk-1})^t L_1 L_2 \ldots L_{m-4} L(n)^{kt}}{L_1 L_2 \ldots L_{m-4} L(m)}. g_{n,m,k,t}(M_n)$$

$$= \frac{-(ab)^{m-1} (y_2^k a^{m-tnk+tk-1})^t L(n)^{tk}}{|a|^{2(m-1)}(L_1 L_2 \ldots L_{m-4})(L_1 L_2 \ldots L_{m-4})^{tk} L(m)}.$$
using (6) we obtain

\[
\frac{(-1)^{tk+1}((-1)^k((ab)^n-1y_2)^{tk}b^{m-tn+n+1-k}t)}{(|a|^{2k(m-1)}(L_1L_2\ldots L_{n-4})^{2k})^t}g_{n,m,k,t}(M_n)
\]

\[
= \frac{-(ab)^m-1(y_2^k\bar{a}^{m-tn+n+1-k})^t}{|a|^{2k(m-1)}(L_1L_2\ldots L_{n-4})^{2k}}
\]

thus, setting \( p = \frac{m-tn+n+1-k}{t} \) we have

\[
\frac{(-1)^{2kt+1}(|a||b|^{2k(m-1)}|y_2|^{2k}b^{2pt}n,m,k,t(M_n)}{(|a|^{2k(m-1)}|y_2|^{2k})^t}
\]

\[
= \frac{-(b^p(y_2(b\bar{a})^{n-1})^k)(ab)^m-1(y_2^k\bar{a}^p)^t}{|a|^{2k(m-1)}}
\]

therefore

(10) \[ g_{n,m,k,t}(M_n) = \frac{(b^{m-tn+n+1-k}\bar{a}^{m-1})^k(y_2^k\bar{a}^{m-tn+n+1-k})^t}{|ab|^{2k(m-1)}|y_2|^{2kt}} \]

Evidently, from the definition of the induced bundle we have that

(11) \[ \tilde{g}_{n,m}(M_n(a,b,y_2)) = g_{\tilde{V}_U}^m \circ \tilde{\rho}_n(M_n(a,b,y_2)) = \frac{b^{m-1}(ab)^m-1\bar{a}^{m-1}}{|a||b|^{2k(m-1)}} \]

is also a transition function of \( \tilde{P}_{n,m} \), which coincides with \( g_{n,m,0,1} \).

We have in this way the following:

**Lemma 2.** Let \( n,m \) be integers greater than 1. If \( \tilde{U}_n = \{ M_n(a,b,y_2) \in \tilde{P}_n; a \neq 0 \}, \tilde{V}_n = \{ M_n(a,b,z_n) \in \tilde{P}_n; b \neq 0 \} \) then for all \( k,t \in \mathbb{Z} \) such that \( \frac{m-tn+n+1-k}{t} \in \mathbb{Z} \) the functions \( g_{n,m,k,t} : \tilde{U}_n \cap \tilde{V}_n \rightarrow S^3 \) given by

\[
g_{n,m,k,t}(M_n(a,b,y_2)) = \frac{(b^{m-tn+n+1-k}\bar{a}^{m-1})^k(y_2^k\bar{a}^{m-tn+n+1-k})^t}{|ab|^{2k(m-1)}|y_2|^{2kt}}
\]

are equivalent transition functions (in the sense of equivalent coordinate transformation, Lemma 2.10 from Steenrod [St]) of the bundle \( S^3\ldots \tilde{P}_{n,m} \rightarrow \tilde{P}_n \).

3. **Trivialization of \( \tilde{P}_9 \)**

Here we construct a global section of \( \tilde{P}_9 \) up to a homotopy. To do this we use some elementary results from the theory of nilpotent groups, which we list below, and finally we show how this method can be applied to construct a diffeomorphism between \( Sp(2) \times S^3 \) and \( E_7 \times S^3 \) up to homotopies of powers of commutators.
THE ROLE OF COMMUTATORS IN A NON-CANCELLATION PHENOMENON

Let \( \Gamma \) be a multiplicative group, given elements \( x \) and \( y \) in \( \Gamma \) we recall that (cf. [Ha]) the **commutator** of \( x \) and \( y \) is defined by \( [x, y] = x^{-1}y^{-1}xy \) and for \( x, y \) and \( z \in \Gamma \) we have the following properties:

\[
\begin{align*}
\text{p}_1: & \quad [x, y]^{-1} = [y, x], \\
\text{p}_2: & \quad [xy, z] = [x, z][x, y][y, z], \\
\text{p}_3: & \quad [x, yz] = [x, z][x, y][x, y], \\
\text{p}_4: & \quad xy = yx[x, y], \\
\text{p}_5: & \quad xy = [x^{-1}, y^{-1}]yx.
\end{align*}
\]

Given subsets \( X \) and \( Y \) of a group \( \Gamma \), we define \([X, Y]\) as the subgroup of \( \Gamma \) generated by every \([x, y] \in \Gamma \) such that \( x \in X \) and \( y \in Y \).

A group \( \Gamma \) is called **nilpotent of class** \( \leq r \) if there are subgroups \( \Gamma_0, \Gamma_1, \ldots, \Gamma_r \) of \( \Gamma \) such that

\[
\begin{align*}
\text{(12)} & \quad \Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_r = \{1\}, \\
\text{(13)} & \quad \Gamma_i \text{ is a normal subgroup of } \Gamma, \quad i = 1, 2, \ldots, r, \\
\text{(14)} & \quad [\Gamma_i, \Gamma] \subseteq \Gamma_{i+1}, \quad i = 0, 1, 2, \ldots, r - 1.
\end{align*}
\]

The series of subgroups (12) satisfying (13) and (14) is called **central series** or **central chain**. Observe that (14) is equivalent to \( \frac{\Gamma_i}{\Gamma_{i+1}} \subseteq \text{center}(\frac{\Gamma}{\Gamma_{i+1}}) \), in particular

\[
\text{(15)} \quad \Gamma \text{ is nilpotent of class } \leq r \implies \Gamma_{r-1} \subseteq \text{center}(\Gamma).
\]

If \( G_2 \) and \( G_3 \) are nilpotent groups of classes \( \leq 2 \) and \( \leq 3 \) respectively with central chains \( G_2 = \Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 = \{1\} \) and \( G_3 = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 = \{1\} \), then the following formulas are easily obtained from properties \( \text{p}_1, \ldots, \text{p}_5 \):

\[
\begin{align*}
\text{N1: } & \quad [x, y] = [y^{-1}, x] = [y, x^{-1}] = [x^{-1}, y^{-1}] \quad \forall x, y \in G_2, \\
\text{N2: } & \quad [x, y]^n = [x^n, y] \quad \forall x, y \in G_2 \text{ and } \forall n \in \mathbb{Z}, \\
\text{N3: } & \quad (xy)^n = y^n[x, y]^\varphi(n)x^n \quad \forall x, y \in G_2, \text{ where } \varphi(n) = \binom{n+1}{2} = \frac{n(n+1)}{2}, \\
\text{N4: } & \quad [x, y]^n = [x^n, y]^\varphi(n-1) \quad \forall x, y \in G_3 \text{ and } n \in \mathbb{N}, \\
\text{N5: } & \quad [x^n, y] = [x, y]^n[x, [y, x]]^\varphi(n-1) \quad \forall x, y \in G_3 \text{ and } n \in \mathbb{N}, \\
\text{N6: } & \quad \text{Given } x, y \in G_3 \text{ and } n \in \mathbb{Z} \text{ we have} \\
& \quad \text{i) if } n \geq 0 \quad (xy)^n = y^n[x^{-1}, y, x]a(n)[x, y]^a(n-1)[x, y]^\varphi(n)x^n, \\
& \quad \text{ii) if } n < 0 \quad (xy)^n = y^n[y^{-1}, x^{-1}, y^{-1}]a(n)[x^{-1}, y^{-1}, x^{-1}]a(n-1)[x^{-1}, y^{-1}]^\varphi(n)x^n, \\
& \quad \text{where } a(n) = \binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}, \\
\text{N7: } & \quad [[x, y], z][[y, z], x][[z, x], y] = 1 \quad \forall x, y, z \in G_3.
\end{align*}
\]

**Remark 1.** The function \( \varphi \) in N4, N5 and N6 is the same as that of N3.
To trivialize $\tilde{P}_9$ let us consider the diffeomorphisms $\alpha : U \cap V \longrightarrow S^3 \times S^3 \times (0, \frac{\pi}{2})$ and $\beta : S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow U \cap V$ given by $\alpha(a) = \left(\frac{a}{|a|}, \frac{b}{|b|}, \cos^{-1}|a|\right)$ and $\beta(A, B, \theta) = (\cos \theta A, \sin \theta B)$, which are mutual inverses (cf. [R]).

Let $S_0^0 : U \longrightarrow \tilde{P}_9^{-1}(U)$ and $S_i^0 : V \longrightarrow \tilde{P}_9^{-1}(V)$ be the sections given in Remark 2. Above $\Gamma$ is nilpotent of class $\leq 2$ and has a central chain $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k = \{1\}$ such that $\Gamma_i$ be the group of homotopy classes of maps $f \in \Gamma$ such that $f|P_i$ is nullhomotopic, then we have

**Theorem 3** (G. W. Whitehead [W1]). The group $[S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}, G]$ has the central chain $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k = \{1\}$ and

$$\frac{\Gamma_{i-1}}{\Gamma_i} \cong \prod_{|I|=i} \pi_{n(I)}(G),$$

where $I \subseteq \{1, 2, 3, \ldots, k\}$, $|I|$ = cardinality of $I$ and $n(I) = \sum_{i \in I} n_i$.

If $\Gamma = [S^3 \times S^3, S^3]$, then $G = S^3$, $k = 2$, $n_1 = n_2 = 3$ and by the result above $\Gamma$ is nilpotent of class $\leq 2$ and has a central chain $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 = \{1\}$ such that

$$\frac{\Gamma_0}{\Gamma_1} \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \frac{\Gamma_1}{\Gamma_2} \cong \mathbb{Z} \cong \mathbb{Z}_{12}.$$

**Remark 2.** Recently M. Mimura and H. Ōshima (cf. [MO]) described the group structure of $[S^m \times S^n, S^k, \mu_r]$ for $m, n, k \in \{1, 3, 7\}$ and of $[E_\alpha, E_\beta, \mu_0^{(r)}]$ for $\alpha, \beta \in \pi_6(S^3)$, where $\mu_r(x, y) = xy[x, y]^r$ and $\mu_0(x, y) = xy$ are the usual complex, quaternionic or Cayley multiplications and $\mu_0^{(r)}$ is defined similarly for convenient multiplications $\mu_0^{(0)}$. Thus, for example $[S^3 \times S^3, S^3, \mu_0]$ is the group generated by the projections $p_1, p_2 : S^3 \times S^3 \longrightarrow S^3$ and with relations $p_1[p_1, p_2] = [p_1, p_2]p_1, p_2[p_1, p_2] = [p_1, p_2]p_2, [p_1, p_2]^{12} = 1.$
Given \( f : S^3 \times S^3 \to S^3 \) we denote by \( \widetilde{f} : S^3 \times S^3 \to S^3 \) the map \( \widetilde{f}(x,y) = \overline{f(x,y)} \) (quaternionic conjugation of \( f \)) and as in the remark above \( p_i : S^3 \times S^3 \to S^3 \) \((i = 1, 2)\) are the projections.

We know that, if \( f, g \in \Gamma = [S^3 \times S^3, S^3] \), then \( f, g \) is the homotopy class of the product of \( f \) by \( g \) in \( S^3 \) with this we observe that \( \overline{p_i} = p_i^{-1} \) in \( \Gamma \) \((i = 1, 2)\) and from N3 we have that \((p_1\overline{p_2})^n = \overline{p_2^n [\overline{p_2}, \overline{p_1}]^{\varphi(n)}} p_1^n \) so \( p_2^n (p_1\overline{p_2})^n \overline{p_1} = [\overline{p_2}, \overline{p_1}]^{\varphi(n)} \) in \( \Gamma \).

We observe that \( p_2^{n-1}(p_1\overline{p_2})^{n-1}\overline{p_1} = g_{VU}^n \circ \beta \), and so we conclude

\[
g_{VU}^n \circ \beta \simeq [\overline{p_2}, \overline{p_1}]^{\varphi(n)}.\]

Thus, \( g_{VU}^0 \circ \beta \simeq [\overline{p_2}, \overline{p_1}]^{36} \), as \([\overline{p_2}, \overline{p_1}] \in \Gamma \simeq \mathbb{Z}_{12}\) it follows that \( g_{VU}^0 \circ \beta \simeq 1 \).

Let \( F : S^3 \times S^3 \times [0, \frac{\pi}{2}] \to S^3 \) be a smooth homotopy such that

\[
F(A, B, \theta) = \begin{cases} 
B^8(AB)^8A^8 & \text{if } \theta \in [0, \frac{\pi}{6}] \\
1 & \text{if } \theta \in [\frac{\pi}{6}, \frac{\pi}{2}],
\end{cases}
\]

then \( S : S^3 \to \tilde{P}_9 \) given by

\[
S \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) = \begin{cases} 
S_0^3 \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2} \\
S_0^3 \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) (F \circ \alpha) \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) & \text{if } \frac{\pi}{12} \leq \cos^{-1} |a| \leq \frac{5\pi}{12} \\
S_0^3 \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) | \leq 0 \cos^{-1} |a| \leq \frac{\pi}{12}
\end{cases}
\]

is a global section of \( \tilde{P}_9 \).

A diffeomorphism \( \Phi : S^3 \times S^3 \to \tilde{P}_9 \) is given by

\[
\Phi \left( \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right), \nu \right) = M_9(a, b, w_1v, w_2v, w_3v, w_4v, w_5v, w_6v, w_7v, w_8v, w_9v)
\]

where

\[
w_1 = \begin{cases} 
-(b\overline{a})^7\overline{b}^8|b|^{-14}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2} \\
-(b\overline{a})^7\overline{b}^8|b|^{-14}(L_1L_2L_3L_4L_5)L(9)^{-1}(F \circ \alpha)\left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) & \text{if } \frac{\pi}{12} \leq \cos^{-1} |a| \leq \frac{5\pi}{12} \\
-(b\overline{a})^8|a|^{-2}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12}
\end{cases}
\]

\[
w_2 = \begin{cases} 
(b\overline{a})^8\overline{b}^8|b|^{-16}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2} \\
(b\overline{a})^8\overline{b}^8|b|^{-16}(L_1L_2L_3L_4L_5)L(9)^{-1}(F \circ \alpha)\left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) & \text{if } \frac{\pi}{12} \leq \cos^{-1} |a| \leq \frac{5\pi}{12} \\
b^8(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12}
\end{cases}
\]

\[
w_k = \begin{cases} 
-(b\overline{a})^{9-k}\overline{b}^8|b|^{-2(k-9)}\frac{(L_1L_2L_3L_4L_5)(L_9)(L_9)(L_9)(L_9)(L_9)}{L_9(L_9)(L_9)(L_9)(L_9)} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2} \\
-(b\overline{a})^{9-k}\overline{b}^8|b|^{-2(k-9)}\frac{(L_1L_2L_3L_4L_5)(L_9)(L_9)(L_9)(L_9)(L_9)}{L_9(L_9)(L_9)(L_9)(L_9)}(F \circ \alpha)\left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) & \text{if } \frac{\pi}{12} \leq \cos^{-1} |a| \leq \frac{5\pi}{12} \\
-(b\overline{a})^{k-1}\overline{a}^8|a|^{-2(k-9-k)}\frac{(L_1L_2L_3L_4L_5)(L_9)(L_9)(L_9)(L_9)(L_9)}{L_9(L_9)(L_9)(L_9)(L_9)} & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12}
\end{cases}
\]
for $3 \leq k \leq 6$,

$$w_7 = \begin{cases} 
- (b)^{2}b^{8}|b|^{-4}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\
- (b)^{2}b^{8}|b|^{-4}(L_1L_2L_3L_4L_5L(9))^{-1}(F \circ \alpha)\left(\frac{a}{b}\right) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\
- (ab)^{6}a^{8}|a|^{-12}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12},
\end{cases}$$

$$w_8 = \begin{cases} 
- (b)^{6}b^{8}|b|^{-2}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\
- (b)^{6}b^{8}|b|^{-2}(L_1L_2L_3L_4L_5L(9))^{-1}(F \circ \alpha)\left(\frac{a}{b}\right) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\
- (ab)^{7}a^{8}|a|^{-14}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12},
\end{cases}$$

$$w_9 = \begin{cases} 
- b^{8}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\
- b^{8}(L_1L_2L_3L_4L_5L(9))^{-1}(F \circ \alpha)\left(\frac{a}{b}\right) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\
- (ab)^{8}a^{8}|a|^{-16}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}.
\end{cases}$$

The reasoning employed above shows us that if the transition function $g^n_{UV}$ is null-homotopic then the bundle $\tilde{P}_n$ is trivial.

Let us consider the following commutative diagram:

$$\begin{array}{c}
S^3 \times S^3 \\
\downarrow \wedge \downarrow \omega \\
S^6 = S^3 \wedge S^3 \xrightarrow{\text{id}} S^3 \wedge S^3,
\end{array}$$

diagram 3

that is, $\omega$ is defined here by $\omega(A \wedge B) = [\tilde{B}, \tilde{A}] = BAB\tilde{A}$.

**Remark 3.** It is well known that $\omega$ defined above is a generator of $\pi_6(S^3)$ (cf. [J], [Mc] or remark 2). With the aid of this fact and (16) above we note that the transition functions $g^n_{UV} : U \cap V \to S^3$, $g^n_{UV}(\frac{a}{b}) = \frac{b^{n-1}(ab)^{n-1}a^{n-1}}{|a||b|^{2(n-1)}}$ of $\tilde{P}_n$ are such that $g^n_{UV} \circ \beta : S^3 \times S^3 \to S^3$ all factor through $S^3 \wedge S^3$ where $\beta$ is the diffeomorphism between $S^3 \times S^3 \times (0, \frac{\pi}{2})$ and $U \cap V$ given above, that is, there exists $\omega_n : S^3 \wedge S^3 \to S^3$ such that $g^n_{UV} \circ \beta = \omega_n \circ \wedge$. Moreover if the equivalence class of the transition function $g^n_{UV}$ classifies the bundle $\xi$ then the homotopy class of $\omega_n$ in $\pi_6(S^3)$ also classifies the same bundle $\xi$. We have thus the following homotopy-commutative diagram:

$$\begin{array}{c}
S^3 \times S^3 \\
\downarrow \wedge \downarrow \omega \\
S^3 \wedge S^3 \to S^3,
\end{array}$$

diagram 4

where $\omega_n(A \wedge B) \simeq [\tilde{B}, \tilde{A}]^{\varphi(n-1)}$. 

http://escholarship.lib.okayama-u.ac.jp/mjou/vol43/iss1/2
Remark 4. Recently Carlos E. Duran [D] has exhibited the following a priori smooth formula for the Blakers-Massey element \( \omega : S^6 \subseteq \text{Im}(H) \oplus H \longrightarrow S^3 \),
\[
\omega \left( \frac{p}{u} \right) = \begin{cases} 
\frac{\overline{u}}{|u|} \exp(\pi p) \frac{u}{|u|} & \text{if } u \neq 0 \\
-1 & \text{if } u = 0,
\end{cases}
\]
where \( \exp(\theta p) = \cos(\theta |p|) + \sin(\theta |p|) \frac{p}{|p|} \).

4. The Diffeomorphism \( Sp(2) \times S^3 = E_7 \omega \times S^3 \)

Now we try writing explicitly (in terms of transition functions) a diffeomorphism \( Sp(2) \times S^3 = E_7 \omega \times S^3 \) following the same steps as above.

Lemma 3. There exists a diffeomorphism \( \delta_n : S^3 \times S^3 \times S^3 \times (0, \pi/2) \longrightarrow \tilde{U}_n \cap \tilde{V}_n \) for all \( n \in \mathbb{N} \), \( n \geq 2 \).

Proof. We define \( \delta_n \) as follows:

\[
\delta_n(A, B, C, \theta) = M_n(\cos \theta A, \sin \theta B, (\cos^{n-1} \theta)(l_1 l_2 \ldots l_{n-4})l(n)^{-1} C),
\]

where \( l_k = L_k \) changing \( a \) by \( \cos \theta A \), \( b \) by \( \sin \theta B \) (\( 1 \leq k \leq n-4 \)) and \( l(n) = L(n) \) making the same changes. As examples we have

\[
\delta_2(A, B, C, \theta) = \begin{pmatrix} 
\cos \theta A & -\sin \theta A \bar{B}C \\
\sin \theta B & \cos \theta C
\end{pmatrix},
\]

\[
\delta_3(A, B, C, \theta) = \begin{pmatrix} 
\cos \theta A & -\sin^3 \theta B & -\cos \theta \sin \theta \sqrt{1 + \sin^2 \theta A \bar{B}C} \\
\sin \theta B & \cos \theta \sin^2 \theta B \bar{A} \bar{B} & \cos^2 \theta \sqrt{1 + \sin^2 \theta C} \\
0 & \cos \theta \sqrt{1 + \sin^2 \theta A} & -\cos^{-1} \theta \sin \theta A \bar{B}C
\end{pmatrix},
\]

\[
\delta_4(A, B, C, \theta) = \begin{pmatrix} 
\cos \theta A & -\sin^3 \theta B l^{-1} & 0 & -\cos^2 \theta \sin \theta A \bar{B}C l^{-1} \\
\sin \theta B & \sin^2 \theta \cos \theta B \bar{A} \bar{B} l^{-1} & 0 & -\cos \theta \sin^2 \theta (AB)^2 C l^{-1} \\
0 & \cos^3 \theta A l^{-1} & -\sin \theta B & -\cos \theta \sin^2 \theta (AB)^3 C l^{-1} \\
0 & \cos^2 \theta \sin \theta A \bar{B} A l^{-1} & \cos \theta A & -\sin^3 \theta (AB)^3 C l^{-1}
\end{pmatrix},
\]

where \( l = l(4) = \sqrt{\sin^4 \theta + \cos^4 \theta} \).

We can easily verify then

\[
\delta_n^{-1}(M_n(a, b, y_2)) = \begin{pmatrix} a \\ b \\ y_2 \end{pmatrix},\]

and with this the Lemma is proved. \( \square \)
**Lemma 4.** Let \( f : S^3 \times S^3 \times S^3 \to S^3 \) and \( p : S^3 \times S^3 \times S^3 \to S^3 \times S^3 \) be continuous functions.

i) If \( f = c_2 \circ p \), where \( c_2(x, y) = [x, y] = x^{-1}y^{-1}xy \), then the order of \( f \) in \([S^3 \times S^3 \times S^3, S^3]\) is a divisor of 12,

ii) If \( f = c_3 \circ p \), where \( c_3(x, y) = [x, [x, y]] \) or \([y, [x, y]]\), then \( f \simeq 1 \).

**Proof.** It is suffices to observe that, by Theorem 3 applied to the group \([S^3 \times S^3, S^3]\) we have that \( c_2 \) represents a homotopy class of order 12, and \( c_3 \simeq 1 \).

With this Lemma, we note that for \( x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3] \) we have

\[
[x, y] = x^{-1}y^{-1}xy = x^{-1}xy^{-1}[y^{-1}, x]y = y^{-1}y[y^{-1}, x][[y^{-1}, x], y] = [y^{-1}, x].
\]

In a similar manner as in N1 we obtain

**Lemma 5.** Given \( x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3] \), then

\[
[x, y] = [y^{-1}, x] = [y, x^{-1}] = [x^{-1}, y^{-1}].
\]

With the aid of the last two lemmas we observe that N4, N5 and N6 applied to the group \( G_3 = [S^3 \times S^3 \times S^3, S^3] \) transform to Lemma 6 and Theorem 4 bellow.

**Lemma 6.** Given \( x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3] \), then

\[
[x, y]^n = [x^n, y] = [x, y^n], \quad n \in \mathbb{Z}.
\]

**Proof.** For \( n \geq 0 \) the result follows directly from N1 and N5. If \( n < 0 \) we have

\[
[x, y]^n = [y, x]^{-n} = [y^{-n}, x] = [y, x^{-n}],
\]

but \([y^{-n}, x] = [(y^n)^{-1}, x] = [x, y^n]\) and \([y, x^{-n}] = [y, (x^n)^{-1}] = [x^n, y]\), hence the result follows.

**Theorem 4.** Given \( x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3] \), then

\[
(x y)^n = y^n[x, y]^\varphi(n) x^n, \quad n \in \mathbb{Z}.
\]

We observe that \([S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}), S^3] = [S^3 \times S^3 \times S^3, S^3]\). It follows from Theorem 3 that \( \Gamma = [S^3 \times S^3 \times S^3, S^3] \) is a nilpotent group of class \( \leq 3 \) with central chain \( \Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 = \{1\} \) such that

\[
\frac{\Gamma_0}{\Gamma_1} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \frac{\Gamma_1}{\Gamma_2} \simeq \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}, \quad \frac{\Gamma_2}{\Gamma_3} = \Gamma_2 \simeq \mathbb{Z}_3.
\]

We denote by \( A, B, C : S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) \to S^3 \) the projections \( A(x_1, x_2, x_3, \theta) = x_1, B(x_1, x_2, x_3, \theta) = x_2, C(x_1, x_2, x_3, \theta) = x_3 \), and by \( \bar{A}, \bar{B}, \bar{C} \) the quaternionic conjugates of \( A, B, C \) respectively. To simplify the notation we set \( p = \frac{m-ikn+ik-1}{l} \) and using the results above we obtain
### The Role of Commutators in a Non-Cancellation Phenomenon

**Theorem 5.** The transition function \(g_{n,m,k,t} \circ \delta_n\) can be written as a product of simple and double commutators, namely,

\[
g_{n,m,k,t} \circ \delta_n \simeq [C, B]^{e_1} [B, A]^{e_2} [A, C]^{e_1} [[B, C], A]^{f} \quad \text{for certain } e_1, e_2, f \in \mathbb{Z}.
\]

**Proof.**

\[
g_{n,m,k,t} \circ \delta_n = (B^p(C(B \bar{A})^{n-1})^k t)(A \bar{B})^{m-1}(C^k \bar{A}^t)\]

(T. 4)\[
\simeq (C(B \bar{A})^{n-1})^{tk} [B^p, (C(B \bar{A})^{n-1})^k \phi(t)] B^{pt}(A \bar{B})^{m-1} \bar{A}^{pt}\]

(L. 4)\[
\simeq [B^p, (C(B \bar{A})^{n-1})^k \phi(t)] (C(B \bar{A})^{n-1})^{tk} B^{pt}(A \bar{B})^{m-1} \bar{A}^{pt}\]

(T. 4 and L. 6)\[
\simeq [B, C(B \bar{A})^{n-1}]^k \phi(t) p(B \bar{A})^{tk(n-1)} [C, (B \bar{A})^{n-1}] \phi(tk) C^{tk} B^{pt}\]

((p3, p4 and L. 6)\[
\simeq (B(B \bar{A})^{n-1})^k \phi(t) p(A \bar{B})^{tk(n-1)} [C, (B \bar{A})^{n-1}] \phi(tk) C^{tk} B^{pt}\]

((p2, p3 and T. 4)\[
\simeq [B, B \bar{A})^{(n-1) k \phi(t)} p(B, C \bar{A})^{k \phi(t)} p(B, B \bar{A})^{(n-1) k \phi(t)} p\]

([C, B \bar{A})^{\phi(tk)(n-1)} C^{tk} B^{pt} (B \bar{A})^{(n-1) k \phi(t)} p\]

(p2 and p4)\[
\simeq [B, B \bar{A})^{(n-1) k \phi(t)} p(B, C \bar{A})^{k \phi(t)} p(B, B \bar{A})^{(n-1) k \phi(t)} p\]

([C, A \bar{A})^{\phi(tk)(n-1)} C^{tk} B^{pt} (B \bar{A})^{(n-1) k \phi(t)} p\]

((p1 and L. 6)\[
\simeq [C, B \bar{A})^{k \phi(t) p + \phi(tk-1)(n-1)} C^{tk} B^{pt} (B \bar{A})^{(n-1) k \phi(t)} p + \phi(tk-1)(n-1)\]

([C, A \bar{A})^{k \phi(t) p + \phi(tk-1)(n-1)} C^{tk} B^{pt} (B \bar{A})^{(n-1) k \phi(t)} p + \phi(tk-1)(n-1)\]

([A, B \bar{A})^{k \phi(t) p + \phi(tk-1)(n-1)} C^{tk} B^{pt} (B \bar{A})^{(n-1) k \phi(t)} p + \phi(tk-1)(n-1)\]

We showed thus that for \(k, t \in \mathbb{Z}\) such that \(m - tk + 1 \in \mathbb{Z}\), we have

\[
g_{n,m,k,t} \circ \delta_n \simeq [C, B]^{e_1} [B, A]^{e_2} [A, C]^{e_1} [[B, C], A]^{f} \quad \text{for certain } e_1, e_2, f \in \mathbb{Z}.
\]
where \( e_1 = \frac{(m-n+p)tk}{2}, \ e_2 = \frac{(m-kn+k)p}{2}, \ e_3 = \frac{(p+t(p+k)-1)tk(n-1)}{2}, \ e_4 = \frac{(2m-tp-1)t^2 pk}{2} \) and \( p = p(m, n, k, t) = \frac{m-tkn+tk-1}{tk(n-1)} \).

Now if \( \land : S^3 \times S^3 \times S^3 \to S^3 \times S^3 \times S^3 = S^9 \) is the natural projection, then \( [[B, C], A] = \land^*(\omega \circ \Sigma^3 \omega) \), where \( \omega \) is the generator of \( \pi_6(S^3) \) given in Remark 3 above (cf. [B1]), and as \( \omega \circ \Sigma^3 \omega \) is a generator of \( \pi_6(S^3) \cong \mathbb{Z}_3 \) it follows that \( [[B, C], A] \) in \( [S^3 \times S^3 \times S^3, S^3] \) has order 3. The same occurs with \( [[C, A], B] \) and \( [[A, B], C] \) and as none of them is nullhomotopic we conclude with the aid of N7 that \( [[B, C], A] = [[C, A], B] = [[A, B], C] \) in \( [S^3 \times S^3 \times S^3, S^3] \). Thus we have finally

\[
g_{n,m,k,t} \circ \delta_n \simeq [C, B]^{e_1} [B, A]^{e_2} [A, C]^{e_1} [[B, C], A]^f,
\]

where \( f = e_3 - e_4 \).

**Remark 5.** The choice of the partial sections \( S^3_U \) for the bundles \( S^3 \cdots \tilde{P}_n \to S^7 \) given in page 77 enabled us to write the homotopy class of the corresponding transition functions as a power of a unique commutator of weight 2. We would like to have partial sections for the bundles \( S^3 \cdots \tilde{P}_{n,m} \to \tilde{P}_n \) for which the homotopy class of the corresponding transition functions could be expressed as power of a unique commutator of weight 3 as is suggested by the obstruction in the Hilton-Loitberg formula (Theorem 1). This however, cannot be realized with our choice of transition functions:

If we suppose that \( [C, B]^{e_1} [B, A]^{e_2} [A, C]^{e_1} [[B, C], A]^f \simeq [[B, C], A]^r \) for some \( r \) then \( g = [C, B]^{e_1} [B, A]^{e_2} [A, C]^{e_1} [[B, C], A]^f \simeq [[B, C], A]^s \) \( (s = r - f) \), and so \( g \in [\Gamma_1, \Gamma] \subseteq \Gamma_2 \) which implies that \( g|P_2 \simeq 1 \) where \( P_2 = X_1 \cup X_2 \cup X_3 \), \( X_1 = \{1\} \times S^3 \times S^3 \), \( X_2 = S^3 \times \{1\} \times S^3 \), \( X_3 = S^3 \times S^3 \times \{1\} \) and this implies \( g|X_i \simeq 1 \) \( (i = 1, 2, 3) \) in other words, \( [C, B]^{e_1}, [B, A]^{e_2}, [A, C]^{e_1} : S^3 \times S^3 \to S^3 \) are all nullhomotopic, which gives \( e_1 \equiv e_2 \equiv 0 \) mod 12 for every \( m, n, k, t \) such that \( m-n^kt+kt^2-1 \in \mathbb{Z} \), but this is not true for example if \( m = 5, n = 2, k = 1 \) and \( t = 2 \) or \( m = 14, n = 23, k = 2 \) and \( t = 13 \). We do not know if there exist such transition functions.

It follows from Theorem 2 that \( E_{1,7} = \tilde{P}_{14,23} \) and \( E_{7,1} = \tilde{P}_{14,23} \).

From Theorem 3 and Theorem 5 choosing \( k = 19 \) and \( t = 1 \) we have that \( g_{14,23,19,1} \circ \delta_{14} \) is homotopic to

\[
(17) \quad ([C, B]^{12})^{-171} ([B, A]^{12})^{2100} ([A, C]^{12})^{-171} ([B, C], A)^{3})^{174591} \simeq 1.
\]

Also, choosing \( k = 5 \) and \( t = -13 \) we obtain that \( g_{23,14,5,-13} \circ \delta_{23} \) is homotopic to

\[
(18) \quad ([C, B]^{12})^{325} ([B, A]^{12})^{-5772} ([A, C]^{12})^{325} ([B, C], A)^{3})^{-22437350} \simeq 1.
\]

Thus, following the same steps of the trivialization of \( \tilde{P}_0 \), we can exhibit, up to a homotopy of the above commutator powers to the constant 1, the
diffeomorphisms
\[ \tilde{P}_{23} \times S^3 = \tilde{P}_{23,14} \quad \text{and} \quad \tilde{P}_{14} \times S^3 = \tilde{P}_{14,23}. \]

Let \( H : S^3 \times S^3 \times S^3 \times [0, \frac{\pi}{2}] \rightarrow S^3 \) be a smooth homotopy such that
\[
H(A, B, C, \theta) = \begin{cases} 
g_{23,14,5,13} \circ \delta_{23}(A, B, C, \theta) & \text{if } \theta \in [0, \frac{\pi}{6}] \\
1 & \text{if } \theta \in [\frac{\pi}{6}, \frac{\pi}{2}].
\end{cases}
\]

By remembering that \( s_{\tilde{U}_{23}}^{5,13} : \tilde{U}_{23} \rightarrow \tilde{P}_{23,14} \) and \( s_{\tilde{V}_{23}}^{5,13} : \tilde{V}_{23} \rightarrow \tilde{P}_{23,14} \)
are partial sections of \( \tilde{P}_{23,14} \) over \( \tilde{U}_{23} \) and \( \tilde{V}_{23} \) respectively given by
\[
s_{\tilde{U}_{23}}^{5,13}(M_{23}(a, b, y_2)) = (M_{23}(a, b, y_2), M_{14}(a, b, Y_2(5, -13))),
\]
\[
s_{\tilde{V}_{23}}^{5,13}(M_{23}(a, b, z_23)) = (M_{23}(a, b, z_23), M_{14}(a, b, Z_{14}(5, -13))),
\]
where
\[
Y_2(5, -13) = (y_2^5 \bar{a}^{-111})^{-13} \frac{L_1 L_2 \ldots L_{10} L (23)^{-65}}{(L_1 L_2 \ldots L_{19})^{-65} L(14)},
\]
\[
Z_{14}(5, -13) = (-1)^{-64}(z_23^5 \bar{B}^{-111})^{-13} \frac{(L_1 L_2 \ldots L_{19} L(23))^{-65}}{L_1 L_2 \ldots L_{10} L(14)},
\]
we can then construct a global section \( s_{23,14} : \tilde{P}_{23} \rightarrow \tilde{P}_{23,14} \) given by
\[
s_{23,14}(M_{23}) = \begin{cases} 
s_{\tilde{U}_{23}}^{5,13}(M_{23}) & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\
s_{\tilde{V}_{23}}^{5,13}(M_{23})(H \circ \delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\
s_{\tilde{U}_{23}}^{5,13}(M_{23}) & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}.
\end{cases}
\]

Therefore, a diffeomorphism \( \Phi_{23,14} : \tilde{P}_{23} \times S^3 \rightarrow \tilde{P}_{23,14} \) is given by
\[
\Phi_{23,14}(M_{23}(a, b, x_1, x_2, \ldots, x_{23}), q) = (M_{23}(a, b, x_1, x_2, \ldots, x_{23}), M_{14}(a, b, y_1 q, y_2 q, \ldots, y_{14} q)),
\]
where if \( \theta = \theta(a) = \cos^{-1}|a| \) then
\[
y_1 = \begin{cases} 
(b\bar{a})^{12}|b|^{-24}(L_1 L_2 \ldots L_{10})^2 Z_{14} & \text{if } \frac{5\pi}{12} \leq \theta \leq \frac{\pi}{2} \\
(b\bar{a})^{12}|b|^{-24}(L_1 L_2 \ldots L_{10})^2 Z_{14}.(H \circ \delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12} \\
-(\bar{a}b)|a|^{-2} Y_2 & \text{if } 0 \leq \theta \leq \frac{5\pi}{12}.
\end{cases}
\]
\[
y_2 = \begin{cases} 
-(b\bar{a})^{13}|b|^{-26}(L_1 L_2 \ldots L_{10})^2 Z_{14} & \text{if } \frac{5\pi}{12} \leq \theta \leq \frac{\pi}{2} \\
-(b\bar{a})^{13}|b|^{-26}(L_1 L_2 \ldots L_{10})^2 Z_{14}.(H \circ \delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12} \\
Y_2 & \text{if } 0 \leq \theta \leq \frac{5\pi}{12}.
\end{cases}
\]
Let now $\theta$ be a smooth homotopy such that $G : S^3 \times S^3 \times S^3 \times [0, \frac{\pi}{2}] \rightarrow S^3$ be a smooth homotopy such that

$$G(A, B, C, \theta) = \begin{cases} g_{14, 23, 19, 1} \circ \delta_{14}(A, B, C, \theta) & \text{if } \theta \in [0, \frac{\pi}{6}] \\ 1 & \text{if } \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]. \end{cases}$$

Following the same steps of the construction of $\Phi_{23, 14}$ we have

$$s_{\tilde{U}_{14}}^{19,1}(M_{14}(a, b, y_2)) = (M_{14}(a, b, y_2), M_{23}(a, b, \mathcal{Y}_2(19, 1))),$$

$$s_{\tilde{V}_{14}}^{19,1}(M_{14}(a, b, z_{14})) = (M_{14}(a, b, z_{14}), M_{23}(a, b, \mathcal{Z}_{23}(19, 1)))$$

are partial sections of $\tilde{P}_{14, 23}$ over $\tilde{U}_{14}$ and $\tilde{V}_{14}$ respectively, where

$$\mathcal{Y}_2(19, 1) = \mathcal{Y}_2(19, 1)(a, b, y_2) = y_{19}^{12} a^{-225} \frac{L_1 L_2 L_3 \cdots L_{19} L(14)}{(L_1 L_2 \cdots L_{10})^{19} L(23)},$$

$$\mathcal{Z}_{23}(19, 1) = \mathcal{Z}_{23}(19, 1)(a, b, z_{14}) = -z_{19}^{12} a^{-225} \frac{(L_1 L_2 \cdots L_{10} L(14))^{19}}{L_1 L_2 L_3 \cdots L_{19} L(23)}.$$
THE ROLE OF COMMUTATORS IN A NON-CANCELLATION PHENOMENON

If \( M_{14} = M_{14}(a, b, x_1, x_2, \ldots, x_{14}) \in \tilde{P}_{14} \) then a global section, \( s_{14,23} : \tilde{P}_{14} \rightarrow \tilde{P}_{14,23} \) is given by

\[
\begin{align*}
\text{if } & \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2}, \\
\text{then } & \begin{cases} 
\bar{S}_{14}^{19}(M_{14}) \\
\bar{S}_{14}^{19}(M_{14}).(G \circ \delta_{14}^{-1})(M_{14}) \\
\bar{S}_{14}^{19}(M_{14})
\end{cases}
\end{align*}
\]

and a diffeomorphism \( \Phi_{14,23} : \tilde{P}_{14} \times S^3 \rightarrow \tilde{P}_{14,23} \) is given by

\[
\Phi(M_{14}(a, b, x_1, x_2, \ldots, x_{14}), q) = (M_{14}(a, b, x_1, \ldots, x_{14}), M_{23}(a, b, r_1.q, r_2.q, \ldots, r_{23}.q))
\]

with

\[
\begin{align*}
r_1 &= \begin{cases} 
(b\bar{a})^{21}|b|^{-2}(L_1L_2\ldots L_{19})^2Z_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2}, \\
-(ab)|a|^{-2}Y_2 & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12},
\end{cases} \\
r_2 &= \begin{cases} 
-(b\bar{a})^{22}|b|^{-2}(L_1L_2\ldots L_{19})^2Z_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2}, \\
-(ab)^{20}a(-40)(L_1L_2\ldots L_{19})^{-2}Y_2 & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12},
\end{cases} \\
r_k &= \begin{cases} 
(b\bar{a})^{23-k}|b|^{-2(23-k)}(L_1L_2\ldots L_{21-k})^2Z_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2}, \\
-(ab)^{21}|a|^{-2(k-1)}(L_{23-(k+1)}L_{23-k}\ldots L_{19})^{-2}Y_2 & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12},
\end{cases}
\end{align*}
\]

for \( 3 \leq k \leq 20 \),

\[
\begin{align*}
r_{21} &= \begin{cases} 
(b\bar{a})^{2}|b|^{-4}Z_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2}, \\
-(ab)^{20}a(-40)(L_1L_2\ldots L_{19})^{-2}Y_2 & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12},
\end{cases} \\
r_{22} &= \begin{cases} 
(b\bar{a})b^{-2}Z_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2}, \\
-(ab)^{21}|a|^{-2}(L_1L_2\ldots L_{19})^{-2}Y_2 & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12},
\end{cases} \\
r_{23} &= \begin{cases} 
-x_{14}^{19}b^{-225}(L_1L_2\ldots L_{19}L_{14})^{19}L_{12}\ldots L_{19}L_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1} |a| \leq \frac{\pi}{2}, \\
-x_{14}^{19}b^{-225}(L_1L_2\ldots L_{19}L_{14})^{19}L_{12}\ldots L_{19}L_{23} & \text{if } \frac{\pi}{12} \leq \cos^{-1} |a| \leq \frac{5\pi}{12}, \\
-(ab)^{22}|a|^{-2}(L_1L_2\ldots L_{19})^{-2}Y_2 & \text{if } 0 \leq \cos^{-1} |a| \leq \frac{\pi}{12},
\end{cases}
\end{align*}
\]
\[
\mathcal{Y}_2 = x_2^{19} a^{-225} \frac{L_1 L_2 \ldots L_{19} L(14)^{19}}{(L_1 L_2 \ldots L_{10})^{19} L(23)},
\]
\[
\mathcal{Z}_{23} = -x_1^{19} b^{-225} \frac{(L_1 L_2 \ldots L_{10} L(14))^{19}}{L_1 L_2 \ldots L_{19} L(23)}.
\]

We have with this

\[
\tilde{P}_{23} \times S^3 \xrightarrow{\Phi_{23,14}} \tilde{P}_{23,14} \xrightarrow{c} \tilde{P}_{14,23} \xrightarrow{\Phi_{14,23}^{-1}} \tilde{P}_{14} \times S^3,
\]

where \( c(M_2, M_{11}) = (M_{11}, M_2) \).

**Remark 6.** We also observe that the same procedure provides the trivialization of the bundle \( \tilde{P}_{2,11} \), choosing \( k = 11 \) and \( t = 5 \).

**Exotic Actions**

Non-cancellation phenomena related to products \( M \times G = N \times G \) of non equivalent spaces \( M \) and \( N \) by a group \( G \) can be seen as exotic actions of the group on, say, \( M \times G \) with quotient \( N \). We have treated here two such cases were \( G = S^3 \). We showed precisely how the specific diffeomorphisms and, equivalently, the corresponding exotic actions depend on the homotopy commutativity of certain powers of commutators in \( S^3 \).

In the case of the exotic actions treated above we have \( S^3 \) acting freely on \( P \times S^3 \), where \( P \) can be considered as a parameter space on which \( S^3 \) acts in a standard way and that parametrizes a complicated action of \( S^3 \) on itself, the second factor, so that the action on the product is free and the quotient is not equivalent to \( P \). Investigating properties of prospective \( P \)'s, like for example, how small such a compact \( P \) can be, etc., seems like an interesting way of looking at some classical questions.

On the other hand, specifying explicit homotopies between powers of commutators and constants seems to be a problem of geometric nature [CR], [RC].

**References**


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