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## C-Totally Real Submanifolds of $S^7(1)$ with Non-Negative Sectional Curvature

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## C-TOTALLY REAL SUBMANIFOLDS OF $S^7(1)$ WITH NON-NEGATIVE SECTIONAL CURVATURE

FRANKI DILLEN\* and LUC VRANCKEN\*

**1. Introduction.** In this paper, we study 3-dimensional, compact, minimal C-totally real (connected) submanifolds  $M$  in  $S^7(1)$ . The Sasakian structure on  $S^7$  is recalled in section 2. Fundamental formulas for such submanifolds are given in section 3; for more details, see [B], [Y-K]<sub>1</sub>, [Y-K]<sub>2</sub>.

Let  $K$  denote the sectional curvature function on  $M$ . Then Van Lindt, Verheyen and Verstraelen proved the following: if  $K > 0$ , then  $M$  is totally geodesic in  $S^7$ . The main purpose of this paper is to prove the two following results.

**Corollary 4.1.** *If  $M$  is a 3-dimensional, compact, minimal, C-totally real submanifold of  $S^7$  and if all the sectional curvatures  $K$  of  $M$  satisfy  $0 \leq K \leq 1$  then either  $K \equiv 1$  ( $M$  is totally geodesic) or  $K \equiv 0$  ( $M$  is flat).*

**Main theorem.** *Let  $x: M^3 \rightarrow S^7$  be a C-totally real, minimal immersion of a 3-dimensional compact Riemannian manifold  $M$ . If the sectional curvatures  $K$  of  $M$  satisfy  $K \geq 0$ , then either*

- (i)  *$M$  is simply connected and  $x$  is congruent to  $i: S^3 \rightarrow S^7$  (i.e.  $M$  is totally geodesic in  $S^7$ ),*
- (ii)  *$M$  is a covering of  $T^3$  with covering map  $\pi$  and  $x$  is congruent to  $j \circ \pi: M \rightarrow S^7$ ,*
- (iii)  *$M$  is a covering of  $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$  with covering map  $\pi$  and  $x$  is congruent to  $k \circ \pi: M \rightarrow S^7$ ,*

*where the immersions  $i$ ,  $j$  and  $k$  are defined in section 5.*

**2. The Sasakian structure on  $S^{2n+1}$ .** We consider the unit sphere  $S^{2n+1}$  in  $\mathbb{C}^{n+1}$ . Using the natural complex structure  $J$  of  $\mathbb{C}^{n+1}$ , we define a tangent vector field  $\xi$ , a 1-form  $\eta$ , and a (1,1) tensor field  $\phi$  on  $S^{2n+1}$  as follows:

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$\langle , \rangle$  is the induced metric from  $\mathbb{C}^{n+1}$  on  $S^{2n+1}$  (So  $S^{2n+1}$  has constant sectional curvature 1)

$\xi_p = J_p N$ , where  $N$  is the unit normal in  $p$ , for all  $p \in S^{2n+1}$

$\eta(X) = \langle X, \xi \rangle$

$\phi = s \circ J$ ,

where  $s$  denotes the orthogonal projection from  $T_p \mathbb{C}^{n+1}$  on  $T_p S^{2n+1}$ . Using these definitions, we obtain for all tangent vector fields  $X$  and  $Y$  on  $S^{2n+1}$  that

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi,$$

$$(2.2) \quad \phi\xi = 0,$$

$$(2.3) \quad \eta(\phi X) = 0,$$

$$(2.4) \quad \eta(\xi) = 1,$$

$$(2.5) \quad N(X, Y) = -2 d\eta \otimes \xi,$$

$$(2.6) \quad \langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y),$$

where  $N$  is defined by  $N(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y]$ . These formulas imply that  $(\phi, \xi, \eta, \langle , \rangle)$  determines a Sasakian structure on  $S^{2n+1}$  with constant  $\phi$ -sectional curvature 1. Therefore, we also have ([B], [Y-K]<sub>2</sub>)

$$(2.7) \quad \tilde{\nabla}_X \xi = \phi X$$

and

$$(2.8) \quad (\tilde{\nabla}_X \phi)(Y) = -\langle X, Y \rangle \xi + \eta(Y)X,$$

where  $\tilde{\nabla}$  denotes the Levi Civita connection of  $\langle , \rangle$  and  $X, Y$  are tangent vector fields on  $S^{2n+1}$ .

**3. C-totally real submanifolds of  $S^{2n+1}$ .** A Riemannian manifold  $M^m$ , isometrically immersed in  $S^{2n+1}$ , is called a C-totally real submanifold of  $S^{2n+1}$ , if  $\xi$  is a normal vector field on  $M$ . A direct consequence of this definition is that  $\phi(TM) \subset T^\perp M$  (i.e. that  $M$  is an anti-invariant submanifold of  $S^{2n+1}$ ), since for  $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} \langle \phi X, Y \rangle &= (1/2) \{ \langle \phi X, Y \rangle - \langle X, \phi Y \rangle \} \\ &= (1/2) \{ \langle \tilde{\nabla}_X \xi, Y \rangle - \langle X, \tilde{\nabla}_Y \xi \rangle \} \\ &= -(1/2) \langle \xi, [X, Y] \rangle = 0. \end{aligned}$$

In this paper we consider the case  $\dim M = n$ . We denote the Levi Civita connection of  $M$  by  $\nabla$ . The formulas of Gauss and Weingarten are then given by

$$(3.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$(3.2) \quad \tilde{\nabla}_X \alpha = -A_\alpha X + \nabla_X^\perp \alpha,$$

where  $X$  and  $Y$  are tangent vector fields on  $M$  and  $\alpha$  is a normal vector field on  $M$ . The second fundamental form  $h$  is related to  $A_\alpha$  by

$$(3.3) \quad \langle h(X, Y), \alpha \rangle = \langle A_\alpha X, Y \rangle.$$

From (3.1) and (3.2), we find that

$$(3.4) \quad \nabla_X^\perp \phi Y = \phi \nabla_X Y - \langle X, Y \rangle \xi$$

and

$$(3.5) \quad \phi h(X, Y) = -A_{\phi Y} X.$$

If we denote the curvature tensors of  $\nabla$  and  $\nabla^\perp$  by  $R$  and  $R^\perp$ , respectively, then the equations of Gauss, Codazzi and Ricci are given by

$$(3.6) \quad R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + A_{h(Y,Z)} X - A_{h(X,Z)} Y$$

$$(3.7) \quad (\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z)$$

$$(3.8) \quad \langle R^\perp(X, Y)\alpha, \beta \rangle = \langle [A_\alpha, A_\beta]X, Y \rangle$$

where  $X, Y, Z$  (respectively  $\alpha, \beta$ ) are tangent (respectively normal) vector fields on  $M$  and  $(\nabla h)$  is defined by  $(\nabla h)(X, Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$ . From (2.7), (3.2) and (3.3), it immediately follows that

$$(3.9) \quad \langle h(X, Y), \xi \rangle = 0.$$

By means of a straightforward calculation, we also obtain for tangent vector fields  $X, Y, Z$  on  $M$  that

$$(3.10) \quad \langle h(X, Y), \phi Z \rangle = \langle h(X, Z), \phi Y \rangle,$$

$$(3.11) \quad \langle (\nabla h)(X, Y, Z), \xi \rangle = -\langle h(Y, Z), \phi X \rangle.$$

Finally, we will use the following proposition from [V-V].

**Proposition 3.1.** *If  $M$  is an  $n$ -dimensional compact, minimal, C-totally*

real submanifold of  $S^{2n+1}$  and if all the sectional curvatures  $K$  of  $M$  satisfy  $K \geq 0$ , then :

$$(1) \quad (\nabla h)(u, v, w) = \langle (\nabla h)(u, v, w), \xi(p) \rangle \xi(p),$$

and

$$(2) \quad R(v, A_{\phi v} v, A_{\phi v} v, v) = 0$$

for all  $p \in M$  and  $u, v, w \in T_p M$ .

**4. 3-dimensional C-totally real submanifolds of  $S^7$ .** In this section,  $M$  will always denote a 3-dimensional, compact, minimal, C-totally real submanifold of  $S^7$ . We now choose an orthonormal basis of  $T_p M$ , at a point  $p$  of  $M$ , in the following way. We define a function  $f$  on  $UM_p$  by

$$f(u) = \langle h(u, u), \phi(u) \rangle,$$

for  $u \in UM_p$ . Let  $e_1$  be an element of  $UM_p$  in which the function  $f$  attains an absolute maximum. By using (3.10), we see that this implies that  $\langle h(e_1, e_1), \phi(u) \rangle = 0$ , for  $u \in UM_p$  with  $\langle u, e_1 \rangle = 0$ . So  $e_1$  is an eigenvector of  $A_{\phi e_1}$ . Now we consider the function  $f_1$  which is nothing else but the function  $f$  restricted to  $B$ , where  $B = \{u \in UM_p \mid \langle u, e_1 \rangle = 0\}$ . If  $f_1$  is constant, then we choose  $e_2$  and  $e_3$  as eigenvectors of  $A_{\phi e_1}$ . If  $f_1$  is not constant, we choose  $e_2$  as an element of  $B$  in which the function  $f_1$  attains its absolute maximum. Therefore  $\langle h(e_2, e_2), \phi(u) \rangle = 0$ , where  $u$  is orthogonal to both  $e_1$  and  $e_2$ . Finally we take  $e_3$  orthonormal to  $e_1$  and  $e_2$ . Since, we can still change the sign of  $e_3$ , we also may assume that  $\langle h(e_1, e_2), \phi e_3 \rangle \geq 0$ . Using this, (3.9), (3.10) and the minimality of  $M$ , we obtain the following expressions for the second fundamental form  $h$  at the point  $p$ .

$$\begin{aligned} h(e_1, e_1) &= a\phi e_1, \\ h(e_2, e_2) &= b\phi e_1 + d\phi e_2, \\ h(e_3, e_3) &= -(a+b)\phi e_1 - d\phi e_2, \\ h(e_1, e_2) &= b\phi e_2 + c\phi e_3, \\ h(e_1, e_3) &= -(a+b)\phi e_3 + c\phi e_2, \\ h(e_2, e_3) &= c\phi e_1 - d\phi e_3, \end{aligned}$$

where  $a \geq d \geq 0$ ,  $c \geq 0$  and  $b \in \mathbb{R}$ . Since  $\{e_1, e_2, e_3\}$  is an orthonormal basis of  $T_p M$ , any vector  $v \in T_p M$  can be written as  $v = v_1 e_1 + v_2 e_2 + v_3 e_3$ , where  $v_1, v_2, v_3 \in \mathbb{R}$ . Therefore, using the Gauss equation, we find

after a very long but straightforward computation\* that the second condition of Proposition 3.1 is equivalent to the following equations in  $a$ ,  $b$ ,  $c$  and  $d$ .

- (4.1)  $-2a^4 - 7a^3b - 9a^2b^2 - a^2c^2 - 2a^2d^2 + a^2 - 5ab^3 - 2abc^2 - 7abd^2 + 2ab - b^4 - b^2c^2 - 5b^2d^2 + b^2 - c^2d^2 - 2d^4 + d^2 = 0,$
- (4.2)  $c(3a^3 + 7a^2b + 5ab^2 + ac^2 - a + b^3 + bc^2 + bd^2 - b) = 0,$
- (4.3)  $5a^3b + 11a^2b^2 - 25a^2c^2 + 6a^2d^2 + a^2 + 7ab^3 - 28abc^2 + 26abd^2 + b^4 - 3b^2c^2 + 22b^2d^2 - b^2 - 4c^4 + 14c^2d^2 + 4c^2 + 12d^4 - 6d^2 = 0,$
- (4.4)  $c(-5a^2b - 5ab^2 + 5ac^2 - a - 2bd^2) = 0,$
- (4.5)  $-4a^2b^2 - 3ab^3 + 22abc^2 - 15abd^2 - 2ab + b^4 - 3b^2c^2 - 21b^2d^2 - b^2 - 4c^4 - 33c^2d^2 + 4c^2 - 18d^4 + 9d^2 = 0,$
- (4.6)  $bc(2ab - b^2 - c^2 - 3d^2 + 1) = 0,$
- (4.7)  $b^2(ab - b^2 - c^2 + 1) = 0,$
- (4.8)  $cd(3ab + 3b^2 + c^2 + 3d^2 - 1) = 0,$
- (4.9)  $d(6a^3 + 21a^2b + 30ab^2 + 9ac^2 + 6ad^2 - 5a + 15b^3 + 3bc^2 + 12bd^2 - 7b) = 0,$
- (4.10)  $cd(-6a^2 - 15ab - 15b^2 - 11c^2 - 12d^2 + 7) = 0,$
- (4.11)  $d(-11a^2b - 31ab^2 - 11ac^2 - 18ad^2 + 7a - 24b^3 - 32bc^2 - 36bd^2 + 16b) = 0,$
- (4.12)  $cd(2ab + 2b^2 + 8c^2 + 9d^2 - 4) = 0,$
- (4.13)  $bd(-ab + b^2 + 5c^2 - 1) = 0,$
- (4.14)  $12a^4 + 38a^3b + 44a^2b^2 + 14a^2c^2 + 6a^2d^2 - 6a^2 + 22ab^3 + 14abc^2 + 17abd^2 - 10ab + 4b^4 + 7b^2d^2 - 4b^2 - 4c^4 - 25c^2d^2 + 4c^2 + d^2 = 0.$
- (4.15)  $c(-12a^3 - 23a^2b - 19ab^2 - 11ac^2 - 6ad^2 + 7a - 8b^3 - 8bc^2 - 20bd^2 + 8b) = 0,$
- (4.16)  $-9a^3b - 21a^2b^2 + 27a^2c^2 - 21a^2d^2 + a^2 - 24ab^3 - 63abd^2 + 12ab - 12b^4 - 45b^2d^2 + 12b^2 + 12c^4 + 27c^2d^2 - 12c^2 + d^2 = 0,$
- (4.17)  $c(9a^2b + 5ab^2 - 3ac^2 + 6ad^2 - a + 8b^3 + 8bc^2 + 28bd^2 - 8b) = 0.$

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\*For this computation, the authors used a computer.

$$(4.18) \quad 2a^2b^2 - 6ab^3 - 14abc^2 + abd^2 + 2ab + 4b^4 - b^2d^2 - 4b^2 - 4c^2 - 17c^2d^2 + 4c^2 + d^2 = 0,$$

$$(4.19) \quad cd(-7ab - 3b^2 + 5c^2 - 1) = 0,$$

$$(4.20) \quad d(-18a^3 - 53a^2b - 61ab^2 - 11ac^2 + 7a - 18b^3 + 30bc^2 + 2b) = 0,$$

$$(4.21) \quad cd(6a^2 + 17ab + 21b^2 - 3c^2 - 1) = 0,$$

$$(4.22) \quad d(-a^2b + 3ab^2 + 5ac^2 - a - 2b^3 - 18bc^2 + 2b) = 0,$$

$$(4.23) \quad -18a^4 - 51a^3b - 53a^2b^2 - 33a^2c^2 + 9a^2 - 24ab^3 - 24abc^2 + 12ab - 4b^4 - 8b^2c^2 + 4b^2 - 4c^4 + 4c^2 = 0,$$

$$(4.24) \quad ac(9a^2 + 8ab + 8b^2 + 8c^2 - 4) = 0,$$

$$(4.25) \quad a^3b - 5a^2b^2 - 17a^2c^2 + a^2 + 8ab^3 + 8abc^2 - 4ab - 4b^4 - 8b^2c^2 + 4b^2 - 4c^4 + 4c^2 = 0.$$

In order to solve these equations, we consider the following cases :

*case 1:*  $a \neq 0, b \neq 0, c \neq 0, d \neq 0$ . A contradiction follows if we compare (4.7) with (4.13).

*case 2:*  $a \neq 0, b \neq 0, c \neq 0, d = 0$ . In this case,  $f_1$  is identically zero. Therefore, we took  $e_2$  as an eigenvector of  $A_{\neq e_1}$ . This implies that  $c = 0$ .

*case 3:*  $a \neq 0, b \neq 0, c = 0$ . Then the equation (4.7) becomes

$$(4.26) \quad a = (b^2 - 1)/b.$$

Substituting this value for  $a$  in (4.23) gives

$$b^2 = 1/3 \text{ or } b^2 = 3/5.$$

Using the fact that  $a \geq 0$ , we deduce from (4.26) that either

$$b = -1/\sqrt{3} \text{ and } a = 2/\sqrt{3},$$

or

$$b = -\sqrt{3}/\sqrt{5} \text{ and } a = 2/(\sqrt{3}\sqrt{5}).$$

So, we have to consider two subcases.

*subcase 3a:*  $a = 2/\sqrt{3}, b = -1/\sqrt{3}, c = 0$ . From (4.1) we deduce that either  $d = 0$  or  $d = \sqrt{2}/\sqrt{3}$ . Then, after a straightforward calculation one sees that in both cases all the other equations are also satisfied.

*subcase 3b:*  $a = 2/(\sqrt{3}\sqrt{5}), b = -3/\sqrt{5}, c = 0$ . Here, it follows

from (4.16) that  $d = 2/\sqrt{15}$ . If we put  $u = (\sqrt{2}/2)(e_2 - e_1)$ , we see that  $\langle h(u, u), \phi(u) \rangle = (\sqrt{2}/4)(3\sqrt{3}/\sqrt{5}) > 2/\sqrt{15}$ . This implies that  $e_1$  is not an absolute maximum.

case 4:  $a \neq 0, b = 0, c = 0$ . This is not possible, since (4.25) implies  $a = 0$ .

case 5:  $a \neq 0, b = 0, c \neq 0, d = 0$ . Applying the same argument as in case 2, we obtain a contradiction.

case 6:  $a \neq 0, b = 0, c \neq 0, d \neq 0$ . First, we deduce from (4.22) that  $c^2 = 1/5$ . Then, it follows from (4.21) that  $a = 2/\sqrt{15}$ . From (4.8) we then find that  $d = 2/\sqrt{15}$ . Now, we put  $u = -1/\sqrt{5} |e_1 + e_2 - \sqrt{3}e_3|$ . Then we notice that  $\langle h(u, u), \phi(u) \rangle = (-1/5\sqrt{5})(-50/\sqrt{15}) > 2/\sqrt{15}$ .

case 7:  $a = 0$ . This implies that  $f$  is identically zero. By linearization, we then deduce that also  $b = c = d = 0$ .

By combining this with Proposition 3.1, we immediately obtain the following lemma.

**Lemma 4.1.** *If  $M$  is a 3-dimensional, compact, minimal, C-totally real submanifold of  $S^7$  and if all the sectional curvatures  $K$  of  $M$  satisfy  $K \geq 0$ , then, for each point  $p$  of  $M$ , there exists an orthonormal basis  $\{e_1, e_2, e_3\}$  of  $T_pM$  such that either*

$$(4.27) \quad (i) \quad \begin{aligned} h(e_1, e_1) &= h(e_2, e_2) = h(e_3, e_3) = 0, \\ h(e_1, e_2) &= h(e_1, e_3) = h(e_2, e_3) = 0. \end{aligned}$$

$$(4.28) \quad (ii) \quad \begin{aligned} h(e_1, e_1) &= (2/\sqrt{3})\phi e_1, \\ h(e_2, e_2) &= -(1/\sqrt{3})\phi e_1 + (\sqrt{2}/\sqrt{3})\phi e_2, \\ h(e_3, e_3) &= -(1/\sqrt{3})\phi e_1 - (\sqrt{2}/\sqrt{3})\phi e_2, \\ h(e_1, e_2) &= -(1/\sqrt{3})\phi e_2, \\ h(e_1, e_3) &= -(1/\sqrt{3})\phi e_3, \\ h(e_2, e_3) &= -(\sqrt{2}/\sqrt{3})\phi e_3. \end{aligned}$$

or

$$(4.29) \quad (iii) \quad \begin{aligned} h(e_1, e_1) &= (2/\sqrt{3})\phi e_1, \\ h(e_2, e_2) &= h(e_3, e_3) = -(1/\sqrt{3})\phi e_1, \\ h(e_1, e_3) &= -(1/\sqrt{3})\phi e_3, \\ h(e_1, e_2) &= -(1/\sqrt{3})\phi e_2. \end{aligned}$$



$$h(e_2, e_3) = 0.$$

Let  $M$  be like in Lemma 4.1. Then we have the following proposition.

**Proposition 4.1.** *Let  $p \in M$ . Then :*

- (a) *if (4.27) holds,  $K(p) \equiv 1$ ,*
- (b) *if (4.28) holds,  $K(p) \equiv 0$ ,*
- (c) *if (4.29) holds, then  $0 \leq K(p) \leq 4/3$ , where  $K(p) = 0$  for every plane through  $e_1$  and  $K(p) = 4/3$  only for the plane through  $e_2$  and  $e_3$ .*

*Proof.*

(a) In this case,  $h \equiv 0$ , so  $p$  is a totally geodesic point. From the Gauss equation we obtain that  $K(p) \equiv 1$ .

(b) From the Gauss equation and (4.28) we find that

$$\begin{aligned} R(e_1, e_2)e_2 &= R(e_1, e_3)e_3 = R(e_2, e_3)e_3 = 0 \\ R(e_1, e_2)e_3 &= R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0. \end{aligned}$$

Linearization then implies that  $R_p \equiv 0$  and therefore  $K(p) \equiv 0$ .

(c) From the Gauss equation and (4.29), we obtain that

$$\begin{aligned} R(e_1, e_2)e_3 &= R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0, \\ R(e_1, e_2)e_2 &= R(e_1, e_3)e_3 = 0, \\ R(e_2, e_3)e_3 &= (4/3)e_2. \end{aligned}$$

Let  $\sigma$  be any plane section of  $T_pM$ . Then we can find an orthonormal basis  $\{X, Y\}$  of  $\sigma$  such that  $X = \cos \theta e_2 + \sin \theta e_3$  and  $Y = \sin \varphi e_1 - \cos \varphi \sin \theta e_2 + \cos \varphi \cos \theta e_3$  where  $\theta, \varphi \in \mathbb{R}$ . Then

$$\begin{aligned} R(X, Y, Y, X) &= \cos^2 \theta R(e_2, Y, Y, e_2) + 2 \cos \theta \sin \theta R(e_2, Y, Y, e_3) \\ &\quad + \sin^2 \theta R(e_3, Y, Y, e_3) \\ &= (4/3) \cos^4 \theta \cos^2 \varphi + (8/3) \cos^2 \varphi \cos^2 \theta \sin^2 \theta \\ &\quad + (4/3) \sin^4 \theta \cos^2 \varphi = (4/3) \cos^2 \varphi. \end{aligned}$$

Let  $M$  be as in Lemma 4.1. The connectedness of  $M$  and Proposition 4.1 imply that if  $M$  satisfies condition (4.27) (respectively (4.28), (4.29)) in a point  $p$  of  $M$ , then  $M$  satisfies condition (4.27) (respectively (4.28), (4.29)) in every point of  $M$ . The next statement follows then easily from Proposition 4.1.

**Corollary 4.1.** *If  $M$  is a 3-dimensional, compact, minimal, C-totally*

real submanifold of  $S^7$  and if all the sectional curvatures  $K$  of  $M$  satisfy either  $0 \leq K \leq 1$  or  $0 \leq K < 4/3$ , then either  $K \equiv 1$  ( $M$  is totally geodesic) or  $K \equiv 0$  ( $M$  is flat) on  $M$ .

In the following proposition, we study more closely the case (c) of Proposition 4.1.

**Proposition 4.2.** *Let  $M$  be a 3-dimensional, compact, minimal, C-totally real submanifold of  $S^7$  with  $K$  not constant and satisfying  $K \geq 0$ . Then there exists globally a unique tangent vector field  $E_1$  and locally tangent vector fields  $E_2$  and  $E_3$  such that*

- (a)  $\{E_1, E_2, E_3\}$  is a local orthonormal frame,
- (b) For any  $p \in M$ ,  $f$  attains its maximum value at  $E_1(p)$ ,
- (c)  $h(E_1, E_1) = (2/\sqrt{3})\phi E_1$ ,  
 $h(E_2, E_2) = h(E_3, E_3) = -(1/\sqrt{3})\phi E_1$ ,  
 $h(E_1, E_3) = -(1/\sqrt{3})\phi E_3$ ,  
 $h(E_1, E_2) = -(1/\sqrt{3})\phi E_2$ ,  
 $h(E_2, E_3) = 0$ .
- (d)  $\nabla_{E_1} E_1 = \nabla_{E_2} E_1 = \nabla_{E_3} E_1 = 0$ .

*Proof.* Since  $K \geq 0$  and  $K$  is not constant, it immediately follows from Proposition 4.1 and Lemma 4.1 that the vector field  $E_1(p)$ , where  $E_1(p)$  is the unique point at which  $f$  attains its maximum at  $p$ , is well defined on the whole of  $M$  and is also differentiable. This proves (a), (b) and (c). In order to prove (d), we use the first part of Proposition 3.1. From

$$(\nabla h)(E_1, E_1, E_1) \equiv 0 \pmod{\xi},$$

we obtain

$$\nabla_{E_1} E_1 = 0.$$

The other claims follow then from

$$(\nabla h)(E_2, E_1, E_1) \equiv 0 \pmod{\xi} \text{ and } (\nabla h)(E_3, E_1, E_1) \equiv 0 \pmod{\xi}.$$

**Theorem 4.1.** *Let  $M$  be a 3-dimensional, compact, minimal, C-totally real submanifold of  $S^7$  with sectional curvatures  $K$  satisfying  $K \geq 0$ . Let  $\tilde{M}$  be the universal covering of  $M$ . Then either*

$$\tilde{M} \cong S^3(1),$$

or

$$\tilde{M} \cong \mathbb{R}^3$$

or

$$\tilde{M} \cong \mathbb{R} \times S^2(\sqrt{3}/2),$$

where  $S^m(r)$  denotes the  $m$ -dimensional sphere with radius  $r$ .

*Proof.* If the sectional curvature  $K$  of  $M$  is constant, we immediately obtain that  $\tilde{M} \cong \mathbb{R}^3$  or  $\tilde{M} \cong S^3(1)$ . Therefore, we may assume that  $K$  is not constant. Then, we know from Proposition 4.2 that there exists a globally defined vector field  $E_1$  on  $M$ . Now we define 2 distributions  $T_0$  and  $T_1$  on  $M$  by

$$\begin{aligned} T_0 : p &\mapsto T_0|_p = \text{vect}\{E_1(p)\}, \\ T_1 : p &\mapsto T_1|_p = \{v \in T_p M \mid \langle v, e_1 \rangle = 0\}. \end{aligned}$$

From (d) of Proposition 4.2 we know that the distributions  $T_0$  and  $T_1$  satisfy  $\nabla_{T_0} T_0 \subset T_0$ ,  $\nabla_{T_1} T_0 \subset T_0$ . From this we easily deduce that also  $\nabla_{T_0} T_1 \subset T_1$  and  $\nabla_{T_1} T_1 \subset T_1$ . Let  $\tilde{T}_0, \tilde{T}_1$  denote the corresponding distributions on  $\tilde{M}$ . Clearly  $\tilde{T}_0$  and  $\tilde{T}_1$  also satisfy  $\nabla_{\tilde{T}_0} \tilde{T}_0 \subset \tilde{T}_0$ ,  $\nabla_{\tilde{T}_1} \tilde{T}_1 \subset \tilde{T}_1$ ,  $\nabla_{\tilde{T}_0} \tilde{T}_1 \subset \tilde{T}_1$  and  $\nabla_{\tilde{T}_1} \tilde{T}_0 \subset \tilde{T}_0$ . Then, we know from the de Rham decomposition theorem that

$$\tilde{M} \cong \mathbb{R} \times S^2(\sqrt{3}/2).$$

This completes the proof.

From the proof of Theorem 4.1, we see that we can choose the orientation of  $\mathbb{R} \times S^2(\sqrt{3}/2)$  such that the vectorfield  $\tilde{E}_1$  on  $\mathbb{R} \times S^2(\sqrt{3}/2)$  is given by

$$(4.30) \quad \tilde{E}_1(\alpha, \beta) = (1, 0)_{\alpha, \beta}.$$

In the next section, we will also need the following proposition about flat,  $C$ -totally real submanifolds.

**Proposition 4.3.** *Let  $M$  be a 3-dimensional, flat, compact, minimal  $C$ -totally real submanifold of  $S^7$ . Then there exists globally an orthonormal frame  $\{E_1, E_2, E_3\}$  such that*

- (a) *For any  $p \in M$ ,  $f$  attains its maximum value at  $E_1(p)$  and  $f_1$*

attains its maximum value at  $E_2(p)$  ;

- (b)  $h(E_1, E_1) = (2/\sqrt{3})\phi E_1,$   
 $h(E_2, E_2) = -(1/\sqrt{3})\phi E_1 + (\sqrt{2}/\sqrt{3})\phi E_2,$   
 $h(E_3, E_3) = -(1/\sqrt{3})\phi E_1 - (\sqrt{2}/\sqrt{3})\phi E_2,$   
 $h(E_1, E_2) = -(1/\sqrt{3})\phi E_2,$   
 $h(E_1, E_3) = -(1/\sqrt{3})\phi E_3,$   
 $h(E_2, E_3) = -(\sqrt{2}/\sqrt{3})\phi E_3$
- (c)  $\nabla_{E_i} E_j = 0$  for  $i, j \in \{1, 2, 3\}.$

Furthermore, we can choose Euclidean coordinates on  $\mathbb{R}^3$ , the universal covering of  $M$ , such that the corresponding orthonormal frame  $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$  on  $\mathbb{R}^3$  has the following form :

$$\begin{aligned} \tilde{E}_1(x, y, z) &= (1, 0, 0)_{(x,y,z)}, \\ \tilde{E}_2(x, y, z) &= (0, 1, 0)_{(x,y,z)}, \\ \tilde{E}_3(x, y, z) &= (0, 0, 1)_{(x,y,z)}. \end{aligned}$$

*Proof.* Let  $p \in M$ . First, we notice that for every basis, chosen as in the beginning of this section (4. 28) holds.

Next, we can write  $v \in UM_p$  as  $v = \alpha e_1 + \beta e_2 + \gamma e_3$ , where  $\alpha^2 + \beta^2 + \gamma^2 = 1$ . A straightforward computation then shows that

$$\begin{aligned} h(v, v) &= ((2/\sqrt{3})\alpha^2 - (1/\sqrt{3})\beta^2 - (1/\sqrt{3})\gamma^2)\phi e_1 \\ &\quad + ((\sqrt{2}/\sqrt{3})\beta^2 - (\sqrt{2}/\sqrt{3})\gamma^2 - (2/\sqrt{3})\alpha\beta)\phi e_2 \\ &\quad + (-(2/\sqrt{3})\alpha\gamma - (2\sqrt{2}/\sqrt{3})\beta\gamma)\phi e_3 \end{aligned}$$

and

$$\begin{aligned} \|h(v, v)\|^2 &= (4/3)\alpha^4 + \beta^4 + \gamma^4 + 2\beta^2\gamma^2 + 4\sqrt{2}\alpha\beta\gamma^2 - (4\sqrt{2}/3)\alpha\beta^3 \\ &= (4/3)(\alpha^2 + \beta^2 + \gamma^2)^2 - (1/3)(\beta^4 + \gamma^4 + 2\beta^2\gamma^2) \\ &\quad - (1/3)(4\sqrt{2}\alpha(\beta^3 - 3\beta\gamma^2)) - (1/3)(8\alpha^2(\beta^2 + \gamma^2)). \end{aligned}$$

Then we define

$$\begin{aligned} (4. 31) \quad h(\alpha, \beta, \gamma) &= 8\alpha^2(\beta^2 + \gamma^2) + 4\sqrt{2}\alpha(\beta^3 - 3\beta\gamma^2) + (\beta^4 + \gamma^4 + 2\beta^2\gamma^2), \\ g(\beta, \gamma) &= -8\gamma^2(3\beta^2 - \gamma^2)^2. \end{aligned}$$

Now, we look at  $h$  as a quadratic function in  $\alpha$  with discriminant  $g$ . This implies that  $h(\alpha, \beta, \gamma) \geq 0$  and  $h(\alpha, \beta, \gamma) = 0$  only if either

$$(4. 32) \quad \gamma = 0 \text{ and } h(\alpha, \beta, 0) = 0$$

or

$$(4.33) \quad \gamma^2 = 3\beta^2 \text{ and } h(\alpha, \beta, \pm\sqrt{3}\beta) = 0.$$

From (4.31) it is immediately clear that (4.32) is equivalent with

$$(4.34) \quad \begin{aligned} \gamma = 0; \beta = 0; \alpha = \pm 1 \\ \gamma = 0; \beta = -2\sqrt{2}\alpha; \alpha = \pm 1/3 \end{aligned}$$

and (4.33) is equivalent with either (4.34) or

$$\alpha = \pm 1/3; \beta = \sqrt{2}\alpha; \gamma = \pm\sqrt{6}\alpha.$$

This implies that  $\|h(v, v)\|^2 \leq 4/3$  and  $\|h(v, v)\|^2 = 4/3$  if and only if either

$$v = \pm e_1,$$

or

$$v = \pm((1/3)e_1 + (\sqrt{2}/3)e_2 \pm (\sqrt{6}/3)e_3),$$

or

$$v = \pm((1/3)e_1 - (2\sqrt{2}/3)e_2).$$

From this, we see that  $e_1, -((1/3)e_1 + (\sqrt{2}/3)e_2 \pm (\sqrt{6}/3)e_3), -(1/3)e_1 + (2\sqrt{2}/3)e_2$  are the unique vectors in which  $f$  attains its maximal value. Therefore, we can define a differentiable vectorfield  $E_1$  on the whole of  $M$  such that  $f$  attains an absolute maximum in  $E_1(p)$ . As a matter of fact, we have 4 possible choices for  $E_1$ .

Now, we put  $u = \cos \theta e_2 + \sin \theta e_3$ . Then

$$\begin{aligned} \langle h(u, u), \phi u \rangle &= (\sqrt{2}/\sqrt{3}) \cos^3 \theta - (3\sqrt{2}/\sqrt{3}) \sin^2 \theta \cos \theta \\ &= (\sqrt{2}/\sqrt{3}) \cos \theta \{ \cos^2 \theta - 3 \sin^2 \theta \} \\ &= (\sqrt{2}/\sqrt{3}) \cos \theta \{ 4 \cos^2 \theta - 3 \}. \end{aligned}$$

From this, we see that  $\langle h(u, u), \phi(u) \rangle$  is maximal if and only if either

$$u = e_2 \text{ or } u = -(1/2)e_2 \pm (\sqrt{3}/2)e_3.$$

This proves (a) and (b). In order to prove (c), we use the first part of Proposition 3.1. From

$$(\nabla h)(E_i, E_1, E_1) \equiv (\nabla h)(E_i, E_2, E_2) \equiv 0 \pmod{\xi},$$

we see that  $\nabla_{E_i} E_1 = \nabla_{E_i} E_2 = \nabla_{E_i} E_3 = 0$ .

The rest of the proof follows then immediately from the de Rham decomposition theorem.

**5. Some examples.** In this section, we give three examples of C-totally real, minimal, 3-dimensional submanifolds of  $S^7$  satisfying  $K \geq 0$ . Using the results of section 4, we then prove that these examples are basically the only ones.

**Example 5.1.** If  $i$  denotes the inclusion map of

$$S^3 = \{x \in S^7(1) \mid x = x_1 e_1 + x_3 e_3 + x_5 e_5 + x_7 e_7\},$$

where  $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$  is the standard basis of  $\mathbb{R}^8$ , then  $i: S^3 \rightarrow S^7$  is a C-totally real, totally geodesic embedding.

**Example 5.2.**[Y-K]<sub>1</sub> Let  $\tilde{j}: \mathbb{R}^3 \rightarrow S^7$  be a minimal immersion represented by

$$\begin{aligned} j(u_1, u_2, u_3) &= (1/2)(\cos u_1, \sin u_1, \cos u_2, \sin u_2, \cos u_3, \sin u_3, \cos u_4, \sin u_4), \end{aligned}$$

where  $u_4 = -(u_1 + u_2 + u_3)$ . Then  $\tilde{j}(\mathbb{R}^3)$  is a compact, flat minimal, C-totally real submanifold of  $S^7$ . We will denote  $\tilde{j}(\mathbb{R}^3) = T^3$  and the embedding from  $T^3$  into  $S^7$  by  $j$ .

**Example 5.3.** Let  $\tilde{k}: \mathbb{R} \times S^2(\sqrt{3}/2) \rightarrow S^7: (u, y_1, y_2, y_3) \mapsto (x_1, x_2, \dots, x_8)$ , where  $y_1^2 + y_2^2 + y_3^2 = 3/4$  and

$$\begin{cases} x_1 = (1/4)(\cos \sqrt{3} u + 2\sqrt{3} y_3 \cos(1/\sqrt{3}) u), \\ x_2 = (1/4)(\sin \sqrt{3} u - 2\sqrt{3} y_3 \sin(1/\sqrt{3}) u), \\ x_3 = (\sqrt{3}/4)(\sin \sqrt{3} u + (2/\sqrt{3}) y_3 \sin(1/\sqrt{3}) u), \\ x_4 = (\sqrt{3}/4)(-\cos \sqrt{3} u + (2/\sqrt{3}) y_3 \cos(1/\sqrt{3}) u), \\ x_5 = y_2 \cos(1/\sqrt{3}) u, \\ x_6 = -y_2 \sin(1/\sqrt{3}) u, \\ x_7 = y_1 \cos(1/\sqrt{3}) u, \\ x_8 = -y_1 \sin(1/\sqrt{3}) u. \end{cases}$$

Furthermore, we consider the standard product metric on  $\mathbb{R} \times S^2(\sqrt{3}/2)$ . Then,  $k$  is a C-totally real, isometric, minimal immersion from  $\mathbb{R} \times S^2(\sqrt{3}/2)$  into  $S^7$ . Furthermore, we see that

$$\tilde{k}(u, y_1, y_2, y_3) = \tilde{k}(u', y'_1, y'_2, y'_3)$$

if and only if

$$u \equiv u' \pmod{2\pi\sqrt{3}}$$

and

$$y_i = y'_i \quad \text{for } i = 1, 2, 3.$$

Therefore, we obtain an embedding  $k$  from  $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$  into  $S^7(1)$ .

**Main theorem.** *Let  $x: M^3 \rightarrow S^7$  be a  $C$ -totally real, isometric, minimal immersion of a 3-dimensional compact Riemannian manifold  $M$ . If the sectional curvatures  $K$  of  $M$  satisfy  $K \geq 0$ , then either*

- (i)  *$M$  is simply connected and  $x$  is congruent to  $i: S^3 \rightarrow S^7$  (i.e.  $M$  is totally geodesic in  $S^7$ )*

or

- (ii)  *$M$  is a covering of  $T^3$  with covering map  $\pi$  and  $x$  is congruent to  $j \circ \pi: M \rightarrow S^7$ .*

or

- (iii)  *$M$  is a covering of  $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$  with covering map  $\pi$  and  $x$  is congruent to  $k \circ \pi: M \rightarrow S^7$ .*

*Proof.* By Theorem 4.1 we only have to consider the 3 following cases.

*case 1:* The universal covering of  $M$  is  $S^3$ . In this case  $M$  is totally geodesic in  $S^7$ . This completes the proof in this case.

*case 2:* The universal covering of  $M$  is  $\mathbb{R}^3$ . Then, we have the following diagram :

$$\begin{array}{ccc}
 & & M^3 \xrightarrow{x} S^7 \\
 & \nearrow \pi_1 & \\
 \mathbb{R}^3 & & \\
 & \searrow \pi_2 & \\
 & & T^3 \xrightarrow{j} S^7.
 \end{array}$$

Take a point  $p \in \mathbb{R}^3$  and a small neighbourhood of  $p$  such that  $\pi_1|_U$  and  $\pi_2|_U$  are injective. So, we obtain an isometry  $\pi: \pi_1(U) \rightarrow \pi_2(U)$ . By Proposition 4.3, we may assume that  $\pi_*(E_i) = E_i^*$ , where  $i \in \{1, 2, 3\}$  and  $\{E_1, E_2, E_3\}$  (respectively  $\{E_1^*, E_2^*, E_3^*\}$ ) is chosen on  $M$  (resp.  $T^3$ ) as is indicated in Proposition 4.3. Then, we define a map  $\tilde{\pi}: N_{\pi_1}(U) \rightarrow N_{\pi_2}(U)$  by

$$\tilde{\pi}(\phi X) = \phi \pi_* X \text{ and } \tilde{\pi}(\xi) = \xi \circ \pi.$$

Using the fact that both  $x$  and  $j$  are  $C$ -totally real immersions, we see that we can apply the rigidity theorem ([Sp], p. 72–80). Therefore, the following diagram commutes :

$$\begin{array}{ccccc}
 & & M & \xrightarrow{x} & S^7 \\
 & \nearrow \pi_1 & & & \downarrow A \\
 U & & \odot & & \\
 & \searrow \pi_2 & T & \xrightarrow{j} & S^7
 \end{array}$$

where  $A$  is an isometry of  $S^7$ . Since minimal immersions from an analytic manifold in spheres are analytic, this implies

$$\begin{array}{ccccc}
 & & M & \xrightarrow{x} & S^7 \\
 & \nearrow \pi_1 & & & \downarrow A \\
 \mathbb{R}^3 & & \odot & & \\
 & \searrow \pi_2 & T & \xrightarrow{j} & S^7
 \end{array}$$

In particular, since  $j$  is an embedding, we obtain that the following map

$$j^{-1} \circ A \circ x : M \rightarrow T$$

is a covering map of  $T$ .

*case 3:* The universal covering of  $M$  is  $\mathbb{R} \times S^2(\sqrt{3}/2)$ . By using Proposition 4.2 and (4.30), we obtain in the same way as in the previous case that the following diagram commutes :

$$\begin{array}{ccccc}
 & & M & \xrightarrow{x} & S^7 \\
 & \nearrow \pi_1 & & & \searrow A \\
 \mathbb{R} \times S^2(\sqrt{3}/2) & & \odot & & \\
 & \searrow \pi_2 & S^1(\sqrt{3}) \times S^2(\sqrt{3}/2) & \xrightarrow{k} & S^7
 \end{array}$$

where  $A$  is an isometry. Since  $k$  is an embedding, this implies that  $M$  is a covering of  $S^1(\sqrt{3}) \times S^2(\sqrt{3}/2)$  with covering map  $k^{-1} \circ A \circ x$ .

**Remark.** The condition that  $M$  is minimal may everywhere be replaced by the condition that the mean curvature vector  $H$  of  $M$  is parallel, since we have the following proposition [Y-K]<sub>2</sub>.

**Proposition 5.1.** If  $M$  is an  $n$ -dimensional  $C$ -totally real submanifold



of  $S^{2n+1}$ , then the following conditions are equivalent

- (i)  $M$  is minimal
- (ii) the mean curvature vector  $H$  of  $M$  is parallel.

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