A NOTE ON CERTAIN METRICS ON $\mathbb{R}^4_+$

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This work is a continuation of the papers [1], [2], [3] and [4], in which we studied the metrics on $R^4_+ = R^3 \times R_+$ with the canonical coordinates \{x_1, x_2, x_3, x_4\} as

\[
(1.1) \quad ds^2 = \frac{1}{x_4 x_4} \sum_{b,c=1}^{3} \left( \delta_{bc} - \frac{ax_b x_c}{1 + ar^2} \right) dx_b dx_c - \frac{1}{x_4 x_4 (1 + ax_4 x_4)} dx_4 dx_4
\]

and

\[
(1.2) \quad ds^2 = \frac{1}{x_4 x_4} \sum_{b,c=1}^{3} \left( \frac{8}{(x_3 + 3r)^2} \left( r^2 \delta_{bc} - x_b x_c \right) + \frac{x_b x_c}{r^2 (1 + ar^2)} \right) dx_b dx_c
\]

\[- \frac{1}{x_4 x_4 (1 + ax_4 x_4)} dx_4 dx_4,
\]

where $r^2 = \sum_{b=1}^{3} x_b x_b$ and $a$ is a constant.

They are derived as special ones from the metric on $R^4_+$:

\[
ds^2 = \frac{1}{u_4 u_4} \sum_{i,j=1}^{4} F_{ij} du_i du_j, \quad F_{ij} = F_{ji},
\]

where $u_1 = r, u_2 = \theta, u_3 = \phi, u_4 = x_4$ and

\[
x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta,
\]

and $(r, \theta, \phi)$ are the polar coordinates of $R^3$, which satisfies the Einstein condition and

\[
F_{ij} = F_{ij}(u_1, u_2) \quad \text{except for} \quad F_{44} = F_{44}(u_1, u_2, u_4)
\]

and

\[
F_{12} = F_{a\lambda} = 0 \quad (\alpha = 1, 2; \lambda = 3, 4).
\]

The metric (1.1) is the one such that

\[
\frac{\partial F_{11}}{\partial u_2} = \frac{\partial F_{22}}{\partial u_2} = 0 \quad \text{and} \quad F_{33} = \psi(u_1) \sin 2u_2,
\]

and, as was proved in [4], any geodesic of this metric is a plane curve in $R^3$. And the metric (1.2) is the one essentially depending on the longitude $\phi$ and any geodesic of this metric is not plane in $R^3$ in general. For a geodesic $(x_i(t))$ of the metric (1.2), we have by (1.19) and (1.20) in [4]

\[
(1.3) \quad \frac{d^2 x_b}{dt^2} + B \frac{dx_b}{dt} + Ax_b + C \delta_{3b} = 0
\]
\[
\frac{d^2 x_4}{dt^2} - \frac{1 + a x_4 x_4}{x_4} \left\{ \frac{1}{1 + ar^2} \left( \frac{dr}{dt} \right)^2 + \frac{8r^2}{(x_3 + 3r)^2} \left( \sum \frac{dx_b \cdot dx_b}{dt \cdot dt} \right) \right\} + \left( - \left( \frac{dr}{dt} \right)^2 \right) + \left( - \frac{2}{x_4} + \frac{1}{x_4(1 + ax_4 x_4)} \right) \left( \frac{dx_4}{dt} \right)^2 = 0,
\]

where

\[ B := -2 \frac{d}{dt} \log \left( \frac{x_3 + 3r}{r} \right), \quad C := \frac{1}{x_3 + 3r} \left( \sum \frac{dx_b \cdot dx_b}{dt \cdot dt} - \frac{dr \cdot dr}{dt \cdot dt} \right) \]

and

\[ A := \left\{ -\frac{3}{r^2} + \frac{1}{r^2(1 + ar^2)} + \frac{8(1 + ar^2)}{(x_3 + 3r)^2} + \frac{3}{r(x_3 + 3r)} \right\} \frac{dr \cdot dr}{dt \cdot dt} \]

\[ + \left\{ \frac{3}{r(x_3 + 3r)} - \frac{2(1 + ar^2)}{(x_3 + 3r)^2} \right\} \sum \frac{dx_b \cdot dx_b}{dt \cdot dt} + \frac{2}{r(x_3 + 3r)} \frac{dr \cdot dx_3}{dt \cdot dt} \]

\[ = \left\{ -\frac{3}{r^2} + \frac{1}{r^2(1 + ar^2)} + \frac{6(1 + ar^2)}{(x_3 + 3r)^2} + \frac{6}{r(x_3 + 3r)} \right\} \frac{dr \cdot dr}{dt \cdot dt} \]

\[ + \left\{ \frac{3}{r} - \frac{2(1 + ar^2)}{x_3 + 3r} \right\} C + \frac{2}{r(x_3 + 3r)} \frac{dr \cdot dx_3}{dt \cdot dt}, \]

from which we obtain

\[ \frac{A}{C} = \frac{3}{r} - \frac{2(1 + ar^2)}{x_3 + 3r} + \frac{\Phi \cdot dr \cdot dx_2}{dt \cdot dx_1} + \frac{2 \cdot dr \cdot dx_3}{r \cdot dt \cdot dx_1}, \]

where

\[ \Phi = -\frac{2 + 3ar^2}{r^2(1 + ar^2)}(x_3 + 3r) + \frac{6(1 + ar^2)}{x_3 + 3r} + \frac{6}{r}. \]

The curve \((x_b(t))\) in \(R^3\) for the metric (1.2) is plane if and only if

\[ \frac{d}{dt} \left( \frac{A}{C} \right) \left( x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \right) = 0 \]

by Proposition 2 in [4].

In this work we shall try to express these metrics in more concrete forms.

When \(a > 0\), we put \(a = 1/\alpha^2\) (\(\alpha > 0\)). Then we have

\[ \frac{dx_4 dx_4}{x_4 x_4(1 + ax_4 x_4)} = \frac{\alpha^2 dx_4 dx_4}{x_4 x_4(\alpha^2 + x_4 x_4)} \quad (x_4 > 0) \]

and

\[ \frac{\alpha dx_4}{x_4 \sqrt{\alpha^2 + x_4 x_4}} = -\frac{1}{2} \frac{d}{dt} \log \frac{\sqrt{\alpha^2 + x_4 x_4} + \alpha}{\sqrt{\alpha^2 + x_4 x_4} - \alpha}. \]

Putting as
\[
\log \frac{\alpha^2 + x_4 x_4 + \alpha}{x_4} = \zeta, \quad \text{i.e.,} \quad \frac{\sqrt{\alpha^2 + x_4 x_4 + \alpha}}{x_4} = e^\zeta,
\]
we have
\[
e^{-\zeta} = \frac{x_4}{\sqrt{\alpha^2 + x_4 x_4 + \alpha}} = \frac{\sqrt{\alpha^2 + x_4 x_4 - \alpha}}{x_4},
\]
from which we obtain
\[
e^{\zeta} - e^{-\zeta} = \frac{\alpha}{x_4}, \quad \frac{e^{\zeta} + e^{-\zeta}}{2} = \frac{\sqrt{\alpha^2 + x_4 x_4}}{x_4},
\]
hence
\[
x_4 = \frac{\alpha}{\sinh \zeta}, \quad \frac{x_4}{\sqrt{\alpha^2 + x_4 x_4}} = \frac{1}{\cosh \zeta}.
\]
Using these expressions, (1.1) and (1.2) are expressed as
\[
(1.1') \quad ds^2 = \frac{1}{\alpha^2} \sinh^2 \zeta \sum_{b,c=1}^3 \left( \delta_{bc} - \frac{x_b x_c}{\alpha^2 + r^2} \right) dx_b dx_c - d\zeta d\zeta \quad (\zeta > 0)
\]
and
\[
(1.2') \quad ds^2 = \frac{1}{\alpha^2} \sinh^2 \zeta \sum_{b,c=1}^3 \left( \frac{8}{(x_3 + 3r)^2} \left( r^2 \delta_{bc} - x_b x_c \right) + \frac{\alpha^2 x_b x_c}{r^2 (\alpha^2 + r^2)} \right) dx_b dx_c - d\zeta d\zeta.
\]
And (1.4) is expressed as
\[
(1.4') \quad \frac{d^2 \zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2 + r^2)} \left( \frac{dr}{dt} \right)^2 + \frac{4r^2 \sinh 2\zeta}{\alpha^2 (x_3 + 3r)} C = 0.
\]
Next, when \( a < 0 \) we put \( a = -1/\alpha^2 (\alpha > 0) \). Then we have
\[
\frac{dx_4 dx_4}{x_4 x_4 (1 + ax_4 x_4)} = \frac{\alpha^2 dx_4 dx_4}{x_4 x_4 (\alpha^2 - x_4 x_4)}.
\]
For \( 0 < x_4 < \alpha \), we have
\[
\frac{\alpha^2 dx_4 dx_4}{x_4 x_4 (\alpha^2 - x_4 x_4)} = \left( d \log \frac{\sqrt{\alpha^2 - x_4 x_4} + \alpha}{x_4} \right)^2.
\]
Putting
\[
\log \frac{\sqrt{\alpha^2 - x_4 x_4} + \alpha}{x_4} = \zeta, \quad \text{i.e.,} \quad \frac{\sqrt{\alpha^2 - x_4 x_4} + \alpha}{x_4} = e^\zeta,
\]
we have
\[
e^{-\zeta} = \frac{\alpha - \sqrt{\alpha^2 - x_4 x_4}}{x_4},
\]
from which we obtain
\[
\frac{e^\zeta + e^{-\zeta}}{2} = \frac{\alpha}{x_4}, \quad \frac{e^\zeta - e^{-\zeta}}{2} = \frac{\sqrt{\alpha^2 - x_4x_4}}{x_4},
\]

hence
\[
x_4 = \frac{\alpha}{\cosh \zeta}, \quad \frac{x_4}{\sqrt{\alpha^2 - x_4x_4}} = \frac{1}{\sinh \zeta}.
\]

Therefore (1.1) and (1.2) can be written in this case as
\[
(1.1') \quad ds^2 = \frac{\cosh^2 \zeta}{\alpha^2} \sum_{b,c=1}^3 \left( \delta_{bc} + \frac{x_bx_c}{\alpha^2 - r^2} \right) dx_b dx_c - d\zeta d\zeta,
\]

and
\[
(1.2') \quad ds^2 = \frac{\cosh^2 \zeta}{\alpha^2} \sum_{b,c=1}^3 \left( \frac{8}{(x_3 + 3r)^2} (r^2 \delta_{bc} - x_bx_c) + \frac{\alpha^2 x_bx_c}{r^2(\alpha^2 - r^2)} \right) dx_b dx_c
\]

\[-d\zeta d\zeta
\]

where \(a = -1/\alpha^2\) and \(\zeta > 0\). For a geodesic \((x_i(t))\) of the metric (1.2'), the equation (1.4) becomes as follows:
\[
\frac{d^2 x_4}{dt^2} - \frac{\alpha^2 - x_4x_4}{\alpha^2 x_4} \left\{ \frac{\alpha^2}{\alpha^2 - r^2} \left( \frac{dr}{dt} \right)^2 + \frac{8r^2}{x_3 + 3r} C \right\} - \frac{\alpha^2 - 2x_4x_4}{x_4(\alpha^2 - x_4x_4)} \left( \frac{dx_4}{dt} \right)^2 = 0.
\]

Using the relation \(x_4 = \alpha / \cosh \zeta\), we obtain
\[
\frac{dx_4}{dt} = -\frac{\alpha \sinh \zeta}{\cosh^2 \zeta} d\zeta,
\]

\[
\frac{d^2 x_4}{dt^2} = -\frac{\alpha \sinh \zeta}{\cosh^2 \zeta} \frac{d^2 \zeta}{dt^2} - \frac{1 - \sinh^2 \zeta}{\cosh^4 \zeta} \frac{d\zeta}{dt} \frac{d\zeta}{dt}
\]

and (1.4) is reduced to the equation
\[
(1.4') \quad \frac{d^2 \zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2 - r^2)} \left( \frac{dr}{dt} \right)^2 + \frac{4r^2 \sinh 2\zeta}{\alpha^2(x_3 + 3r)} C = 0.
\]

Last for \(x_4 > \alpha\), we have
\[
\frac{\alpha^2 dx_4 dx_4}{x_4x_4(\alpha^2 - x_4x_4)} = -\alpha^2 \frac{dx_4 dx_4}{x_4x_4(x_4x_4 - \alpha^2)} = -\alpha^2 \left( \frac{dx_4}{x_4\sqrt{x_4x_4 - \alpha^2}} \right)^2
\]

and
\[
\frac{dx_4}{x_4\sqrt{x_4x_4 - \alpha^2}} = \frac{1}{\alpha} d\tan^{-1} \frac{\sqrt{x_4x_4 - \alpha^2}}{\alpha}.
\]

Putting
\[ \zeta = \tan^{-1} \frac{\sqrt{x_4 x_4 - \alpha^2}}{\alpha}, \quad \text{i.e.,} \quad \tan \zeta = \frac{\sqrt{x_4 x_4 - \alpha^2}}{\alpha}, \]

we have
\[ 1 + \tan^2 \zeta = \frac{1}{\cos^2 \zeta} = \frac{x_4 x_4}{\alpha^2}, \quad x_4 = \frac{\alpha}{\cos \zeta} = \alpha \sec \zeta \quad \left( 0 < \zeta < \frac{\pi}{2} \right). \]

Therefore we obtain
\[
(1.1''') \quad ds^2 = \frac{\cos^2 \zeta}{\alpha^2} \sum_{b,c=1}^{3} \left( \delta_{bc} + \frac{x_b x_c}{\alpha^2 - r^2} \right) dx_b dx_c + d\zeta d\zeta
\]

and
\[
(1.2''') \quad ds^2 = \frac{\cos^2 \zeta}{\alpha^2} \sum_{b,c=1}^{3} \left( \frac{8}{(x_3 + 3r)^2} (r^2 \delta_{bc} - x_b x_c) + \frac{\alpha^2 x_b x_c}{r^2(\alpha^2 - r^2)} \right) dx_b dx_c
\]
\[ + d\zeta d\zeta \]

where \( a = -1/\alpha^2 \) and \( 0 < \zeta < \pi/2 \). For a geodesic \((x_i(t))\) of the metric \((1.2'')\), by means of the equalities:
\[ x_4 = \frac{\alpha}{\cos \zeta}, \quad \frac{dx_4}{dt} = \frac{\alpha \sin \zeta}{\cos^2 \zeta} \frac{d\zeta}{dt}, \]

and
\[ \frac{d^2 x_4}{dt^2} = \frac{\alpha \sin \zeta}{\cos^2 \zeta} \frac{d^2 \zeta}{dt^2} + \frac{\alpha(1 + \sin^2 \zeta)}{\cos^3 \zeta} \left( \frac{d\zeta}{dt} \right)^2 \]

the equation (1.4) is reduced to the expression for this case as follows:
\[
(1.4''') \quad \frac{d^2 \zeta}{dt^2} + \frac{\sin 2\zeta}{2(\alpha^2 - r^2)} \left( \frac{dr}{dt} \right)^2 + \frac{4r^2 \sin 2\zeta}{\alpha^2(x_3 + 3r)} C = 0.
\]

Thus we have a proposition as follows.

**Proposition A.** For any geodesic \((x_i(t))\) in \( R^4_+ \) of the metric (1.2), the function \( \zeta(t) \) defined as above for the expressions (1.2'), (1.2'') and (1.2''') of (1.2) satisfies very simple equations:
\[
(1.4') \quad \frac{d^2 \zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2 + r^2)} \left( \frac{dr}{dt} \right)^2 + \frac{4r^2 \sinh 2\zeta}{\alpha^2(x_3 + 3r)} C = 0,
\]
\[
(1.4'') \quad \frac{d^2 \zeta}{dt^2} + \frac{\sinh 2\zeta}{2(\alpha^2 - r^2)} \left( \frac{dr}{dt} \right)^2 + \frac{4r^2 \sinh 2\zeta}{\alpha^2(x_3 + 3r)} C = 0
\]

and
\[
(1.4'''') \quad \frac{d^2 \zeta}{dt^2} + \frac{\sin 2\zeta}{2(\alpha^2 - r^2)} \left( \frac{dr}{dt} \right)^2 + \frac{4r^2 \sin 2\zeta}{\alpha^2(x_3 + 3r)} C = 0
\]

respectively, where
$C = \frac{1}{x_3 + 3r} \left( \sum_b \frac{dx_b}{dt} \frac{dx_b}{dt} - \frac{dr}{dt} \frac{dr}{dt} \right).$

**Remark B.** By means of specializing the quantities in (1.4')~(1.4'''), we may construct special theories on the orbit of the geodesics.

Finally, regarding Proposition A we consider the case in which $C = 0$. $C = 0$ means that

$$\sum_b \frac{dx_b}{dt} \frac{dx_b}{dt} = \frac{dr}{dt} \frac{dr}{dt}$$

or

$$\left| \left( \frac{dx_b}{dt} \right) \right| = \left| \frac{dr}{dt} \right|.$$

Since $r^2 = \sum_b x_b x_b$, we obtain

$$\frac{dr}{dt} = \sum_b \frac{x_b x_b}{r} \frac{dx_b}{dt} = \left( \frac{dx_b}{dt} \right) \cos \theta$$

where $\theta$ denotes the angle between two vectors $(x_b)$ and $\left( \frac{dx_b}{dt} \right)$ in $R^3$. Hence we have $\cos \theta = 1$ or $-1$, and so we may put $x_b(t) = a_b(1 \pm t)$ and $r^2 = W^2(1 \pm t)^2$, where $W = \sqrt{\sum_b a_b a_b}$.

Hence, (1.4'), (1.4'') and (1.4''') become for this case as follows:

(1.5') \[ \frac{d^2 y}{dt^2} + \frac{\sinh 2y}{2(\beta^2 + (1 \pm t)^2)} = 0, \]

(1.5'') \[ \frac{d^2 y}{dt^2} + \frac{\sinh 2y}{2(\beta^2 - (1 \pm t)^2)} = 0 \]

and

(1.5''') \[ \frac{d^2 y}{dt^2} + \frac{\sin 2y}{2(\beta^2 - (1 \pm t)^2)} = 0 \]
respectively, where \( y = \zeta \) and \( \beta = \alpha/W \). We solve these differential equations as follows. First, we set

\[
(1.7) \quad Q(y) = \int_{1}^{y} \frac{1}{\sinh 2y} dy \quad \text{for} \quad y > 0
\]

and another auxiliary function as follows:

\[
(1.8) \quad P(y) = \int_{0}^{y} \frac{1}{\cosh y} dy \quad \text{for} \quad y \geq 0.
\]

Then we have

\[
dQ(y) \frac{dt}{dt} = \frac{dQ}{dy} \frac{dy}{dt} = \frac{1}{\sinh 2y} \frac{dy}{dt}
\]

and

\[
d^2Q(y) \frac{dt^2}{dt^2} = \frac{1}{\sinh 2y} \frac{d^2y}{dt^2} + \frac{2 \cosh 2y \frac{dy}{dt} \frac{dy}{dt}}{\sinh^2 2y} \frac{dt}{dt}
\]

\[
= \frac{1}{\sinh 2y} \frac{d^2y}{dt^2} - 2 \cosh 2y \left( \frac{d}{dt} Q(y) \right)^2,
\]

hence we obtain

\[
\frac{1}{\sinh 2y} \frac{d^2y}{dt^2} = \frac{d^2Q(y)}{dt^2} + 2 \cosh 2y \left( \frac{dQ}{dt} \right)^2
\]

\[
= \frac{d^2Q}{dt^2} + 2 \left( \frac{dQ}{dt} \right)^2 + 4 \left( \sinh y \frac{dQ}{dt} \right)^2
\]

\[
= \frac{d^2Q}{dt^2} + 2 \left( \frac{dQ}{dt} \right)^2 + \left( \frac{dP}{dt} \right)^2.
\]

Therefore \((1.5') \sim (1.5'')\) become as follows.

\[
(1.6') \quad \frac{d^2Q}{dt^2} + 2 \left( \frac{dQ}{dt} \right)^2 + \frac{1}{2(\beta^2 + (1 \pm t)^2)} = 0,
\]

\[
(1.6'') \quad \frac{d^2Q}{dt^2} + 2 \left( \frac{dQ}{dt} \right)^2 + \frac{1}{2(\beta^2 - (1 \pm t)^2)} = 0
\]

and

\[
(1.6''') \quad \frac{d^2Q}{dt^2} + 2 \left( \frac{dQ}{dt} \right)^2 + \frac{1}{2(\beta^2 - (1 \pm t)^2)} = 0
\]

respectively. Now, since

\[
\frac{dQ}{dy} = \frac{1}{\sinh 2y} = \frac{1}{2 \cosh y \sqrt{\cosh^2 y - 1}},
\]
we have the relation
\[ \left\{ \left( \frac{dP}{dy} \right)^2 + 2 \left( \frac{dQ}{dy} \right)^2 \right\}^2 = 4 \left( \frac{dQ}{dy} \right)^2 \left( 1 + \left( \frac{dQ}{dy} \right)^2 \right). \]

Thus we obtain
\[ \left( \frac{dP}{dy} \right)^2 = 2 \frac{dQ}{dy} \left\{ \sqrt{1 + \left( \frac{dQ}{dy} \right)^2} - \frac{dQ}{dy} \right\} \]

and so
\[ \left( \frac{dP}{dt} \right)^2 = 2 \frac{dQ}{dt} \left\{ \sqrt{\left( \frac{dy}{dt} \right)^2 + \left( \frac{dQ}{dt} \right)^2} - \frac{dQ}{dt} \right\}. \]

Therefore (1.6') becomes
\[ \frac{d^2Q}{dt^2} + 2 \frac{dQ}{dt} \sqrt{\left( \frac{dy}{dt} \right)^2 + \left( \frac{dQ}{dt} \right)^2} + \frac{1}{2(\beta^2 + (1 \pm t)^2)} = 0 \]

and (1.6'') and (1.6'''') become analogous expressions. They are essentially equivalent to (1.5') \sim (1.5'''').

**Proposition.** The solution of (1.5') is given as follows. Setting \( y(0) = 0, \)
\[ y(t) = \sum_{n=1}^{\infty} a_n t^n, \]
and \( y^m = \sum_{n=m}^{\infty} b_{mn} t^n \) with \( b_{mn} = 0 \) for \( n < m, \) we have
\[ (\beta^2 - 1)a_2 = 0, \]
\[ 6(\beta^2 - 1)a_3 = 4a_2 + a_1 = 0, \]
\[ 12(\beta^2 - 1)a_4 = 12a_3 - a_2 = 0, \]
\[ 40(\beta^2 - 1)a_5 = 48a_4 - 10a_3 + \frac{4}{3}a_1^3 = 0, \]
\[ 30(\beta^2 - 1)a_6 = 40a_5 - 11a_4 + 2a_1^2 a_2 = 0, \]
\[ 84(\beta^2 - 1)a_7 = 120a_6 - 38a_5 \]
\[ + 4(a_1^2 a_3 + a_1 a_2^2) + \frac{4}{15}a_1^5 = 0, \]
\[ 2(\beta^2 - 1)(n + 2)(n + 1)a_{n+2} = 4(n + 1)na_n + \sum_{m=0}^{n} \frac{1}{(2m + 1)!} b_{(2m+1)n} = 0, \]
from which we obtain in Case I : \( \beta^2 \neq 1, \)
\[ a_2 = 0, \ a_3 = -\frac{1}{6(\beta^2 - 1)}a_1, \ a_4 = \mp \frac{1}{6(\beta^2 - 1)^2}a_1, \]
\[ a_5 = -\left( \frac{1}{5(\beta^2 - 1)^3} + \frac{1}{24(\beta^2 - 1)^2} \right)a_1 - \frac{1}{30(\beta^2 - 1)}a_1^3, \]
\[ a_6 = \frac{1}{30(\beta^2 - 1)} (\pm 40a_5 + 11a_4) \]
\[ = \frac{\mp 1}{30(\beta^2 - 1)} \left( \left( \frac{7}{2(\beta^2 - 1)^2} + \frac{8}{(\beta^2 - 1)^3} \right)a_1 + \frac{4}{3(\beta^2 - 1)^3}a_1^3 \right) \ldots , \]
and in Case II : $\beta^2 = 1$,

$$a_2 = \pm \frac{1}{4} a_1, \quad a_3 = \pm \frac{1}{12} a_2, \quad a_4 = \pm \frac{1}{36} \left( a_1^3 + \frac{5}{32} a_1 \right),$$

$$a_5 = \frac{7}{80 \times 18} a_1^3 - \frac{11}{32 \times 36} a_1,$$

$$a_6 = \pm \frac{19 \times 11}{60 \times 8 \times 36 \times 32} a_1^3 \pm \frac{13}{60 \times 80 \times 18} a_1^3 \pm \frac{1}{15 \times 30} a_1^5, \ldots,$$

respectively.

\[ \text{REFERENCES} \]


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(Received May 18, 2005)