Tauberian Theorems of Jp → Mp-Type

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TAUBERIAN THEOREMS OF $J_\rho \to M_\rho$-TYPE

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Let $|p_n|$ be a sequence of non-negative numbers with $p_n > 0$. $P_n := p_0 + \cdots + p_n$ for $n = 0, 1, \ldots$, 
\begin{equation}
p(x) := \sum_{n=0}^\infty p_n x^n \quad \text{for real } x
\end{equation}
and, for a given sequence $|s_n|$ of complex numbers let
\begin{equation}
p_s(x) := \sum_{n=0}^\infty p_n s_n x^n \quad \text{for real } x
\end{equation}
and
\[ l_n := \frac{1}{P_n} \sum_{k=0}^n p_k s_k \quad \text{for } n = 0, 1, \ldots \]

We shall say that $|s_n|$ is $M_\rho$-limitable to $\sigma$. and write $M_\rho$-$\lim s_n = \sigma$, if
\[ \lim_{n \to \infty} l_n = \sigma. \]
We shall say that $|s_n|$ is $J_\rho$-limitable to $\sigma$ and write $J_\rho$-$\lim s_n = \sigma$, if the series (1) has radius of convergence 1, the series (2) converges for $0 < x < 1$ and
\[ \lim_{x \to 1-} p_s(x)/p(x) = \sigma. \]
It is known that both, the $M_\rho$-method and the $J_\rho$-method are regular if and only if $P_n \to \infty$. In this case $M_\rho$-$\lim s_n = \sigma$ implies $J_\rho$-$\lim s_n = \sigma$ (see Ishiguro [4]), but the converse is not always true. For many $|p_n|$ however $J_\rho$-$\lim s_n = \sigma$ implies $M_\rho$-$\lim s_n = \sigma$, if $|s_n|$ fulfills an additional condition, which we will call a Tauberian condition of $J_\rho \to M_\rho$-type.

If $p_n := 1$ for $n = 0, 1, \ldots$ then $J_\rho$ is the Abel-method $A_\lambda$ whilst $M_\rho$ is the Cesàro-method $C_\lambda$, and the following famous theorem of Hardy and Littlewood [3] (see [2, Theorem 94]) holds: $s_n = O_\lambda(1)$ is a Tauberian condition of $A_\lambda \to C_\lambda$-type. (For the definition of $O_\lambda$, $O$ and $o$ see [2, p. 149].) But $s_n = O_\lambda(1)$ is a Tauberian condition of $J_\rho \to M_\rho$-type for many other sequences $|p_n|$ (see, for example, [5, Theorem 4.1 and pp. 72/73] and [16]). Most of these results are special cases of

Theorem A ([16, Satz 4.3]). Let $P_n \to \infty$ and
\begin{equation}
np_n = O(P_n).
\end{equation}
Then $s_n = O_\lambda(1)$ is a Tauberian condition of $J_\rho \to M_\rho$-type.

If $p_n := (n+1)^{-1}$ for $n = 0, 1, \ldots$ then $J_\rho$ and $M_\rho$ are the logarithmic
methods $L$ and $l$ respectively, and the following theorem of Kokhanovskii [7, Theorem 1] holds: $s_n = O_L(\ln(n+1))$ is a Tauberian condition of $L \to l$-type. Now for $p_n := (n+1)^{-1}$ the condition $s_n = O_L(\ln(n+1))$ is equivalent to $np_n s_n = O_L(P_n)$, and this is a Tauberian condition of $J_p \to M_{p^*}$-type for many other sequences $|p_n|$ (see, for example, Kokhanovskii [8, Theorem 1] and Mikhalin [11, Theorem 5]). All these results are special cases of

**Theorem B** (Borwein [1, Theorem 2]). Let $P_n \to \infty$ and

\[(4) \quad np_n = o(P_n).\]

Then $np_n s_n = O_L(P_n)$ is a Tauberian condition of $J_p \to M_{p^*}$-type.

If (3) holds, then $np_n s_n = O_L(P_n)$ is implied by $s_n = O_L(1)$. So it is natural to ask, whether (4) in Theorem B can be replaced by (3). The answer is yes, and is a consequence of the following theorem because (3) is strictly stronger than

\[(5) \quad 1 \leq P_n / P_m \to 1 \quad \text{as} \quad 1 < n/m \to 1 \quad (m \to \infty).\]

**Theorem 1.** Let $P_n \to \infty$ and (5). Then $np_n s_n = O_L(P_n)$ is a Tauberian condition of $J_p \to M_{p^*}$-type.

For the proof of Theorem 1, which contains Theorems A and B as special cases, we need some lemmas.

**Lemma 2.** Let $P_n \to \infty$ and $P_{2n} = O(P_n)$. Then

a) $p(x)/p(x^2) = O(1)$ as $x \to 1$.

b) $p(t^{1/n}) = O(P_n)$ as $n \to \infty$ for every $t \in (0, 1)$.

Lemma 2a) was proved by Kratz and Stadtmüller [9, Lemma 2, proof of (vii) ⇒ (viii)]. To prove b) let $t \in (0, 1)$ be fixed. By a) there exists $K > 0$ such that $p(x) < Kp(x^2)$ for all $x \in (0, 1)$. Now we choose $m \in \mathbb{N}$ such that $\varepsilon := K^{1/2} < 1$. Then

$p(t^{1/n}) \leq P_{mn} + \sum_{k=mn-1}^m p_k t^{k/2n} \leq P_{mn} + t^{m/2} p(t^{1/2n}) \leq P_{mn} + K^{m/2} p(t^{1/n}),$

and therefore, $P_{2n} = O(P_n)$ implies

\[
\frac{p(t^{1/n})}{P_n} \leq \frac{P_{mn}}{P_n} \frac{1}{1-\varepsilon} = O(1) \quad \text{as} \quad n \to \infty.
\]
Lemma 3. Let $P_n \to \infty$, (5) and $np_n s_n = O_P(P_n)$. Then
\[ p_s(x)/p(x) = O(1) \text{ as } x \to 1 - \text{ implies } t_n = O(1). \]

Proof. We choose $K > 0$ with $np_n s_n > -KP_n$ for $n = 0, 1, ...$ and define the non-negative sequence $|\gamma_n|$ by
\[ \gamma_n := \begin{cases} |s_n| & \text{if } np_n = 0 \\ Kp_n/np_n & \text{if } np_n \neq 0. \end{cases} \]
Then $s_n \geq -\gamma_n$ for $n = 0, 1, ...$ and $np_n \gamma_n = O(P_n)$. Now, following Borwein [1. proof of Theorem 2], we choose $t \in (0, 1)$ fixed and obtain
\[ \frac{1}{p(t^{1/n})} \sum_{k=0}^{n} p_k s_k = O(1) \text{ as } n \to \infty. \]
From this, and because $P_n/P_n = O(1)$ is a consequence of (5) (see [13]), $t_n = O(1)$ follows by Lemma 2b).

Special cases of Lemma 3 have been given by Kokhanovskii [7. Theorem 1] and [8, Lemma 1].

Lemma 4. Let $P_n \to \infty$, (5), $np_n s_n = O_P(P_n)$ and $t_n = O(1)$. Then
\[ \liminf (t_n - t_m) \geq 0 \text{ as } Q_n/Q_m \to 1 \text{ (}n > m \to \infty\text{)} \]
where $Q_n := P_0 + ... + P_n$ for $n = 0, 1, ...$.

Proof. We have
\[ t_n - t_{n-1} = \frac{p_n}{P_n} (s_n - t_{n-1}) \text{ for } n = 1, 2, ... \]
and therefore
\[ t_n - t_m = \sum_{\nu=0}^{n} \frac{p_{\nu} s_{\nu}}{P_{\nu}} \frac{1}{\nu} - \sum_{\nu=m+1}^{n} \frac{p_{\nu} t_{\nu-1}}{P_{\nu}} \text{ for } n > m > 0. \]
Hence, if $K > 0$ is a constant such that $np_n s_n \geq -KP_n$ and $|t_n| < K$, we get
\[ t_n - t_m \geq -K \ln \frac{n}{m} - K\left(\frac{P_n}{P_m} - 1\right) \text{ for } n > m > 0. \]
Now $Q_n/Q_m \to 1 \text{ (}n > m \to \infty\text{)}$ implies $n/m \to 1$ and thus $P_n/P_m \to 1$ by
(5), so (6) follows from (7).

Special cases of Lemma 4 have been given by Kokhanovskii [7, Lemma 1] and [8, Lemma 3].

**Lemma 5 ([15, Satz 3.9]).** Let \( P_n \to \infty \) and (5). Then

\[
\liminf (s_n - s_m) \geq 0 \quad \text{as} \quad P_n/P_m \to 1 \quad (n > m \to \infty)
\]

is a Tauberian condition of \( J_\rho \to c \)-type.

Now we are able to give the

**Proof of Theorem 1.** Let \( |s_n| \) be a sequence with \( np_n s_n = O_t(P_n) \) and \( J_\rho \)-lim \( s_n = \sigma \). Then \( t_n = O(1) \) by Lemma 3 and \( J_\rho \)-lim \( t_n = \sigma \). Therefore (6) holds by Lemma 4. Now (5) implies

\[
1 \leq Q_n/Q_m \to 1 \quad \text{as} \quad 1 < n/m \to 1 \quad (m \to \infty)
\]

(see [15, Nr. 5]). and so we get \( \lim t_n = \sigma \), i.e. \( M_\rho \)-lim \( s_n = \sigma \), by Lemma 5.

Since \( np_n s_n = O_t(P_n) \), the \( s_n \) in Theorem 1 have to be real. But clearly Theorem 1 holds for complex \( s_n \), if we replace \( O_t \) by \( O \). Thus we have the following corollary, which, for example, contains theorems of Kokhanovskii [6, Theorem 2], Teslenko [14, Theorem 2] and Mikhalin [10, Corollary 1] as special cases.

**Corollary 6.** Let \( P_n \to \infty \) and (5). Then \( np_n s_n = O(P_n) \) is a Tauberian condition of \( J_\rho \to M_\rho \)-type.

**References**


[6] A. P. Kokhanovskii: Tauberian theorems for semicontinuous logarithmic methods of sum-
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