Some Generalizations of Boolean Rings

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Throughout, \( R \) will represent a ring with center \( C \). Let \( N \) denote the set of nilpotents in \( R \), and \( N^* \) the subset of \( N \) consisting of all elements in \( R \) which square to zero. Let \( E \) be the set of idempotents in \( R \). If \( E \subseteq C \) then \( R \) is called normal. In case \( R \) has 1, we denote by \( U \) the multiplicative group of units of \( R \). Following [10], \( R \) is called \((E,N)\) representable, if each \( x \in R \) can be written uniquely in the form \( x = e + a \), where \( e \in E \) and \( a \in N \). Given \( x \in R \), we define inductively \( x^{(n)} = x, x^{(n+1)} = x^{(n-1)} \cdot x \), where \( x \cdot y = x + y + xy \). In [8], Hirano, Komatsu, Tominaga and Yaqub considered the following condition which arose, presumably, in connection with logic: 

\((*) \quad \text{for any } x, y \in R, (x+xy) \cdot (y+yx) = 0 \) if and only if \( x = y \),

and proved that \( R \) satisfies \((*)\) if and only if \( R \) is commutative, \( R/N \) is a Boolean ring and \( a^{(n)} = 0 \) for all \( a \in N \) (see Theorem 1 below). Obviously, every Boolean ring satisfies the condition \((*)\). If \( R \) has 1, then \((*)\) becomes \((*)' \quad \text{for any } x, y \in R, (1+x+xy)(1+y+yx) = 1 \) if and only if \( x = y \). Recently, GROSEN [4] gave a number of characterizations of a ring with 1 in which the condition \((*)'\) holds.

An element \( x \) in \( R \) is called strongly regular, if there exist \( y, y' \in R \) such that \( x^2y = x = y'x^2 \). As is well-known, if \( x \) is strongly regular, there exists (uniquely) \( z \in R \) such that \( x^2z = x, z^2x = z \) and \( xz = zx \); furthermore, \( z \) commutes with every element which commutes with \( x \). We denote by \( S \) the set of strongly regular elements in \( R \). A ring \( R \) is called a \( B'-ring \) if \( S = E \). Obviously, every Boolean ring is a \( B'-ring \).

A ring \( R \) is called \( s\)-unital if \( x \in Rx \cap xR \) for all \( x \in R \), or equivalently if for each finite subset \( F \) of \( R \) there exists \( e \in R \) such that \( ex = x = xe \) for all \( x \in F \) (see [6]). Following [11], \( R \) is called an \( s^*\)-unital ring if for each \( x \in R \) there exist \( e', e'' \in E \) such that \( xe' = x = e''x \), or equivalently if for each finite subset \( F \) of \( R \) there exists \( e \in E \) such that \( eFe = F \) (see [11, Corollary 7]). As is easily seen, every \( s\)-unital \( \pi\)-regular ring is \( s^*\)-unital. In what follows, we shall use freely this fact. A ring \( R \) is a \( cs^*\)-unital ring if for each \( x \in R \) there exists a central idempotent \( e \) such that \( ex = x \).

A ring \( R \) is called an \( I\)-ring (resp. \( N\)-ring) if every element of \( R \) is expressible as a product of elements in \( E \) (resp. \( N \)): \( R \) is called an \( NI\)-ring (or \( I\)-ring) if every element of \( R \) is expressible as a product of elements in
E ∪ N (see [1] and [7]). Needless to say, every Boolean ring is an I-ring.

Our present objective is to improve several results of Grosen obtained in [4, § 5] and the main theorems of Abu-Khuzam [1] and reprove the main theorems of [2].

First, as preliminaries, we state the following lemmas.

Lemma 1 ([10, Theorem 4]). The following are equivalent:

1) $R$ is $(E-N)$ representable.

2) $R$ is normal, and every element of $R$ can be written as a sum of an idempotent and a nilpotent element.

3) $R$ is normal and $x - x^2 ∈ N$ for every $x ∈ R$.

4) $R$ is normal. $N$ is an ideal and $R/N$ is a Boolean ring.

Lemma 2 ([8, Lemma 5]). Let $f(X) = k_1X + k_2X^2 + \cdots + k_nX^n$ be a polynomial in $XZ[X]$ with $(k_1, k_2) = 1$. If $N$ satisfies the identity $f(X) = 0$, then $N$ satisfies the identities $X^3 = 0 = k_1X + (k_2 - k_1)X^2$.

Lemma 3. If $N$ is closed under $*$ (in particular, if $N$ is an ideal) and satisfies the identity $X^{(2)} = 0$, then $N$ is commutative.

Proof. For any $a, b ∈ N$, $a * b = a * (a * b)^{(2)} = b = a^{(2)} * (b * a) = b * a$. whence $ab = ba$ follows.

Lemma 4. (1) If $R$ satisfies the identity $(X + X^3)^{(2)} = 0$, then $8x = 0, x^3 = x^3$ and $x - x^2 ∈ N$ (or $x + x^2 ∈ N$) for all $x ∈ R$, and $a^3 = 0 = a^{(2)}$ for all $a ∈ N$.

(2) If $N$ satisfies the identity $(X + X^3)^{(2)} = 0$, then $4a = 0$ and $a^3 = 0 = a^{(2)}$ for all $a ∈ N$.

Proof. (1) Since $6x^2 + 2x^4 = (x + x^3)^{(2)} + (-x + (-x)^3)^{(2)} = 0$ and $4x^3 = (x + x^3)^{(2)} - (-x + (-x)^3)^{(2)} = 0$, we get $8x = (4x + 4x^3)(2 + x^2) - 2(6x^2 + 2x^4)x = 0$. Further, noting that $2x + 3x^2 + 2x^3 + x^4 = (x + x^3)^{(2)} = 0$, we can easily see that $a^3 = 0 = a^{(2)}$ for all $a ∈ N$ (Lemma 2). Since $(x + x^3)^{(2)} = |(x + x^3)^{(2)} - 2(x + x^3)| = -8(x + x^3)^3 = 0$, we have $(x + x^3)^3 = 0$ (and $(x - x^2)^3 = 0$) by the above, and therefore $x^5 - x^3 = (x + x^3)^3 - (x + x^3)^3 = 0$.

(2) By the proof of (1), we obtain $8a = 0$ and $a^3 = 0 = a^{(2)}$ for all $a ∈ N$. Hence $4a = -2a^2 = a^3 = 0$. 

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Lemma 5. Let \( x \in R \). If \( 2x \in N \) and \( x^n - x^{n+1} \in N \) for some integers \( n > 0 \) and \( k \geq 0 \), then \( x - x^2 \in N \).

Proof. As is easily seen,

\[
(x-x^2)^x_n = (x^n - x^{n+2})x^x + \sum_{i=1}^{x^x} (-1)^i \binom{2^x}{i} x^{i+1} + 2x^{x^2 + n} \in N.
\]

Hence \( x - x^2 \in N \).

Lemma 6. The following are equivalent:
1) \( R \) is normal.
2) If \( e, f \in E \) and \( e - f \in N^* \), then \( e = f \).
In particular, if (*) holds in \(-E\), then \( R \) is normal.

Proof. If \( e, f \in E \), \( ef = fe \) and \( e - f \in N^* \), then \( e - f = (e - f)^3 = 0 \). Conversely, suppose 2). Let \( e \in E \) and \( x \in R \). Then \( f = e - ex(1 - e) \in E \) and \( e - f = ex(1 - e) \in N^* \). Hence we have \( ex = exe \); similarly, \( xe = exe \). This proves that \( R \) is normal. Now, let \( e, f \in E \). Then \((-e + (-e) (-f)) - (-f + (-f)(-e)) = ef + fe - e - f \). This enables us to see the latter assertion.

Corollary 1. Suppose that \( x^y - y^x \in N \cap C \) for all \( x, y \in R \). Then \( x - x^2 \in N \) for all \( x \in R \), and \( R \) is normal.

Proof. If \( x \in N \), clearly \( x - x^2 \in N \). If \( x \in R \setminus N \), then \( (x-x^2)x^3 = x^2 - (x^2)^2 \in N \). Thus \( x - x^2 \in N \) for all \( x \in R \). Now, let \( e, f \in E \) and \( e - f \in N^* \). Then \( ef + fe = e + f \) and \( ef - fe \in C \), and so \( e = e(ef + fe - f)e + e = efe = efe + |e(ef - fe) - (ef - fe)e| = -efe + ef + fe = -e + e + f = f \). Hence \( R \) is normal, by Lemma 6.

Lemma 7. Let \( R \) be a ring with 1. If \( U \subseteq E + N \), then \( 2 \in N \). If, furthermore, \( R \) is normal and for each \( x \in R \setminus U \) there exist integers \( n > 0 \) and \( k \geq 0 \) such that \( x^n - x^{n+1} \in N \), then \( x - x^2 \in N \) for all \( x \in R \).

Proof. Let \(-1 = e + a, e \in E \) and \( a \in N \). Then \(-1 + a = e = 1 \), since \(-1 + a \in U \). Hence \( 2 = -a \in N \). If \( R \) is normal, then \( u - u^2 \in N \) for any \( u \in U \). Now, the latter assertion is clear, by Lemma 5.

We are now ready to complete the proof of our first theorem.
Theorem 1. The following are equivalent:
1) $R$ satisfies $(\ast)$. 
2) $R$ is commutative, $x - x^2 \in N$ for all $x \in R$ (or $R/N$ is a Boolean ring), and $a^{(2)} = 0$ for all $a \in N$. 
3) $R$ is normal, $x - x^2 \in N$ for all $x \in R$, and $a^{(3)} = 0$ for all $a \in N$. 
4) $R$ is $(E-N)$ representable and $a^{(2)} = 0$ for all $a \in N$. 
5) $R$ is normal and satisfies the identity $(X + X^2)^{(2)} = 0$. 
6) $R$ is normal, $N$ satisfies the identity $(X + X^2)^{(2)} = 0$, and $x - x^2 \in N$ for all $x \in R$. 
7) $R$ is normal, $2R \subseteq N$, for each $x \in R$ there exist integers $n \geq 0$ and $k \geq 0$ such that $x^n - x^{n+2k} \in N$, and $a^{(2)} = 0$ for all $a \in N$. 
8) $N$ satisfies the identity $(X + X^2)^{(1)} = 0$, and $x^2y - y^2x \in N \cap C$ for all $x, y \in R \setminus N$.

Proof. By Lemma 6, (1) implies (5). 
3) $\Leftrightarrow$ 4) $\Rightarrow$ 2). By Lemma 1, and Lemmas 1 and 3, respectively. 
5) $\Leftrightarrow$ 7) $\Leftrightarrow$ 3). By Lemma 4 (1), and Lemma 5, respectively. 
8) $\Leftrightarrow$ 6) $\Rightarrow$ 3). By Corollary 1, and Lemma 4 (2), respectively. 
1) $\Leftrightarrow$ 8). We have seen that 1) implies 3) and 2). Hence $x^2y - y^2x = (x^2 - x)y - (y^2 - y)x \in N$ for all $x, y \in R$.

2) $\Leftrightarrow$ 1). Let $x, y \in R$, and put $a = x + xy, b = y + yx$. Obviously, $x + x^2 \in N$, and $(x + x^2)^{(2)} = 0$. Conversely, if $a \cdot b = 0$ then $a^2 + (a + a^2)b = a(a \cdot b) = 0$, and so $a^2 = -(a + a^2)b \in N$. This implies that $a \in N$. Hence $y \cdot x = 0 \cdot b = a^{(2)}b = a \cdot (a \cdot b) = a \cdot 0 = x + xy$, whence $y = x$ follows.

The next includes [4, Theorems 5.5, 5.6 and Corollaries 5.1, 5.3, 5.7] and improves [4, Theorems 5.14, 5.15 and Corollary 5.6].

Corollary 2. Let $R$ be a ring with 1. Then the following are equivalent:
1) $R$ satisfies $(\ast)$. 
2) $R$ is commutative, $R/N$ is a Boolean ring, and $u^2 = 1$ for all $u \in U$ (or $(1 + a)^2 = 1$ for all $a \in N$). 
3) $R$ is normal, $x - x^3 \in N$ for all $x \in R$, and $u^2 = 1$ for all $u \in U$. 
4) $R$ is $(E-N)$ representable and $u^2 = 1$ for all $u \in U$. 
5) $R$ is normal and satisfies the identity $(X + X^2)^{(3)} = 0$. 
6) $R$ is normal, $N$ satisfies the identity $(X + X^2)^{(2)} = 0$, and $x - x^2 \in N$ for all $x \in R$. 
7) $R$ is normal, $2 \in N$, and for each $x \in R$ there exist integers $n \geq 0$.
and \(k \geq 0\) such that \(x^n - x^{n+k} \in N\), and \(u^2 = 1\) for all \(u \in U\).

8) \(N\) satisfies the identity \((x + x^2)^2 = 0\), and \(x^2y - y^2x \in N \cap C\) for all \(x, y \in R \setminus N\).

9) \(R\) is normal. \(U\) satisfies the identity \((x + x^2)^2 = 0\), and \(x - x^2 \in N\) for all \(x \in R\).

10) \(R\) is normal. \(2 \in N\). and for each \(x \in R\) there exists a positive integer \(n\) such that \(x^n = x^{n+2}\) = 0.

11) \(R\) is normal. \(2 \in N\). for each \(x \in R\) there exist integers \(n > 0\) and \(k \geq 0\) such that \(x^n - x^{n+k} \in N\). and if \(u, v \in U\) and \(u-v \in N\) then \(u^2 = v^2\).

12) \(R\) is normal. \(U \subseteq E + N\). for each \(x \in R \setminus U\) there exist integers \(n > 0\) and \(k \geq 0\) such that \(x^n - x^{n+k} \in N\). and if \(u, v \in U\) and \(u-v \in N\) then \(u^2 = v^2\).

Proof. Obviously. 1) \(\Rightarrow\) 11) and 12). and the equivalence of 1) – 10) is clear by Lemma 4 (1) and Theorem 1.

11) (resp. 12)) \(\Rightarrow\) 3). By Lemma 5 (resp. Lemma 7), \(x - x^2 \in N\) for all \(x \in R\). In particular, for each \(u \in U\), we obtain \(1 - u = u^{-1}(u - u^2) \in N\), and so \(1 = u^2\).

Theorem 2. The following are equivalent:

1) \(R\) satisfies \((*)\).

2) \(2R \subseteq N\). and there exists a subset \(A\) of \(R\) containing \(N \cup (-E)\) such that \((*)\) holds in \(A\) and \(R \setminus A \subseteq E + N\).

3) \(R\) is normal. and there exists a subset \(A\) of \(R\) containing \(N\) and satisfying the identity \((x + x^2)^2 = 0\) such that \(R \setminus A \subseteq E + N\).

Proof. By Theorem 1, 1) \(\Rightarrow\) 2) and 3).

2) \(\Rightarrow\) 1). By Lemma 6. \(R\) is normal. and so \(x - x^2 \in N\) for all \(x \in R \setminus A\). Now. let \(x \in A\). Then \((x - x^2)^2 = (x + x^2)^2 - 4x^3 = -2(x + x^2 + 2x^3) \in N\). Hence \(x - x^2 \in N\) for all \(x \in R\). and therefore \(R\) satisfies \((*)\). by Theorem 1 6).

3) \(\Rightarrow\) 1). In view of Theorem 1. it suffices to show that \(x - x^2 \in N\) for all \(x \in R\). First. we consider the case that \(x \in A\). If \(-x \in A\). clearly \(x - x^2 \in N\). If \(-x \in A\). then. by the proof of Lemma 4 (1). \(8x = 0\). and so \(2x \in N\). Hence \((x - x^2)^2 = (x + x^2)^2 - 4x^3 = -2(x + x^2 + 2x^3) \in N\); \(x - x^2 \in N\). Next. we consider the case that \(x \in A\): \(a = x + x^2 \in N\). Since \(2x \in A\) forces a contradiction \(2x = 4a - (2x + 4x^2) \in N \subseteq A\). we see that \(2x \in A\). Then \((4a - 2x)^2 = (2x + 4x^2)^2 = -2(2x + 4x^2) = -2(4a\)
Let $R$ be a ring with 1. A subset $A$ of $R$ is called a weakly normal subset if for each $x \in R$, either $-x$ or $x-1$ is in $A$; a weakly normal subset $A$ of $R$ is called a normal subset if $e, f \in E$ and $e-f \in N^*$ imply $-e, -f \in A$ or $-e, -f \in A$. As is easily seen, if a weakly normal subset $A$ of $R$ satisfies the identity $(X+X^2)^{(2)} = 0$ then $R$ satisfies the same identity; if $(\ast)$ holds in a normal subset $A$ of $R$ then $R$ is normal. (Note that if $e, f \in E$, then $(-e+(-e)(-f)) \cdot (-f+(-f)(-e)) = ef+ef-e-f$ and $(e-1+(e-1)(f-1)) \cdot (f-1+(f-1)(e-1)) = ef+ef-e-f$.)

The next includes [4, Theorems 5.1, 5.2, 5.7, 5.12 and 5.13].

**Corollary 3.** Let $R$ be a ring with 1. Then the following are equivalent:

1) $R$ satisfies $(\ast)$.

2) $2 \in N$, and there exists a subset $A$ of $R$ containing $N \cup (-E)$ such that $(\ast)$ holds in $A$ and $R \setminus A \subseteq E+N$.

3) There exists a subset $A$ of $R$ containing $U \cup (-E)$ such that $(\ast)$ holds in $A$ and $R \setminus A \subseteq E+N$.

4) $R$ is normal, and there exists a subset $A$ of $R$ containing $N$ and satisfying the identity $(X+X^2)^{(2)} = 0$ such that $R \setminus A \subseteq E+N$.

5) There exists a subset $A$ of $R$ satisfying the identity $(X+X^2)^{(2)} = 0$ such that $A \supseteq N$, $(-A) \cap E \subseteq \{0, 1\}$ and every element in $R \setminus A$ is uniquely expressible as $e+a$ with $e \in E$ and $a \in N$.

6) $R$ is normal, and there exists a weakly normal subset $A$ of $R$ satisfying the identity $(X+X^2)^{(3)} = 0$.

7) There exists a normal subset $A$ of $R$ in which $(\ast)$ holds.

**Proof.** Obviously, 1) $\Rightarrow$ 2) $-7)$. By Theorem 2, each of 2) and 4) implies 1). Further, combining Corollary 2 with the remark stated just above, we readily see that each of 6) and 7) implies 1).

3) $\Rightarrow$ 1). Obviously, $8 = (1+1)^2 + 2(1+1^2) = 0$, and so $2 \in N$. Furthermore, $R$ is normal, by Lemma 6. Now, it is easy to see that $x-x^2 \in N$ for all $x \in R$. (See the proof of 2) $\Rightarrow$ 1) of Theorem 2.) Hence $R$ satisfies $(\ast)$, by Corollary 2 9).

5) $\Rightarrow$ 1). By Lemma 6 and Theorem 2.

The next proves the latter part of [2, Theorem 2] and improves [12, Theorem B].
Theorem 3. (1) If for each \( x \in R \) there exists a positive integer \( n \) such that \( x^{n+1} = x^n \), then \( N \) is an ideal of \( R \) and \( R/N \) is a Boolean ring.

(2) Let \( R \) be an \( s \)-unital ring. If for each \( x \in R \) there exists a positive integer \( n \) such that \( x^{n+1} - x^n \in C \), then \( R \) is commutative.

Proof. (1) In the complete matrix ring \( M_2(D) \) over a division ring \( D \) with \( t > 1 \), \((1 + e_{12})^{k-1} \neq (1 + e_{12})^k \) for each positive integer \( k \). Thus, in virtue of the structure theorem of primitive rings, we can easily see that any primitive homomorphic image of \( R \) is a division ring. This shows that \( R/J \) is a reduced ring, where \( J \) is the Jacobson radical of \( R \). Since \( J \) is a nil ideal, we conclude that \( J = N \) and \( R/N \) is a Boolean ring.

(2) In virtue of [6, Proposition 1], we may assume that \( R \) has 1. Let \( x \) be an arbitrary element in \( R \). Then there exists a positive integer \( n \) such that \((1 + x)^{n+1} - (1 + x)^n \in C \). Since \((1 + X)^{n+1} - (1 + X)^n = X - X^2f(X)\) with some \( f(X) \in \mathbb{Z}[X] \), \( R \) is commutative by [5, Theorem 19].

Now, we shall reprove [2, Theorem 1].

Theorem 4. The following are equivalent:

1) \( R \) is a Boolean ring.
2) \( R \) is an \( s \)-unital, \( \pi \)-regular \( B' \)-ring.
3) \( R \) is an \( s \)-unital \( B' \)-ring satisfying the identity \((X + X^2)^{\pi(2)} = 0\).
4) \( R \) is a \( cs^* \)-unital \( B' \)-ring and an \( NI \)-ring.
5) \( R \) is a \( B' \)-ring and an \( I \)-ring.
6) \( R \) is a semiprime \( I \)-ring and \( N^* \) is commutative.
7) \( R \) is a semiprime \( NI \)-ring and \( PI \) ring, and \( N^* \) is commutative.
8) \( R \) is an \( s \)-unital ring, and for each \( x \in R \) there exists a positive integer \( n \) such that \( x^{n+1} = x^n \).

Proof. Obviously, 1) implies 3), 4), 7) and 8).

3) \( \Rightarrow \) 2). By Lemma 4 (1).
4) \( \Rightarrow \) 5). By [7, Lemma 1].

5) \( \Rightarrow \) 1). Let \( a \in N^* \), and choose \( e \in E \) with \( eae = a \) Then \( e - a = (e - a)^2(e + a) \), and \( e - a \in E \), whence \( a = 0 \) follows. Hence \( N = 0 \) and \( E \) is central.

2) \( \Rightarrow \) 1). As above, we see that \( N = 0 \). Now, let \( x \in R \). Then there exists \( y \in R \) such that \( x^nyx^n = x^n \) for some \( n \). Since \( x^ny \) and \( yx^n \) are central idempotents, we obtain \( x^ny = x^n = yx^zn \). As is well-known, there exists \( z \in R \) such that \( xz = zx \) and \( x^{n+1}z = x^n \). Then \((x - x^2z)^n = \)
\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{n+i} z^i = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^n = (x-x)^n = 0, \text{ whence } x = x^2 z.
\]

This proves that \( R \) is strongly regular, and consequently Boolean.

6) \( \Rightarrow \) 1). Let \( e \in E \). Then \( (1-e)ReR(1-e)Re = (1-e)R|eR(1-e)\cdot(1-e)Re \cdot eR(1-e)|e = 0 \), whence \( (1-e)Re = 0 \) follows; similarly, \( eR(1-e) = 0 \). Hence \( E \) is central and \( R \) is Boolean.

7) \( \Rightarrow \) 6). By [9, Theorem 3], \( N = 0 \).

8) \( \Rightarrow \) 1). By Theorem 3 1), it suffices to show that \( N = 0 \). Suppose, to the contrary, that \( N \neq 0 \), and choose a non-zero \( a \) in \( N^* \). Then there exists an idempotent \( e \) such that \( ea = ae = a \). By hypothesis, there exists a positive integer \( n \) such that \( (e+a)^{n+1} = (e+a)^n \). But this forces a contradiction \( a = 0 \).

**Corollary 4** (cf. [2, Lemma 1 (3) and Theorem 2]). If \( R \) is a \( \pi \)-regular \( B^* \)-ring, then for each \( x \in R \) there exists a positive integer \( n \) such that \( x^{n+1} = x^n \).

**Proof.** There exists \( y \in R \) such that \( x^m y x^m = x^m \) for some \( m \). Then \( e' = x^m y \) is an idempotent and \( e'Re' \) is a Boolean ring by Theorem 4 2). Hence \( x^{2m} y = e' x^m e' \) is an idempotent, and so \( x^{2m} = x^{2m} y x^{2m} = (x^{2m} y)^2 x^{2m} = x^{2m} \). This proves that \( e = x^{2m} \) is in \( E \). Again by Theorem 4 2), \( eRe \) is a Boolean ring, and therefore \( x^{2m+1} = ex^2 = (exe)^2 = exe = x^{2m+1} \).

Finally, we state the following which includes [1, Theorems 1, 2 and 3]

**Theorem 5.** Let \( R \) be an \( NI \)-ring.

(1) If \( R \) is Artinian, then \( N \) is a nilpotent ideal of \( R \) and \( R/N \) is the finite direct sum of copies of \( GF(2) \).

(2) If \( R \) is a \( \pi \)-regular \( PI \) ring, then \( N \) coincides with the prime radical of \( R \) and \( R/N \) is a Boolean ring.

(3) If \( N \) is commutative, then \( N \) is a commutative ideal of \( R \) and \( R/N \) is a Boolean ring.

**Proof.** (1) As is well-known, the Jacobson radical \( J \) of \( R \) is nilpotent and \( R/J \) is a finite direct sum of matrix rings over division rings. Then, by [7, Lemma 1], \( R/J \) is a Boolean ring and \( J = N \).

(2) This is [7, Corollary 1].

(3) By [3, Theorem 2], \( N \) is a commutative ideal of \( R \). Since \( R/N \) is
a reduced $I$-ring, it is normal and Boolean.

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