Separated Sets of Torsion Theories

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Dedicated to Prof. Hiroyuki Tachikawa on the occasion of his 60th birthday

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Throughout the following $R$ will denote an associative ring with identity element 1 and $R$-mod will denote the category of unitary left $R$-modules. The frame of all (hereditary) torsion theories on $R$-mod will be denoted by $R$-tors. Notation and terminology concerning $R$-tors will follow [2]. In particular, if $M$ is a left $R$-module then $E(M)$ will denote the injective hull of $M$, $\xi(M)$ will denote the smallest torsion theory on $R$-mod relative to which $M$ is torsion and $\chi(M)$ will denote the largest torsion theory on $R$-mod relative to which $M$ is torsionfree. If $\tau \in R$-tors then a nonzero left $R$-module $N$ is $\tau$-cocritical if $N$ is $\tau$-torsionfree but every proper homomorphic image of $N$ is $\tau$-torsion. A $\tau$-cocritical left $R$-module $N$ is uniform and has the property, which we will use repeatedly, that $\chi(N') = \chi(N)$ for every nonzero submodule $N'$ of $N$. If $\sigma \leq \tau$ in $R$-tors we say that $\tau$ is a generalization of $\sigma$. The generalization is proper if $\sigma < \tau$.

The notion of a separated set of torsion theories on $R$-mod was considered briefly in Chapter 29 of [2]. We expand the definition given there as follows: if $\sigma$ is a torsion theory on $R$-mod then a set $U$ of generalizations of $\sigma$ in $R$-tors is $\sigma$-separated if and only if $\tau \wedge [\vee (U \setminus \tau)] = \sigma$ for each $\tau \in U$. The empty set is trivially $\sigma$-separated for every torsion theory $\sigma$. This relation was studied in the context of modular lattices (under the name of "independence") in [3] and [4]. It is straightforward to show that the following result is true:

1. Proposition. If $\sigma \in R$-tors then the following conditions on a nonempty set $U$ of generalizations of $\sigma$ are equivalent:

   1. $U$ is $\sigma$-separated;
   2. For any partition $U = U' \cup U^*$ of $U$, $\sigma = (\vee U') \wedge (\vee U^*)$;
   3. Every nonempty subset of $U$ is $\sigma$-separated;
   4. Every finite nonempty subset of $U$ is $\sigma$-separated.

   The torsion theory $\sigma$ has finite dimension if and only if every $\sigma$-sepa-
rated set of generalizations of \( \sigma \) is finite.

2. Proposition. If \( \sigma \in R\text{-tors} \) and if \( |U_i| i \in \Omega | \) is a chain of \( \sigma \)-separated sets of generalizations of \( \sigma \) then \( U = \bigcup |U_i| i \in \Omega | \) is \( \sigma \)-separated.

Proof. If \( Y \) is a finite subset of \( U \) then there exists some \( h \in \Omega \) such that \( Y \subseteq U_h \) and so \( Y \) is \( \sigma \)-separated. Hence, by Proposition 1, \( U \) is \( \sigma \)-separated. \( \Box \)

By Zorn's Lemma, we see that if \( \sigma \in R\text{-tors} \) then any \( \sigma \)-separated set of generalizations of \( \sigma \) is contained in a maximal \( \sigma \)-separated set.

For the purposes of this note we will say that a torsion theory \( \sigma \) on \( R\text{-mod} \) is good if and only if for each torsion theory \( \tau \) on \( R\text{-mod} \) satisfying \( \sigma < \tau \) there exists a \( \sigma \)-cocritical \( \tau \)-torsion left \( R\)-module. By Proposition 2.10 of [2] we know that \( \xi \), the unique minimal element of \( R\text{-tors} \), is always good.

Recall that if \( \sigma \in R\text{-tors} \) then the ring \( R \) is \( \sigma \)-noetherian if and only if the set of all \( \sigma \)-pure left ideals of \( R \) satisfies the ascending chain condition. If \( \sigma \) is a torsion theory on \( R\text{-mod} \) such that \( R \) is \( \sigma \)-noetherian then we claim that \( \sigma \) is good. Indeed, assume that \( \sigma < \tau \) in \( R\text{-tors} \) and let \( N \) be a nonzero \( \tau \)-torsion \( \sigma \)-torsionfree left \( R\)-module. If \( 0 \neq x \in N \) then the set of all \( \sigma \)-pure left ideals of \( R \) containing \( (0 : x) \) is nonempty and so has a maximal element \( H \). Then \( R/H \) is \( \tau \)-torsion and \( \sigma \)-cocritical, proving that \( \sigma \) is good.

Thus we see, in particular, that if the ring \( R \) is left noetherian then every torsion theory on \( R\text{-mod} \) is good.

If \( \sigma < \tau \) in \( R\text{-tors} \) then we say that the torsion theory \( \tau \) is \( \sigma \)-uniform if and only if the set of torsion theories \( \tau' \) satisfying \( \sigma < \tau' \leq \tau \) is closed under taking finite meets. We will denote the set of all \( \sigma \)-uniform torsion theories on \( R\text{-mod} \) by \( \sigma \text{-unif} \).

3. Proposition. Let \( \sigma \) be a good torsion theory on \( R\text{-mod} \) and let \( \tau \) be a proper generalization of \( \sigma \) in \( R\text{-tors} \). Then the following conditions are equivalent:

\begin{itemize}
  \item[(1)] \( \tau \) is \( \sigma \)-uniform;
  \item[(2)] \( \lambda(M) = \lambda(M') \) for all \( \tau \)-torsion \( \sigma \)-cocritical left \( R\)-modules \( M \) and \( M' \).
\end{itemize}

Proof. (1) \( \Rightarrow \) (2): If \( M \) and \( M' \) are \( \tau \)-torsion \( \sigma \)-cocritical left \( R\)-modules we set \( \rho = \sigma \lor \xi(M) \) and \( \rho' = \sigma \lor \xi(M') \). Then \( \sigma \neq \rho, \rho' \leq \tau \) and so, by (1), \( \sigma \neq \rho \land \rho' = \sigma \lor (\xi(M) \land \xi(M')) \). Since \( \sigma \) is good,
there exists a $\sigma$-cocritical left $R$-module $N$ which is $(\rho \land \rho')$-torsion, and so $\sigma \lor \hat{\xi}(N) \leq \rho \land \rho'$. In particular, $N$ is not $[\hat{\xi}(M) \land \hat{\xi}(M')]$-torsion-free and so, by restriction if necessary, we can assume that it is $[\hat{\xi}(M) \land \hat{\xi}(M')]$-torsion. This means that there exist nonzero $R$-homomorphisms from $M$ to $E(N)$ and from $M'$ to $E(N)$ which must indeed be monic since $N$ is $\sigma$-torsion-free and $M$, $M'$ are $\sigma$-cocritical. Since $N$ is cocritical and hence uniform as well, this implies that $\lambda(M) = \lambda(M')$.

(2) $\Rightarrow$ (1): Assume that $\sigma < \tau'$, $\tau'' \leq \tau$. Since $\sigma$ is good we know that there exist $\sigma$-cocritical left $R$-modules $M'$ and $M''$ satisfying $\hat{\xi}(M') \leq \tau'$ and $\hat{\xi}(M'') \leq \tau''$. By (2), this means that $\lambda(M') = \lambda(M'')$ and so we can assume that $E(M') = E(M'')$. If $N = M' \cap M''$ then $N$ is $\sigma$-cocritical and $\sigma \notin \sigma \lor \hat{\xi}(N) \leq \tau' \land \tau''$, proving (1). □

If $\sigma \in R$-tors, let us denote by $M(\sigma)$ the set of all prime torsion theories of $R$-mod of the form $\lambda(M)$, where $M$ is a $\sigma$-cocritical left $R$-module. By Proposition 33.21 of [2] we see that $M(\sigma)$ is contained in the set $P_\sigma(\sigma)$ of all minimal prime generalizations of $\sigma$. However, we need not have equality in general. If $\sigma \leq \tau$ in $R$-tors, let $M(\sigma, \tau)$ be the set of all elements of $M(\sigma)$ of the form $\lambda(M)$, where $M$ is both $\sigma$-cocritical and $\tau$-torsion. The cardinality of $M(\sigma, \tau)$ will be called the $\sigma$-rank of $\tau$. Clearly $M(\sigma, \sigma) = \emptyset$ and, by definition, we see that the set $M(\sigma, \tau)$ is nonempty (and hence the $\sigma$-rank of $\tau$ is nonzero) if $\sigma < \tau$ and $\sigma$ is good. We also note that if $\sigma \leq \tau' \leq \tau$ then $M(\sigma, \tau') \subseteq M(\sigma, \tau)$. By Proposition 3, we see that if $\sigma < \tau$ and $\sigma$ is good then $\tau$ is $\sigma$-uniform if and only if it has $\sigma$-rank equal to 1, i.e., if and only if $M(\sigma, \tau)$ is a singleton $\{\pi\}$. In this case, we will say that the $\sigma$-uniform torsion theory $\tau$ is of type $\pi$.

4. Proposition. Let $\sigma$ be a good torsion theory on $R$-mod. A set $U$ of $\sigma$-uniform torsion theories is $\sigma$-separated if and only if no two elements of $U$ are of the same type.

Proof. Assume that no two elements of $U$ are of the same type. By Proposition 1 it suffices to assume that the set $U$ is nonempty and finite. Let $U = \{\tau_1, \ldots, \tau_n\}$ be a set of $\sigma$-uniform torsion theories and let $M(\sigma, \tau_i) = \{\pi_i\}$ for each $1 \leq i \leq n$. For each such $i$, let $M_i$ be a $\tau_i$-torsion $\sigma$-cocritical left $R$-module satisfying $\lambda(M_i) = \pi_i$. If there exists an index $h$ such that $\tau_h \land [\bigvee_{i \neq h} \tau_i] \neq \sigma$ then there exists a $\sigma$-cocritical left $R$-module $N$ which is $\tau_h \land [\bigvee_{i \neq h} \tau_i]$-torsion. In particular, $N$ is $\tau_h$-torsion and so $\lambda(N) \in M(\sigma, \tau_h) = \{\pi_h\}$. Hence, without loss of generality, we can as-
assume that \( N = M_n \). Since \( M_n \) is \( [\vee_{i \neq h} \tau_i] \)-torsion, there exists an index \( k \neq h \) such that \( M_n \) is not \( \tau_k \)-torsionfree. Replacing \( M_n \) by its \( \tau_k \)-torsion submodule, we can assume that it is \( \tau_k \)-torsion and so \( \pi_n = \chi(M_n) \in M(\sigma, \tau_k) = |\pi_k| \), contradicting the assumption that \( \tau_h \) and \( \tau_k \) are not of the same type.

Now, conversely, assume that \( U \) is \( \sigma \)-separated. If there are two distinct elements \( \tau \) and \( \tau' \) of \( U \) of the same type \( \chi(M) \), then \( \chi(M) \leq \tau \wedge [\vee (U \setminus \tau')] \), contradicting \( \sigma \)-separation. Therefore no two elements of \( U \) are of the same type. \( \Box \)

5. Corollary. If \( \sigma \) is a good torsion theory on \( R \)-mod then any two maximal \( \sigma \)-separated sets of \( \sigma \)-uniform torsion theories have the same cardinality.

Proof. The cardinality of a maximal \( \sigma \)-separated set of \( \sigma \)-uniform torsion theories is clearly equal to the cardinality of \( M(\sigma) \). \( \Box \)

Recall that an independence structure \( \mathcal{E} \) on a nonempty set \( A \) consists of a family of subsets of \( A \) satisfying the following conditions:

1. \( \phi \in \mathcal{E} \);
2. If \( A' \subseteq A'' \in \mathcal{E} \) then \( A' \in \mathcal{E} \);
3. If \( A' \) and \( A'' \) are finite sets in \( \mathcal{E} \) satisfying \(|A'| < |A''|\) then there is a set \( B \in \mathcal{E} \) satisfying \(|A' \subseteq B \subseteq A' \cup A''\) and \(|B| = |A''|\);
4. If every finite subset \( A' \) of a set \( A \) belongs to \( \mathcal{E} \) then \( A \in \mathcal{E} \).

For more information on such structures, refer to [1] or [5].

6. Proposition. If \( \sigma \) is a good torsion theory on \( R \)-mod then the family of all \( \sigma \)-separated sets of \( \sigma \)-uniform torsion theories is an independence structure on \( \sigma \)-unif.

Proof. Condition (1) is true by definition and conditions (2) and (4) follow from Proposition 1. We are therefore left to prove condition (3). Let \( U' \) and \( U'' \) be finite \( \sigma \)-separated sets of \( \sigma \)-uniform torsion theories on \( R \)-mod with \(|U'| < |U''|\). Say \( U' = |\tau_1, \cdots, \tau_k|\), where each \( \tau_i \) is of type \( \pi_i \), and let \( U'' = |\sigma_1, \cdots, \sigma_n|\), where each \( \sigma_i \) is of type \( \pi'_i \). By renumbering if necessary, we can assume that there exists an integer \( 1 \leq t \leq k+1 \) such that \( \pi_i \) and \( \pi'_{i} \) are equal for all \( i < t \) and are not equal for all \( i \geq t \). Then \( Y = |\tau_1, \cdots, \tau_k, \sigma_{k+1}, \cdots, \sigma_n|\) is a set of \( \sigma \)-uniform torsion theories no two of which are of the same type and so the set is \( \sigma \)-separated. Moreover, \( U' \subseteq Y \subseteq U' \cup U'' \) and \(|Y| = |U''|\). \( \Box \)
If \( \sigma < \sigma' \leq \tau \) in \( R\)-tors then \( \tau \) is \( \sigma \)-essential over \( \sigma' \) if and only if \( \sigma \neq \sigma' \wedge \sigma'' \) for all \( \sigma < \sigma' \leq \tau \). Thus, trivially, if \( \sigma < \tau \) then \( \tau \) is \( \sigma \)-essential over itself. Moreover, a torsion theory \( \tau > \sigma \) is \( \sigma \)-uniform if and only if it is \( \sigma \)-essential over every torsion theory \( \sigma' \) satisfying \( \sigma < \sigma' \leq \tau \).

7. Proposition. If \( \sigma < \sigma' \leq \tau \) are torsion theories on \( R\)-mod with \( \sigma \) good then \( \tau \) is \( \sigma \)-essential over \( \sigma' \) if and only if \( M(\sigma, \tau) = M(\sigma, \sigma') \).

Proof. Assume that \( \tau \) is \( \sigma \)-essential over \( \sigma' \). Since \( \sigma' \leq \tau \) we have \( M(\sigma, \sigma') \subseteq M(\sigma, \sigma) \). On the other hand, assume that \( \pi \in M(\sigma, \tau) \) and let \( N \) be a \( \sigma \)-cocritical \( \tau \)-torsion left \( R \)-module satisfying \( \pi = \chi(N) \). Then \( \sigma < \sigma \vee \tilde{\xi}(N) \leq \tau \) so \( \sigma \neq \sigma' \wedge (\sigma \vee \tilde{\xi}(N)) \). Since \( \sigma \) is good, there exists a \( \sigma \)-cocritical left \( R \)-module \( N' \) which is \( [\sigma' \wedge (\sigma \vee \tilde{\xi}(N))] \)-torsion. In particular, there exists a nonzero \( R \)-homomorphism from \( N \) to \( E(N') \), which must be monic since \( N \) and \( N' \) are \( \sigma \)-cocritical. Then the uniformness of \( N' \) implies that \( \pi = \chi(N) = \chi(N') \in M(\sigma, \sigma') \), proving that \( M(\sigma, \tau) = M(\sigma, \sigma') \).

Conversely, assume that \( M(\sigma, \tau) = M(\sigma, \sigma') \) and let \( \sigma < \sigma' \leq \tau \). If \( N \) is a \( \sigma \)-cocritical \( \sigma^* \)-torsion left \( R \)-module then \( \chi(N) \in M(\sigma, \tau) = M(\sigma, \sigma') \) and so \( N \) is \( \sigma^* \)-torsion as well. Thus \( \sigma < \sigma \vee \tilde{\xi}(N) \leq \sigma' \wedge \sigma^* \), proving that \( \tau \) is \( \sigma \)-essential over \( \sigma' \). □

8. Corollary. If \( \sigma \in R\)-tors is good then a \( \sigma \)-essential generalization of a \( \sigma \)-uniform torsion theory is \( \sigma \)-uniform.

Proof. This is a direct consequence of Proposition 3 and Proposition 7. □

9. Corollary. If \( \sigma \in R\)-tors is good and if \( \tau > \sigma \) in \( R\)-tors then there exists a \( \sigma \)-separated set \( U \) of \( \sigma \)-uniform torsion theories on \( R\)-mod such that \( \tau \) is a \( \sigma \)-essential extension of \( \bigvee \ U \).

Proof. Take \( U = \{ \sigma \vee \tilde{\xi}(M) | M \text{ a } \sigma \text{-cocritical } \tau \text{-torsion left } R\text{-module} \} \). This set is \( \sigma \)-separated by Proposition 4. □

10. Proposition. If \( \sigma \) is a good torsion theory on \( R\)-mod then any proper generalization of \( \sigma \) has a unique maximal \( \sigma \)-essential generalization.

Proof. Let \( \sigma < \tau \) in \( R\)-tors and let \( \tau' = \bigvee \{ \sigma' > \sigma | M(\sigma, \tau) = M(\sigma, \sigma') \} \). Clearly \( \tau' \geq \tau \) and so \( M(\sigma, \tau) \subseteq M(\sigma, \tau') \). Conversely, if
Theorem. If $\sigma \leq \tau$ in $R$-tors we will denote the $\sigma$-rank of $\tau$ by $r_\sigma(\tau)$. The torsion theory $\tau$ is $\sigma$-flat if and only if $r_\sigma(\tau') > r_\sigma(\tau)$ for all $\tau' > \tau$ in $R$-tors. In other words, $\tau$ is $\sigma$-flat if and only if for each $\tau' > \tau$ there is an $\tau'$-torsion $\sigma$-cocritical left $R$-module which is $\tau$-torsionfree. (Note that the term "flat" is used here in its combinatoric, rather than algebraic, sense; see [1] or [5].)

11. Proposition. If $\sigma \leq \tau$ in $R$-tors then the family of all $\sigma$-flat generalizations $\tau$ is closed under taking arbitrary meets.

Proof. Let $U$ be a nonempty set of $\sigma$-flat generalizations of $\tau$ and assume that $\tau' > \wedge U$ in $R$-tors. Then there exists an element $\rho$ of $U$ such that $\tau' \not< \rho$ and so $\tau' \lor \rho > \rho$. Thus there is a $\rho$-torsionfree $\sigma$-cocritical left $R$-module $M$ which is $(\tau' \lor \rho)$-torsion and hence not $\tau'$-torsionfree. Replacing $M$ by its $\tau'$-torsion submodule, we can assume that it is $\tau'$-torsion. On the other hand, $M$ is $(\wedge U)$-torsionfree since it is $\rho$-torsionfree. Therefore $r_\sigma(\tau') > r_\sigma(\wedge U)$. □

12. Proposition. Let $\sigma$ be a good torsion theory on $R$-mod and let $\sigma < \tau$. If $M(\sigma, \tau)$ is finite then the maximal $\sigma$-essential generalization of $\tau$ is the meet of all $\sigma$-flat generalizations of $\tau$.

Proof. Let $\tau'$ be the maximal $\sigma$-essential generalization of $\tau$. If $\tau'' > \tau'$ then $\tau''$ is not $\sigma$-essential over $\tau$ and so, by Proposition 7, $M(\sigma, \tau'') \supset M(\sigma, \tau')$. Therefore $r_\sigma(\tau'') > r_\sigma(\tau')$, proving that $\tau'$ is $\sigma$-flat over $\tau$. Now assume that $\rho$ is $\sigma$-flat over $\tau$ and satisfies $\rho < \tau'$. Then $r_\sigma(\tau') > r_\sigma(\rho)$ and so $M(\sigma, \tau) = M(\sigma, \tau') \supset M(\sigma, \rho) \supset M(\sigma, \tau)$, which is a contradiction. Thus, by Proposition 11, $\tau'$ is the meet of all $\sigma$-flat generalizations of $\tau$. □

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(Received May 20, 1988)