On Harada Rings. I

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ON HARADA RINGS. I

To Takasi Nagahara on his 60th birthday

KIYOICHI OSHIRO

In [4]—[6] (cf. [7]), M. Harada has studied the following two conditions:

(*) Every non-small right \( R \)-module contains a non-zero injective submodule.

(*\!* Every non-cosmall right \( R \)-module contains a non-zero projective summand.

In particular, he has studied two rings which are characterized by ideal theoretic conditions: the one is a perfect ring with (*) and the other is a semi-perfect ring with (\(*\)!). In [8] and [9], the author has studied these rings with some additional conditions and introduced Harada (abbreviated \( H \)-) rings and co-Harada rings: A ring \( R \) is a right \( H \)-ring if it is a right artinian ring with (\(*\)), and dually \( R \) is a right co-\( H \)-ring if it satisfies (\(*\)! and the ascending chain condition for right annihilator ideals. In view of results in there, we see that these two rings are ‘companions’ of QF-rings and generalized uniserial rings. Although these classical artinian rings are left-right symmetric, \( H \)- and co-\( H \)-rings are not left-right symmetric. This fact seems to be an interesting phenomenon. However, in the present paper, we shall show a more interesting fact that the left \( H \)-rings and the right co-\( H \)-rings are the same rings. As a by-product of the study of \( H \)-rings, we shall show that a right generalized right QF-3 ring is a generalized uniserial ring.

1. Preliminaries. Throughout this paper, we assume that all rings \( R \) considered are associative rings with identity, all \( R \)-modules are unitary and all homomorphisms between \( R \)-modules are written on the opposite side of scalars. The notation \( M_R \) (resp. \( _R M \)) is used to denote that \( M \) is a right (resp. left) \( R \)-modules. Let \( M \) be an \( R \)-module. We use \( E(M), J(M), S(M) \) and \( Z(M) \) to denote its injective hull, Jacobson radical, socle and singular submodule, respectively. Further, by \( |J_1(M)| \) and \( |S_1(M)| \), we denote its descending Loewy chain and ascending Loewy chain, respectively. i.e., \( J_0(M) = M, J_1(M) = J(M), J_2(M) = J(J_1(M)), \ldots, S_0(M) = 0, S_1(M) = S(M), S_2(M)/S_1(M) = S(M/S_1(M)), \ldots \)
For submodules $A$ and $B$ of an $R$-module $M$ with $A \subseteq B$, the notation $A \subseteq_e B$ stands for 'A is an essential submodule of $B$'; while $A \subseteq_e B$ (in $M$) means 'A is a co-essential submodule of $B$', i.e., $B/A$ is a small submodule of $M/A$. For two $R$-modules $M$ and $N$, we use $M \subseteq_e N$ to stand for there is a monomorphism $f$ from $M$ into $N$; in particular, $M \subseteq_e N$ means that such $f$ exists and $f(M) \subseteq_e N$. The term $ACC$ means the ascending chain condition.

**Definition.** An $R$-module $M$ is an extending (resp. lifting) module if, for any submodule $A$ of $M$, there exists a direct summand $A^*$ of $M$ such that $A \subseteq_e A^*$ (resp. $A^* \subseteq_e A$).

**Definition** ([4]–[7], [10]). An $R$-module $M$ is a small module if it is small in its injective hull, and $M$ is a non-small module if it is not a small module. Dually, $M$ is a cosmall module if, for any projective module $P$ and any epimorphism $f: P \to M$, $\ker f$ is an essential submodule of $P$, and $M$ is a non-cosmall module if it is not a cosmall module.

**Definition** ([1], [2], [7], [11]). A ring $R$ is said to be right QF-3 if it has a minimal faithful right ideal. Right and left QF-3 rings are said to be QF-3 rings.

The following lemma due to Fuller plays an important role in our study.

**Lemma 1.1.** Let $R$ be a left artinian ring and $f$ a primitive idempotent of $R$. Then $_RRe$ is injective if and only if there exists a primitive idempotent $e$ in $R$ such that $(eRe : _RRe)$ is an injective pair, i.e.,

\[ _RRe/J(_RRe) \cong _RS(_RRe) \] and \[ S(eRe)_R \cong fR_R/J(fR_R) \]

Moreover, when this is so, $eR_R$ is also injective.

**Notation.** Let $R$ be a left artinian ring with a complete set of orthogonal primitive idempotents. When $E$ is arranged as $E = \{ e_1, e_2, \ldots, e_n \}$, we can identify $R$ with the matrix ring:

\[
\begin{bmatrix}
e_1Re_1 & \cdots & e_1Re_n \\
\vdots & & \vdots \\
e_nRe_1 & \cdots & e_nRe_n
\end{bmatrix}
\]

For the sake of convenience, we use the terms $e_i$-row and $e_i$-column instead.
of the terms $i$-row and $i$-column, respectively. So we identify $e_i R$ and $Re_i$ with $e_i$-row and $e_i$-column, respectively.

We note that if $R$ is basic and $(e_i R_R, \ k Re_j)$ is an injective pair, then

$$S(e_i R_R) = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}$$

$$S(e_i R_R) = \begin{bmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}$$

**Lemma 1.2** (Rayer [10], cf. [6]). *Let $R$ be a right artinian ring, and let $M$ be a right $R$-module. Then $M$ is a small module if and only if $MS_R = 0$.***

**2. Background.** As mentioned in the introduction, Harada has studied the conditions: (*) Every non-small right $R$-module contains a non-zero injective submodule. (***) Every non-cosmall right $R$-module contains a non-zero projective summand. And he has shown the following two theorems which give ideal theoretic characterizations of right artinian rings with (*) and semi-perfect rings with (***) .

**Theorem 2.1** ([6, Theorem 2, 3]). *A right artinian ring $R$ satisfies (*) if and only if, for any primitive idempotent $e$ in $R$ such that $eR_R$ is non-small, there exists an integer $t \geq 0$ for which

a) $eR_R/S_k(eR_R)$ is injective for $0 \leq k \leq t$, and

b) $eR_R/S_{k+1}(eR_R)$ is a small module.*

**Remark.** A left and right perfect ring with (*) is right artinian ([4, Theorem 5]).
Theorem 2.2 ([5], [6]). A semi-perfect $R$ ring satisfies $(\ast)^*$ if and only if, for a complete set $\{e_i\} \cup \{f_i\}$ of orthogonal primitive idempotents of $R$ such that each $e_iR$ is non-small and each $f_iR$ is small,

a) each $e_iR$ is injective,

b) for each $e_iR$, there exists $t_i \geq 0$ such that $J_t(e_iR)$ is projective for $0 \leq t \leq t_i$ and $J_{t_i-1}(e_iR)$ is a singular module, and

c) for each $f_iR$, there exists $e_i$ such that $f_iR \subseteq e_iR$. 

Remark. In case $R$ is left or right artinian, the condition ‘$J_{t_i+1}(e_iR)$ is singular’ in b) above can be dropped as we see in section 4.

Definition. We call a ring $R$ a right Harada (abbreviated $H$-) ring if $R$ is a right artinian ring with $(\ast)$, and call $R$ a right co-Harada ring if $R$ satisfies $(\ast)^*$ and the ACC for right annihilator ideals. Left and right $H$- (resp. co-$H$-) rings are called $H$- (resp. co-$H$-) rings.

Of course, right $H$-ring and right co-$H$-rings are mutually dual notions, as the following theorems shows:

Theorem 2.3 ([8]). The following conditions are equivalent for a given ring $R$:

1) $R$ is a right $H$-ring.

2) Every injective right $R$-module is a lifting module.

3) $R$ is a right perfect ring with the property that the family of all injective right $R$-modules is closed under taking small covers.

4) Every right $R$-module is expressed as a direct sum of an injective module and a small module.

When this is so, $R$ is a QF-3 ring and satisfies the ACC on left annihilator ideals (cf. [1]).

Theorem 2.4 ([8]). The following conditions are equivalent for a given ring $R$:

1) $R$ is a right co-$H$-ring.

2) Every projective right $R$-module is an extending module.

3) The family of all projective right $R$-modules is closed under taking essential extensions.

4) Every right $R$-module is expressed as a direct sum of a projective
module and a singular module.

When this is so, \( R \) is semi-primary QF-3 and satisfies the ACC on left annihilator ideals.

Remark. 1) Right \( H \)-rings and right co-\( H \)-rings are Morita invariant. 2) QF-rings and generalized uniserial rings are \( H \) and co-\( H \)-rings ([8], [9]). 3) Not all right \( H \)-rings are left \( H \)-rings, and not all right co-\( H \)-rings are left co-\( H \)-rings ([8]). 4) For a local QF-ring \( Q \), the following two rings mentioned in [8] are typical examples of right co-\( H \)-rings and at the same time of left \( H \)-rings:

\[
\begin{bmatrix}
Q & Q \\
J & Q/S
\end{bmatrix}
\]

where \( J = J(Q) \) and \( S = S(Q) \). 5) For an algebra \( R \) over a field of finite dimension, \( R \) is a left \( H \)-ring if and only if it is a right co-\( H \)-ring.

These remarks 4) and 5) led us to conjecture that left \( H \)-rings and right co-\( H \)-rings are the same rings. This conjecture is in fact true. In the next section we prove that right co-\( H \)-rings are left \( H \)-rings and, in section 5, the converse.

3. Right co-\( H \) \( \Rightarrow \) left \( H \). In this section, we show that right co-\( H \)-rings are left \( H \)-rings. As both rings are Morita invariant, we may show that basic right co-\( H \)-rings are left \( H \)-rings. Therefore in this section we assume that \( R \) is a basic right co-\( H \)-ring with a complete set

\[ E = \{ e_{11}, \ldots, e_{1m_1}, \ldots, e_{m_1}, \ldots, e_{mn_1}, \ldots \} \]

of orthogonal primitive idempotents such that

a) each \( e_{ii} R_\ell \) is injective,

b) \( e_{ii} R_\ell \supseteq e_{i2} R_\ell \supseteq \cdots \supseteq e_{mn_i} R_\ell \); more precisely, there exists an isomorphism \( \theta_{j,j-1}^i \) from \( e_{ij} R_\ell \) to \( J(e_{ij-1} R_\ell)_\ell \) for \( j = 1, \ldots, n(i) \) and \( i = 1, \ldots, m \),

c) all \( S(e_{ij} R_\ell)_\ell \) are simple.

Notation. 1) we put

\[ \theta_{j,k}^i = \theta_{k+1,k}^i \theta_{k+2,k+1}^i \cdots \theta_{j-1,j-2}^i \theta_{j,j-1}^i \]

for \( 0 \leq k < j \leq n(i) \). Then \( \theta_{j,k}^i \) is an isomorphism from \( e_{ij} R_\ell \) to \( J_{j-k}(e_{j,k} R_\ell)_\ell \).
2) We define a mapping $\Phi_{j,j-1}^i$ from $\text{End}_R(e_{ij}R_R)$ to $\text{End}_R(e_{i,j+1}R_R)$ by the rule

$$\Phi_{j,j+1}^i(\alpha) = (\theta_{j+1,j}^i)^{-1} \alpha \theta_{j+1,j}^i$$

for $\alpha \in \text{End}_R(e_{ij}R_R)$. Then it is routine to check that $\Phi_{j,j+1}^i$ is a ring epimorphism. For $j < k$, we put

$$\Phi_{j,k}^i = \Phi_{j,k-1,\ldots,j+1}^i \Phi_{j,j+1}^i$$

Then $\Phi_{j,k}^i$ is an epimorphism from $\text{End}_R(e_{ij}R_R)$ to $\text{End}_R(e_{ik}R_R)$.

**Proposition 3.1.** $\Phi_{j,j+1}^i$ is an isomorphism if and only if $e_{ij}R_R$ is not a projective cover of $S(e_{ij}R_R)_R$.

**Proof.** This is clear, since $\ker \Phi_{j,j+1}^i = \{ \alpha \in \text{End}_R(e_{ij}R_R) \mid \text{Im} \alpha \subseteq S(e_{ij}R_R) \}$.

**Remark.** Henceforth we observe $R$ by representing it as

$$R = \begin{bmatrix}
(e_{11}, e_{11}) & \cdots & (e_{m,n,m}, e_{11}) \\
(e_{11}, e_{12}) & \cdots & (e_{m,n,m}, e_{12}) \\
\vdots & \ddots & \vdots \\
(e_{11}, e_{mn,m}) & \cdots & (e_{m,n,m}, e_{mn,m}) \\
e_{11}Re_{11} & \cdots & e_{11}Re_{mn,m} \\
e_{12}Re_{11} & \cdots & e_{12}Re_{mn,m} \\
\vdots & \ddots & \vdots \\
e_{mn,m}Re_{11} & \cdots & e_{mn,m}Re_{mn,m}
\end{bmatrix}$$

where $(e_{ij}, e_{kl}) = \text{Hom}_R(e_{ij}R, e_{kl}R)$.

**Proposition 3.2.** $R$ is left artinian.

**Proof.** We may show that $e_{ef}eRf$ is artinian for all $e, f$ in $E$. We check this fact in four steps.

Step 1. As $R$ is semiprimary QF-3 and each $e_{ij}R_R$ is injective, we see from [1] that $e_{ij}Re_{ij}$ is artinian as a left $e_{ij}Re_{ij}$-module for all $i, j$.

Step 2. Assume that $e_{pq}R_R$ is a projective cover of $S(e_{pq}R_R)_R$. If $p \neq i$, Proposition 3.1 shows

$$\Phi_{i,2}^i \quad \Phi_{2,3}^i \quad \Phi_{m\mid n-1, m\mid l}^i \quad e_{i1}Re_{i1} \quad e_{i2}Re_{i2} \quad \cdots \quad e_{ml}Re_{ml} \quad (\text{as ring})$$

If $p = i$, Proposition 3.1 also shows

http://escholarship.lib.okayama-u.ac.jp/mjou/vol31/iss1/16
\[ \Phi_{1,2} e_{11} R e_{11} \sim e_{12} R e_{12} \sim \cdots \sim e_{1q} R e_{1q}, \]
\[ \Phi_{q+1,q+2}^{\pm} e_{L,q+1} R e_{L,q+1} \sim \cdots \sim e_{L,n} R e_{L,n}, \]
\[ \Phi_{q,q+1}^{\pm} : e_{1q} R e_{1q} \to e_{L,q+1} R e_{L,q+1} \] is a ring epimorphism.

Since \( e_{11} R e_{11} \) is artinian as a left \( e_{11} R e_{11} \)-module, \( e_{i1} R e_{1j} \) is artinian as a left \( e_{i1} R e_{1j} \)-module for all \( i \).

Step 3. We observe \( e_{i1} R e_{kt} \) for \( i \neq k \). Put \( f_j = e_{ij}, f_1 = e_{11}, g_l = e_{kt}, g_1 = e_{kt} \). Then note that \( f_j R e_{kt} \) becomes a left \( f_j R e_{kfj} \)-module by the epimorphism \( \Phi_{i,j} f_j R e_{kt} \to f_j R e_{kj} \). We define a mapping \( \zeta \) from \( f_j R e_{kt} = (g_1, f_i) \) to \( f_j R e_{kt} = (g_1, f_i) \) by the rule: \( a \to \theta_{k,i} a \) for \( a \in (g_1, f_i) \). Then it is easy to check that \( \zeta \) is a left \( f_j R e_{kj} \)-homomorphism. Moreover, noting \( i \neq k \), we see that it is an isomorphism. On the other hand, the mapping \( \eta : f_j R e_{kt} = (g_1, f_i) \to f_j R e_{kt} = (g_1, f_i) \) given by the rule: \( a \to \alpha \theta_{k,i} a \) for \( \alpha \in (g_1, f_i) \) is a left \( f_j R e_{kj} \)-epimorphism. Hence \( \zeta^{-1} \) gives an epimorphism:

\[ f_j R e_{kt} f_j R e_{kt} \to f_j R e_{kt} f_j R e_{kt}. \]

Since \( f_j R e_{kt} f_j R e_{kt} \) is artinian, it follows that \( f_j R e_{kt} f_j R e_{kt} \) is artinian.

Step 4. Put \( e_1 = e_{11}, \ldots, e_{n1} = e_{n1}, \) and observe \( e_j R e_k \) for \( k \neq j \).

If \( k < j \), then \( e_j R e_k \) becomes a left \( e_k R e_k \)-module by the epimorphism \( \Phi_{k,j} : e_k R e_k \to e_j R e_j \). Consider a mapping \( \lambda : e_j R e_k = (e_k, e_j) \to e_k R e_k = (e_k, e_k) \) given by the rule: \( a \to \theta_{k,j} a \) for \( a \in (e_k, e_j) \). As is easily seen, \( \lambda \) is a left \( e_k R e_k \)-homomorphism and moreover it is a monomorphism. Since \( e_k R e_k e_k R e_k \) is artinian, it follows that \( e_k R e_k e_j R e_j \) is artinian. Next, if \( k > j \), then \( e_k R e_k \) becomes a left \( e_j R e_j \)-module by \( \Phi_{j,k} \), and \( e_k R e_k e_j R e_k \to e_k R e_k e_j R e_k \) by the mapping: \( a \to (\theta_{k,j})^{-1} a \). Hence, in this case, we also see that \( e_k R e_k e_j R e_k \) is artinian, since \( e_k R e_k e_k R e_k \) is artinian.

By Steps 1 – 4, \( e R f \) is artinian as a left \( e R e \)-module for all \( e, f \) in \( E \), so \( R \) is left artinian.

**Lemma 3.3.** Let \( e, f \) be in \( E \) such that \( e R e \) is injective and \( f R e \) is a projective cover of \( S(e R e) \). Put \( X = \text{Hom}_R(f R, S(e R e)) \). Then

1) \( e R e X \) and \( X_{f R e} \) are simple.
2) \( e R e X \simeq e R e S(e R e f R) \) and \( X_{f R e} \simeq S(e R e f R)_{f R e} \).
3) \( S(e R e f R) = S(e R e f R) \).

**Proof.** 1) It is enough to show that \( e R e a = f R e = X \) for any non-zero \( a \) in \( X \). Let \( 0 \neq \beta \in X \). Since \( f R e \) is projective, there exists \( \gamma \) in
fRf satisfying \( \alpha \gamma = \beta \): whence \( X = \alpha fRf \). On the other hand, \( \alpha \) and \( \beta \)
induce isomorphisms: \( fR/J(fR) \cong S(eR_k) \) and \( fR/J(fR) \cong S(eR_k) \). Since
any automorphism of \( S(eR_k) \) is induced from one of \( eR_k \), we can obtain \( \sigma \)
in \( eRe \) such that \( \beta^{-1} \sigma = \sigma \) on \( S(eR_k) \). Then \( \beta = \sigma \alpha \) and it follows that
\( \beta = \sigma \alpha \). As a result, we see \( X = eRe \). 2) and 3) are clear from 1).

**Lemma 3.4.** Let \( f \) and \( g \) in \( E \), and assume that \( fR_k \) is a projective
cover of \( S(e_{i_1}R_k) \) or, equivalently, \( fR/J(fR) \cong S(e_{i_1}R) \).

1) For \( \alpha \) in \( (f, g) \) with \( \text{Im } \alpha \supseteq S(gR) \), there exists \( \beta_k \) in \( (g, e_{i_k}) \) such
that \( \text{Im } \beta_k \alpha = S(e_{i_k}R_k) \) for \( k = 1, \ldots, n(i) \).

2) If \( g \) is in \( E \setminus \{ e_{i_1}, \ldots, e_{i_{n(i)}} \} \), then, for any \( 0 \neq \alpha \in (f, g) \), there
exists \( \beta_k \in (g, e_{i_k}) \) such that \( \text{Im } \beta_k \alpha = S(e_{i_k}R_k) \) for \( k = 1, \ldots, n(i) \).

3) If \( g = e_{i_1} \), then, for any \( 0 \neq \alpha \in (f, g) \), there exists \( \beta_k \in (g, e_{i_k}) \)
such that \( \text{Im } \beta_k \alpha = S(e_{i_k}R_k) \) for \( k = 1, \ldots, t \).

**Proof.** Note that 2) is contained in 1). For convenience's sake, put
\( e_t = e_{i_1}, \ldots, e_{n(i)} = e_{i_{n(i)}} \). Let \( 0 \neq \alpha \in (f, g) \). \( \alpha \) induces an isomorphism \( \tilde{\alpha} \):
\( fR/\ker \alpha \cong \text{Im } \alpha \). Then, note \( \ker \alpha \subseteq J(fR) \). Since \( fR/J(fR) \cong S(e_kR) \),
there exists an \( R \)-homomorphism \( \gamma_k \) from \( \text{Im } \alpha \) onto \( S(e_kR) \). Put \( \theta = \theta_{i_1} \); \( e_kR \cong J_{k-1}(e_kR) \). Since \( e_kR_k \) is injective, there exists \( \delta_k \in (g, e_t) \) which
is an extension of \( \theta \gamma_k \), i.e., \( \delta_k = \theta \gamma_k \) on \( \text{Im } \alpha \).

Now, if \( \gamma_k \) is not a monomorphism, that is, \( \text{Im } \alpha \supseteq S(gR) \), then \( \text{Im } \delta_k \subseteq J_{n(i)}(e_kR_k) \subseteq \theta(e_kR_k) \); so \( \theta^{-1} \delta_k \in (g, e_k) \) with \( \text{Im } \theta^{-1} \delta_k \alpha = S(e_kR) \).
Next, assume \( g = e_t \) and \( \text{Im } \alpha = S(e_tR) \). Then \( \delta_k \) is a monomorphism with
\( \text{Im } \delta_k = J_{t-1}(e_tR) = \theta_{t-1}(e_tR) \). So, \( \theta^{-1} \delta_k \) has a sense as a homomorphism
from \( e_kR_k \) to \( e_kR_k \) for \( k \leq t \), and \( \text{Im } \theta^{-1} \delta_k \alpha = S(e_kR) \). The proof
is completed.

**Proposition 3.5.** Let \( f \) be in \( E \), and assume that \( fR_k \) is a projective
cover of \( S(e_{i_1}R_k) \). Then

1) \( S_k(Rf) = S(e_{i_1}R_k) + \cdots + S(e_{i_k}R_k) \) for \( k = 1, \ldots, n(i) \); whence \( S_k(Rf) \) is a two sided ideal of \( R \).

2) \( S(e_{i_1}R_k)(Rf/S_k(Rf)) = (S(e_{i_1}R_k) + S_k(Rf))/S_k(Rf) \) for \( k = 1, \ldots, n(i) - 1 \).

3) \( S(R_k)(Rf/S_{n(i)}(Rf)) = 0 \)

Therefore \( Rf/S_k(Rf) \) is a non-small left \( R \)-module for \( 1 \leq k < n(i) \) and \( Rf/S_{n(i)}(Rf) \) is a small left \( R \)-module by Lemma 1.2.
Proof. Put $e_k = e_{ik}$ for $k = 1, \ldots, n(i)$. We observe $Rf$ and $e_kR$ by identifying these with $f$-column and $e_k$-row, respectively:

$$
Rf = \begin{bmatrix}
0 & & & \\
\vdots & & & \\
(f, e_{i1}) & 0 & & \\
(f, e_{i2}) & & & \\
\vdots & & & \\
\end{bmatrix}
$$

$$
e_kR = \begin{bmatrix}
(e_{i1}, e_k) & \cdots & (f, e_k) & \cdots & (e_{m1}, e_k)
\end{bmatrix}
$$

We put $X_k = \text{Hom}_R(fR, S(e_kR))$. Then, by Lemma 3.4,

$$
S(e_kR) = \begin{bmatrix}
0 & \cdots & 0 & X_k & 0 & \cdots & 0
\end{bmatrix}
$$

so we see

$$
S(e_1R_\bar{a}) + \cdots + S(e_kR_\bar{a}) = \begin{bmatrix}
0 & \cdots & 0 & X_k & 0 & \cdots & 0
\end{bmatrix}
$$

On the other hand, using Lemma 3.4, we see

$$
S_k(\overline{a}Rf) = \begin{bmatrix}
0 & \cdots & 0 & X_k & 0 & \cdots & 0
\end{bmatrix}
$$
Hence $S(e_1R_R) + \cdots + S(e_kR_R) = S_k(Rf)$. The proofs of 2) and 3) are easy from 1).

**Proposition 3.6.** The following conditions are equivalent for $f$ in $E$.
1) $fR_R$ is a projective cover of some $S(e_{i1}R_R)$.
2) $\kappa Rf$ is injective.
3) $\kappa Rf$ is a non-small module.

**Proof.** 1) $\Rightarrow$ 2). By Proposition 3.5 and Lemma 3.3, we see that $(e_{i1}R_R, \kappa Rf)$ is an injective pair; whence $\kappa Rf$ is injective (Lemma 1.1). 2) $\Rightarrow$ 3) is clear. Assume that $\kappa Rf$ is a non-small module. Then $S(R_R)Rf \neq 0$ by Lemma 1.2. Hence $(S(e_{i1}R_R) + \cdots + S(e_{im}R_R))Rf \neq 0$ for some $i$. Let $g$ be in $E$ such that $gR_R$ is a projective cover of $S(e_{i1}R_R)$. By Proposition 3.5,

$$S(e_{i1}R_R) + \cdots + S(e_{im}R_R) = \begin{bmatrix} X_1 & \vdots \\ -1 & 0 & \vdots \\ 0 & \vdots \end{bmatrix}$$

where $X_k = \text{Hom}(gR, S(e_{ik}R_R))$, $k = 1, \ldots, n(i)$. Hence if $g \neq f$, we see that $(S(e_{i1}R_R) + \cdots + S(e_{im}R_R))Rf = 0$. a contradiction. Thus $f = g$ and hence $fR_R$ is a projective cover of $S(e_{i1}R_R)$.

Now we are in a position to show the following

**Theorem 3.7.** $R$ is a left $H$-ring.

**Proof.** By Proposition 3.2, $R$ is a left artinian ring. Let $f$ be in $E$, and assume that $\kappa Rf$ is injective. Then, by Proposition 3.6, there exists $e_{i1}$ in $E$ for which $fR_R$ is a projective cover of $S(e_{i1}R_R)$. By Proposition 3.5, $Rf/S_k(Rf)$ is a non-small left $R$-module for $k = 1, \ldots, n(i) - 1$, and $Rf/S_{\kappa Rf}Rf$ is a small left $R$-module. Therefore, the proof is completed if we show that $Rf/S_k(Rf)$ is injective for $k = 1, \ldots, n(i) - 1$. By Proposition 3.5, $S_k(Rf)$ is a two sided ideal of $R$. Here we denote the factor ring $R/S_k(Rf)$ by $\bar{R}$, and $r + S_k(Rf)$ by $\bar{r}$ for $r$ in $R$. We observe $\bar{Rf}$ by iden-
tifying it with

\[
\begin{bmatrix}
  \vdots \\
  (f, e_{i_1}) \\
  0 \\
  (f, e_{i_k}) \\
  (f, e_{i_{k+1}}) \\
  \vdots \\
\end{bmatrix}
\xrightarrow{\phi}
\begin{bmatrix}
  0 \\
  : \\
  0 \\
  X_1 \\
  X_k \\
  : \\
  0 \\
\end{bmatrix}
\]

where \( X_j = \text{Hom}_\frak{R}(fR, S(e_{i_j}R)) \), \( j = 1, \ldots, k \). As is easily seen, \((e_{i_{k+1}}, \frak{R}f, \frak{R}f)\) is an injective pair; whence \( \frak{R}f \) is injective (cf. Lemma 1.1). In order to show that \( \frak{R}f \) is injective as a left \( R \)-module, consider a diagram:

\[
0 \to \frak{R}I \to \frak{R}R \\
\downarrow \phi \\
\frak{R}f
\]

where \( \frak{R}I \subseteq \frak{R}R \) and \( \phi \) is an \( R \)-homomorphism. Put \( \phi_0 = \phi \) and \( I_0 = I \). It is easy to see that \( \phi_0(S_1(Rf)) = 0 \) since

\[
S_1(Rf) = \begin{bmatrix}
  0 \\
  \vdots \\
  0 \\
  X_1 \\
  0 \\
  \vdots \\
  0 \\
\end{bmatrix}
\]

Since \( S_1(Rf) \subseteq I_1 \) and \( \phi_0(S_1(Rf)) = 0 \), \( \phi_0 \) induces an \( R/S_1(Rf) \)-homomorphism \( \phi_0^* : I_1/S_1(Rf) \to \frak{R}f \). We can take a left ideal \( I_2 \) containing \( I_1 + S_2(Rf) \) and \( R/S_1(Rf) \)-homomorphism \( \phi_2 : I_2/S_1(Rf) \to \frak{R}f \) such that the restriction \( \phi_2|I_1/S_1(Rf) \) is \( \phi_0^* \). Since \( I_2 \supseteq S_2(Rf) \) and \( \phi_0(S_2(Rf)) = 0 \), \( \phi_2 \) induces an \( R/S_2(Rf) \)-homomorphism \( \phi_2^* : I_2/S_2(Rf) \to \frak{R}f \), and by the same argument, we can take a left ideal \( I_3 \) containing \( I_2 + S_3(Rf) \) and \( R/S_2(Rf) \)-homomorphism \( \phi_3 : I_3/S_2(Rf) \to \frak{R}f \) which is an extension of \( \phi_2^* \). Continuing this procedure \( k \) times, we obtain left ideals \( I_1, \ldots, I_k \) such that \( S_1(Rf) \subseteq I_1, S_2(Rf) + I_1 \subseteq I_2, \ldots, S_k(Rf) + I_{k-1} \subseteq I_k \) and \( R/S_{i-1}(Rf) \)-homomorphism \( \phi_i : I_i/S_{i-1}(Rf) \to \frak{R}f \) such that \( \phi_0|I_{i-1}/S_{i-1}(Rf) = \phi_{i-1}^* \), where \( \phi_{i-1}^* \) is the
induced homomorphism from $\phi_{k-1}: I_{k-1}/S_{k-1}(Rf) \to \bar{Rf}$.

Here we consider the diagram:

$0 \to I_k/S_{k-1}(Rf) \xrightarrow{\phi_k} \bar{Rf}$

Since $\phi_{k-1}$ is injective, there exists an $R$-homomorphism $\xi: \bar{R} \to \bar{Rf}$ which is an extension of $\phi_k$. Let $\eta: R \to \bar{R}$ be the canonical homomorphism. Then $\xi\eta$ is an extension of $\phi$. This completes the theorem.

4. A remark on $(\ast)^{\ast}$. Let $R$ be a basic semi-perfect ring with a complete set $E = \{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$ of orthogonal primitive idempotents satisfying

a) each $e_{ii}R_R$ is injective,

b) $e_{ij}R_R \simeq J_{j-1}(e_{ii}R_R)R_R$ for $j = 1, \ldots, n(i)$ and $i = 1, \ldots, m$.

For this ring $R$, we show the following:

Theorem 4.1. If $R$ is left or right artinian, then $R$ satisfies $(\ast)^{\ast}$, namely the following condition holds:

c) $J(e_{ii}R_R)_R$ is a singular module for $i = 1, \ldots, m$.

Proof. Since $R$ is a perfect ring, a) and b) implies
d) All $S(e_{ii}R_R)_R$ are non-zero simple modules.

Since $R$ is a one sided artinian ring with a), b) and d), in view of arguments in § 3, we see that all results (except Proposition 3.2) in there are valid for this ring $R$. Now, in order to show c), we may show that $J(e_{ii}R_R)_R S(R_R) = 0$ for all $i$.

Let $f_i$ be in $E$ such that $f_iR_R$ is a projective cover of $S(e_{ii}R_R)_R$ for $i = 1, \ldots, m$. Put

$$X_k = \text{Hom}_R(f_iR, S(e_{ik}R_R))$$

for $i = 1, \ldots, m, k = 1, \ldots, n(i)$. Then, as in § 3, we see

$$S(e_{ik}R_R) = \begin{bmatrix}
0 & & \\
& \ddots & \\
& & 0
\end{bmatrix}$$

for $i = 1, \ldots, m, k = 1, \ldots, n(i)$. Then, as in § 3, we see
Noting this fact, we can easily calculate that \( J(e_{\nu}; R_{\ell}) S(R_{\ell}) = 0 \).

5. Left \( H \Rightarrow \) right \( co-H \). In this section, we show that left \( H \)-rings are right \( co-H \)-rings. Since left \( H \)-rings and right \( co-H \)-rings are Morita invariant, we may show that basic left \( H \)-rings are right \( co-H \). So, henceforth, we assume that \( R \) is a basic left \( H \)-ring with \( E \), a complete set of orthogonal primitive idempotents. Since \( R \) is a semi-primary QF-3 (cf. Theorem 2.3), we can have a partition

\[
E = \{e_{11}, \ldots, e_{1\nu_{11}}\} \cup \cdots \cup \{e_{m1}, \ldots, e_{m\nu_{m1}}\} \cup \{g_{1}, \ldots, g_{l}\}
\]

such that

a) each \( e_{i1}; R_{\ell} \) is injective,

b) \( S(e_{i1}; R_{\ell}) S(e_{\ell1}; R_{\ell}) \) for all \( i \),

c) \( S(g_{R_{\ell}}) \) is not simple for \( g \) in \( G = \{g_{1}, \ldots, g_{l}\} \).

As in \( \S \) 3, we observe \( R \) by identifying it with the matrix ring:

\[
\begin{bmatrix}
(e_{11}, e_{11}) & \cdots & (g_{1}, e_{11}) \\
\cdots & \cdots & \cdots \\
(e_{11}, g_{1}) & \cdots & (g_{1}, g_{1})
\end{bmatrix}
\]

where \((p, q)\) means \( \text{Hom}_{\ell}(p_{R}, q_{R}) \) for \( p, q \) in \( E \).

**Lemma 5.1.** Let \( f \) be in \( E \) and assume that \( f_{R_{\ell}} \) is a projective cover of \( S(e_{i1}; R_{\ell}) \). Then \( (e_{i1}; R_{\ell} : \ell Rf) \) is an injective pair, so \( \ell Rf \) is injective.

**Proof.** We put

\[
A_{i} = \begin{bmatrix}
0 & \cdots & 0 \\
0 & \vdots & 0 \\
0 & \text{Hom}_{\ell}(f_{R}, S(e_{i1}; R_{\ell})) & 0 & \cdots & 0 \\
0 & \vdots & 0 \\
0 & \vdots & 0 \\
\end{bmatrix}
\]

Then we see that \( A_{i} \) is a two sided ideal of \( R \), and furthermore, \( A_{i} = S(e_{i1}; R_{\ell}) \) and \( A_{i} \subseteq S(\ell Rf) \). Since \( A_{i} Rf \neq \hat{0}, \ell Rf \) is non-small (cf. Lemma 1.2); whence \( \ell Rf \) is injective and it follows that \( A_{i} = S(\ell Rf) \). Since \( A_{i} = S(e_{i1}; R_{\ell}) = S(\ell Rf) \), clearly \( (e_{i1}; R_{\ell} : \ell Rf) \) is an injective pair.
Lemma 5.2. Let $i$ be in $\{1, \ldots, m\}$. If $n(i) = 1$, then $S(e_{11}R_\pi R) \not\subset gR_\pi$ for all $g$ in $G$.

Proof. Let $f$ be in $E$ such that $fR_\pi$ is a projective cover of $S(e_{11}R_\pi R)$. Assume that there exists $g$ in $G$ such that $S(e_{11}R_\pi R) \subset gR_\pi$. As in Lemma 5.1, we put

$$A_{i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad A_{g} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then, as we saw in Lemma 5.1, $A_{i} = S(e_{11}R_\pi R) = S(Rf)$ and note that $A_{g}$ is a right ideal of $R$ with $A_{g} \subset S(gR_\pi R)$. Since $A_{g}(Rf/S(Rf)) \neq 0$, $Rf/S(Rf)$ is non-small; whence $Rf/S(Rf)$ is injective as a left $R$-module. Consider the factor ring $R = R/A_{i}(= R/S(Rf))$. Since $Rf$ is injective, $Rf$ is also injective. Hence there must exist $h$ in $E$ such that $(hR_\pi R; Rf)$ becomes an injective pair. As is easily seen, $h \neq e_{11}$. Further, we see from $n(i) = 1$ that $h \not\in E - (G \cup \{e_{11}\})$; whence $h$ must be in $G$. Since $Rf$ is injective, $S(hR_\pi R)$ has the simple socle. However, this shows that $S(hR_\pi R)$ has the simple socle, a contradiction. Thus $S(e_{11}R_\pi R) \not\subset gR$ for all $g$ in $G$.

Lemma 5.3. Let $i$ be in $\{1, \ldots, m\}$ and assume that $n(i) \geq 2$ and $e_{12}R_\pi R$ for $j = 3, \ldots, n(i)$. Then

1) $e_{12}R_\pi R \not\subset gR_\pi$ for all $g$ in $G$,
2) $e_{12}R_\pi R \cong J(e_{11}R_\pi R)$.

Proof. We take $f$ in $E$ such that $fR_\pi$ is the projective cover of $S(e_{11}R_\pi R)$. We put

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Then $A_1 = S(e_{11} R_{\bar{R}}) = S(\bar{x} Rf)$ by the proof of Lemma 5.1 and $A_2$ is a right ideal of $R$ with $A_2 \subseteq S(e_{12} R_{\bar{R}})$. Put $\bar{R} = R/A_1$. Then note that $J(e_{11} R_{\bar{R}})$ and $e_{12} R$ become canonically right $\bar{R}$-module, and there exists a canonical isomorphism: $\bar{e}_{12} \bar{R}_{\bar{R}} \cong e_{12} R_{\bar{R}}$.

Since $A_1 \bar{R}f \neq 0$, $\bar{x} Rf$ is non-small; so $\bar{R}f$ is injective as a left $\bar{R}$-module and hence so is as a left $\bar{R}$-module. Thus there exists $h$ in $E$ for which $(h \bar{R}_{\bar{R}}; \bar{R}f)$ is an injective pair. Then, as is easily seen, $h \neq e_{11}$ and $h \neq e_{kt}$ for any $kt$ with $k \neq i$. Thus $h = e_{12}$ or $h \in G$.

Here, assume that there exists $g$ in $G$ such that $e_{12} R_{\bar{R}} \subseteq g R_{\bar{R}}$. Then we see that $h \neq e_{12}$; whence $h \in G$. However, as in the proof of Lemma 5.2, this implies that $h R_{\bar{R}}$ has the simple socle, a contradiction. Thus $e_{12} R_{\bar{R}} \not\subseteq g R_{\bar{R}}$ for all $g$ in $G$.

Now, if $h$ is in $G$, then $h R_{\bar{R}}$ has the simple socle as above, a contradiction. As a result, $h$ must be $e_{12}$, whence $\bar{e}_{12} \bar{R}_{\bar{R}}$ is injective. Since $\bar{e}_{12} \bar{R}_{\bar{R}} \cong e_{12} R_{\bar{R}} \subseteq e J(e_{11} R_{\bar{R}})_{\bar{R}}$, it follows that $e_{12} R_{\bar{R}} \cong J(e_{11} R_{\bar{R}})_{\bar{R}}$; whence $e_{12} R_{\bar{R}} \cong J(e_{11} R_{\bar{R}})_{\bar{R}}$ as desired.

**Lemma 5.4.** 1) There is a permutation $|e_{\alpha(1)}, \ldots, e_{\alpha(n(i))}|$ of $|e_{12}, \ldots, e_{1n(i)}|$ such that

$$J_{\bar{R}}(e_{11} R_{\bar{R}})_{\bar{R}} \cong e_{\alpha(k)} R_{\bar{R}}$$

for $k = 2, \ldots, n(i)$.

2) $e_{ij} R_{\bar{R}} \not\subseteq g R_{\bar{R}}$ for all $j$ and $g$ in $G$. 

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3) $S(e_{i_1}R_R) \not\subseteq gR_R$ for all $g$ in $G$; whence $G$ must be empty.

Proof. 1) and 2) are shown by the same proof as in the proof of Lemma 5.3. In order to prove 3), we take $f$ in $E$ such that $fR_R$ is a projective cover of $S(e_{i_1}R_R)$. By 1) and 2) (cf. Proofs of Lemma 5.1 and 5.3), we see that

$$S_{m_1}(Rf) = S(e_{i_1}R_R) + \cdots + S(e_{i_{m_1}}R_R)$$

Now, assume that there exists $g$ in $G$ such that $S(e_{i_1}R_R) \subseteq S(gR_R)$. Then $S(gR_R)(Rf/S_{m_1}(Rf)) \neq 0$; whence $Rf/S_{m_1}(Rf)$ is non-small and it is injective as a left $R$-module. Put $\bar{R} = R/S_{m_1}(Rf)$. Then $\bar{R}$ is injective as a left $\bar{R}$-module. So, there exists $h$ in $E$ such that $(\bar{h}R_{\bar{R}}; \bar{g}R_{\bar{R}})$ is an injective pair. Then, as is easily seen, $h$ is in $G$. Since $\bar{h}R_{\bar{R}}$ has the simple socle, $\bar{h}R_{\bar{R}}$ has also the simple socle, a contradiction. Thus 3) holds.

We are now ready to show the following

Theorem 5.5. $R$ is a right co-$H$-ring.

Proof. By Theorem 2.4, $R$ satisfies the ACC on right annihilator ideals. And, by Theorems 2.2 and 4.1 and Lemma 5.4, we see that $R$ satisfies $\ast \ast \ast$. So, $R$ is a right co-$H$-ring.

6. An application of $H$-rings. In this section, we show the following

Theorem 6.1. If $R$ is a right QF-3 and right generalized uniserial ring then $R$ is a generalized uniserial ring.

Remark. A ring $R$ is said to be right (left) generalized uniserial if it is right (left) artinian and, for any primitive idempotent $e$, $eR_R(eRe)$ has a unique composition series. A left and right generalized uniserial ring is said to be simply generalized uniserial.

For a proof of Theorem 6.1, the following two lemmas are needed.

Lemma 6.2. If $R$ is a right QF-3 and right generalized uniserial ring, then $R$ is a right co-$H$- (hence left $H$-) ring.

Proof. This is easily shown by Theorem 2.2.

Lemma 6.3. If $R$ is a right QF-3 and right generalized uniserial ring, then $R$ is a right co-$H$- (hence left $H$-) ring.
ring, then so is the factor ring $R/S(R_R)$.

Proof. We can assume that $R$ is a basic ring. By Lemma 6.1, $R$ is a right co-$H$-ring; so, as in § 3, we observe $R$ by identifying it with the matrix ring:

$$
\begin{bmatrix}
(e_{11}, e_{11}) & \cdots & (e_{mn,m}, e_{11}) \\
\vdots & & \vdots \\
(e_{11}, e_{mm,m}) & \cdots & (e_{mn,m}, e_{mm,m})
\end{bmatrix}
$$

where $E = \{e_{11}, \ldots, e_{1n1}, \ldots, e_{m1}, \ldots, e_{mmm} \}$ is a complete set of orthogonal primitive idempotents such that

a) each $e_{ii} R_R$ is injective,

b) $e_{mi} R_R \subseteq e_{i,n,i+1} R_R \subseteq \cdots \subseteq e_{12} R_R \subseteq e_{11} R_R$

for $i = 1, \ldots, m$.

We put $\bar{R} = R/S(R_R)$, and for $\bar{r}$ in $R$, $\bar{r} = r + S(R_R)$. Then $\bar{R}$ is also a basic and $|\bar{e}_{ij}|, \bar{e}_{ij} \neq 0 |$ is a complete set of orthogonal primitive idempotents. We see that

c) $\bar{e}_{ni} \bar{R} \subseteq \bar{e}_{i,n-1} \bar{R} \subseteq \cdots \subseteq \bar{e}_{12} \bar{R}$

for $i = 1, \ldots, m$.

When $\bar{e}_{ii} \bar{R} \neq 0$, we take $f$ in $E$ such that $fR_R$ is the projective cover of $S(e_{ii} R_R)_R$. Then we can take $(\bar{e}_{ii} \bar{R}; \bar{R} \bar{f})$ is an injective pair. As a result

d) each $\bar{e}_{ii} \bar{R}$ is injective if it is non-zero.

By c) and d), we see that $\bar{R}$ is a right co-$H$-ring; so $\bar{R}$ is QF-3 (cf. Theorems 2.2 and 4.1). As $R$ is clearly right generalized uniserial, this completes the proof.

Proof of Theorem 6.1: We can assume that $R$ is a basic ring. Let $E = \{e_{ij}\}$ be as in Lemma 6.3. We take $f_i$ in $E$ such that $f_i R_R$ is the projective cover of $S(e_{ii} R_R)_R$ for $i = 1, \ldots, m$. Then, by Proposition 3.5,

1) $S(e_{ii} R_R) + \cdots + S(e_{ik} R_R) = S_k R f_i$

for $k = 1, \ldots, n(i)$ and moreover we see

2) $S_k R f_i / S_{k-1} R f_i$ is simple as a left $R$-module for $k = 1, \ldots, n(i)$.

Now, by the induction of the sum of composition lengths of all $e_{ii} R_R$ together with Lemma 6.3, $R/S(R_R)$ is a generalized uniserial ring. Here, in view of 1) and 2) above, this implies that $R$ is left uniserial.
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