On the homotopy groups of rotation groups

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ON THE HOMOTOPY GROUPS OF ROTATION GROUPS

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G. W. Whitehead [10] and B. Eckmann [3] determined the $r$-th homotopy groups $\pi_r(R_n)$ of the rotation groups $R_n$ of the $n$-dimensional euclidean space $E^n$ for some values of $r$; and N.E. Steenrod [7, §§22-24] summarized them and further results for the values of $r$ ranging from 1 to 5. Their calculations are based on the homotopy groups $\pi_r(S^n)$ of the $n$-dimensional spheres $S^n$; and, in recent years, J. Serre [5, 6] and H. Toda [9] have independently determined the groups $\pi_r(S^n)$ for $r$ equal to $n + 3$, $n + 4$ and $n + 5$. Therefore, we can calculate the groups $\pi_r(R_n)$ for $r$ equal to 6, 7 and 8, by the analogous processes developed in [7, §§22-24]. The results are stated as follows:

Theorem 1. i) $\pi_5(R_n) = 0$, $\pi_6(R_n) = 12$, $\pi_7(R_n) = 12 + 12$ and $\pi_8(R_n) = 0$ for $n \geq 5$.

ii) $\pi_5(R_n) = 0$, $\pi_6(R_n) = 2$, $\pi_7(R_n) = 2 + 2$ and $\pi_8(R_n) = \infty$; and $\pi_5(R_n) = \infty + 2$, $\pi_6(R_n) = \infty + 4$, $\pi_7(R_n) = \infty + 4 + \infty$ and $\pi_8(R_n) = \infty + 8$ for $n \geq 9$, or $\pi_5(R_n) = \infty$, $\pi_6(R_n) = \infty$, $\pi_7(R_n) = \infty + \infty$ and $\pi_8(R_n) = \infty$ for $n \geq 9$.

iii) $\pi_5(R_n) = 0$, $\pi_6(R_n) = 2$, $\pi_7(R_n) = 2 + 2$, $\pi_8(R_n) = 24$, $\pi_9(R_n) = 2 + 2$, $\pi_{10}(R_n) = 2 + 2 + 2$, $\pi_{11}(R_n) = 2 + 2 + 2$ and $\pi_{12}(R_n) = 2$ for $n > 10$.

As the corollary, some results for the determinations of the groups $\pi_r(S^n)$ having a non-zero element are obtained by using the map $J: \pi_r(R_n) \to \pi_{r+n}(S^n)$ of G.W. Whitehead [12]:

Theorem 2. $\pi_r(S^n) \neq 0$ for the following values of $r$ and $n$:

<table>
<thead>
<tr>
<th>$r$</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>7</td>
<td>4</td>
</tr>
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</table>

1) Numbers in brackets refer to the references cited at the end of this note.
2) We adopt the conventions that equating a group ∞ or $p$ means it is cyclic of order infinite or $p$, respectively.
1. Preliminaries

We shall use notations analogous to those of [3]. Let $R_{n+1}$ be the rotation group of the $(n + 1)$-dimensional euclidean space $E^{n+1}$ and $S^n$ the unit sphere of $E^{n+1}$. Then $R_{n+1}$ is the bundle space over the base space $S^n$ with the fibre and group $R_n$ and the natural projection $p : R_{n+1} \to S^n$ and, therefore, we can consider the exact homotopy sequence of this fibre bundle $\{ R_{n+1}, p, S^n, R_n, R_i \}$:

\[ \cdots \to \pi_{r+1}(S^n) \xrightarrow{\partial} \pi_r(R_n) \xrightarrow{i_*} \pi_r(R_{n+1}) \xrightarrow{p_*} \pi_r(S^n) \to \cdots, \]

where $i_*$ and $p_*$ are the induced homomorphisms of $i$, the inclusion map of $R_n$ into $R_{n+1}$, and $p$ respectively, and $\partial$ is the composed homomorphism $p_* \circ \partial$ of the isomorphism $p_*^{-1} : \pi_{r+1}(S^n) \to \pi_{r+1}(R_{n+1}, R_n)$ and the boundary homomorphism $\partial : \pi_r(R_{n+1}, R_n) \to \pi_r(R_n)$. The kernel of $i_*$ is $T_{r+1} \pi_r(S^{n-1})$ for $r \leq 2n - 3$, where $T_{n+1} : S^{n-1} \to R_n$ is the characteristic map of this bundle; and $i_* : \pi_r(R_n) \to \pi_r(R_{n+1})$ is isomorphic onto for $n \geq r + 2$, and onto for $n = r + 1$ and moreover $n = r$ if $r$ is even.

Representing $S^3$ by the group of quaternions $q$ of absolute value 1, and let $\rho : S^3 \to R_3$ and $\sigma : S^3 \to R_i$ be the map such that

\[ \rho(q) \cdot q' = qq'q^{-1} \quad \text{and} \quad \sigma(q) \cdot q' = qq', \]

respectively, where $q' + \bar{q}' = 0$, that is $q' \in S^3$, in the former case. Then, the induced homomorphism $\rho_* : \pi_3(S^3) \to \pi_3(R_3)$ is isomorphic onto for $r \geq 2$; and $\pi_r(R_n) \cong \pi_r(S^n) + \pi_r(R_n)$ for $r \geq 1$, where the isomorphism of $\pi_r(S^n)$ into $\pi_r(R_n)$ is given by $\sigma_*$. For $r = 1, 2, 3$ and 4, the groups $\pi_r(R_n)$ and their generators are known as follows.

\begin{enumerate}
  \item $\pi_r(R_n) = 0$, $\pi_r(R_n) = 2$ for $r \geq 3$, and $\pi_r(R_n) = 0$ for $r \geq 2$.
  \item $\pi_1(R_n) = 0$, $\pi_2(R_n) = 2$ for $n \geq 5$, where $\alpha_3$ and $\beta_3$ are the elements represented by $\rho$ and $\sigma$, respectively.
  \item $\pi_3(R_n) = 2 \{ \alpha_3 \}$, $\pi_3(R_n) = 2 + 2 = \{ \alpha_3 \} + \{ \beta_3 \}$, $\pi_4(R_n) = 2 = \{ \beta_3 \}$, and $\pi_5(R_n) = 0$ for $n \geq 5$, where $\alpha_3 = \alpha_3 \cdot \gamma_3$ and $\beta_3 = \beta_3 \cdot \gamma_3$.
\end{enumerate}

1) For the properties of this section, cf. [7], §§7, 17, 22 - 24.
2) We denote by $\{ \alpha \}$ the cyclic group generated by the element $\alpha$.
3) In $\pi_3(R_n)$, the term $\alpha_3$ must be written $i_3 \alpha_3$ precisely, where $i$ is the inclusion map of $R_n$ into $R_3$. From now on, if $i_*$ maps a subgroup $\{ \alpha \}$ of $\pi_r(R_n)$ isomorphically onto a subgroup $\{ i_* \alpha \}$ of $\pi_r(R_{n+1})$, we shall omit the letter $i_*$.
4) $\gamma_3$ is the generator of $\pi_3(S^n)$, cf. 1.2. i), below.
1.2. The following groups $\pi_n(S^n)$ are known explicitly:

i) $\pi_n(S^n) = \infty = \{ e_n \}$ for $n \geq 1$. $\pi_3(S^3) = \infty = \{ e_3 \}$ and $\pi_{n+1}(S^n) = 2 = \{ e_n \}$, for $n \geq 3$, where $e_n = E^{n-2} e_2$.

ii) $\pi_3(S^3) = 2 = \{ e_3 \}$, $\pi_7(S^4) = 12 = \{ e_7 \}$, and $\pi_{n+3}(S^n) = 24 = \{ e_n \}$ for $n \geq 5$, where $e_n = E^{n-2} e_3$ and $e_n = E^{n-4} e_4$.

iii) $\pi_2(S^2) = 12 = \{ e_2 \}$, $\pi_4(S^4) = 2 = \{ e_6 \}$, $\pi_6(S^6) = 2 = \{ e_8 \}$, and $\pi_{n+3}(S^n) = 0$ for $n \geq 6$.

iv) $\pi_2(S^2) = 2 = \{ e_2 \}$, $\pi_4(S^4) = 2 = \{ e_6 \}$, $\pi_6(S^6) = 2 = \{ e_8 \}$, and $\pi_{n+6}(S^n) = 0$ for $n \geq 7$.

1.3. The groups $\pi_n(R_n)$ are calculated without proofs in [7, 24.11].

Now, we shall determine their generators for the use of later.

**Proposition.** $\pi_2(R_3) = 2 = \{ e_2 \}$ and $\pi_2(R_3) = 2 = \{ e_2 \}$, where $c_2 = c_2 \circ c_4$. $\pi_4(R_3) = 2 = \{ e_4 \}$, $\pi_6(R_3) = \infty$ for $n \geq 7$, where $c_2$ is transformed into $2 c_2$ of $\pi_2(S^2)$ by the map $p_2 : \pi_2(R_3) \to \pi_2(S^2)$.

$\pi_2(R_3)$ and $\pi_2(R_3)$ are followed immediately from 1.1.

Consider the bundle $\{ R_1, R_2, S^1, R_4, R_1 \}$ and its homotopy sequence:

$$
\pi_n(R_1) \overset{i_*}{\longrightarrow} \pi_n(R_3) \overset{p_*}{\longrightarrow} \pi_n(S^3) \overset{\delta}{\longrightarrow} \pi_{n-1}(R_3) \overset{i^*}{\longrightarrow} \pi_{n-1}(R_3).
$$

As image $\delta = \text{kernel } i^*$ is cyclic subgroup of $\pi_n(R_3)$ of order 2 [7, 23.9. Theorem] and $\pi_2(S^3) = 2$, $\delta$ is isomorphic onto and hence $i^*$ is onto by exactness. The kernel of $i_*$ is $T_{\pi_2}(S^3)$ and its generator is $T_{\pi_2}(S^3) = (-\alpha_3 + 2 \beta_3 \circ \gamma_3 \circ \gamma_4^3) = \alpha_3 \circ \gamma_3 \circ \gamma_4 = \alpha_5$, and therefore $\pi_2(R_3)$ is cyclic of order 2 generated by the image of $\beta_3$.

In the case $\pi_2(R_3)$, if we consider the sequence: $\pi_2(R_3) \overset{i_*}{\longrightarrow} \pi_2(R_3) \overset{p_*}{\longrightarrow} \pi_2(S^3) \overset{\delta}{\longrightarrow} \pi_2(R_3) \overset{i^*}{\longrightarrow} \pi_2(R_3)$, then image $p_2 = \text{kernel } \delta$ is the infinite cyclic subgroup of $\pi_2(S^3) = \infty$ consisting of all even elements, because $\pi_2(R_3) = 0$ and $\pi_2(R_3) = 2$. On the other hand, kernel $i_* = T_{\pi_2}(S^3) = \{ (\beta_3 \circ \gamma_3 \circ \gamma_4) \}$ is $\{ 0 \}$, and hence $\pi_2(R_3) = \infty$.

The homomorphism $i_* : \pi_2(R_3) \to \pi_2(R_3)$ is onto and its kernel is $T_{\pi_2}(S^3)$. It is known that $p T_{\pi_2}(S^3)$ maps $S^3$ onto $S^3$ with degree 2 [7, 23.4. Theorem], and hence $p T_{\pi_2}$ represents $2 c_2$ of $\pi_2(S^3)$. This shows

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1) Cf. [9], Appendix 2, a) - f).

2) The maps $T_{\pi_2}$ and $T_{\pi_2}$ represent the elements $-\alpha_3 + 2 \beta_3$ and $\beta_3 \circ \gamma_3$, respectively, cf. [7], 23.6. Theorem, and proofs of 24.6. Theorem.
that $T_1$ represents the generator $\delta_2$ of $\pi_1(R_3)$, and therefore $T_{n,\pi_3(S^n)} = \pi_3(R_3)$. Thus we have $\pi_1(R_3) = 0$ and so $\pi_3(R_3) = 0$ for $n \geq 7$.

1.4. For $R_3$ and $R_4$, it follows immediately, from 1.1 and 1.2.

**Proposition.** $\pi_3(R_3) = 12 = \{\alpha_1\}$, $\pi_3(R_3) = 2 = \{\alpha_1\}$ and $\pi_3(R_4) = 2 = \{\alpha_1\}$, where $\alpha_1 = \alpha_1 \circ \mu_3$, $\alpha_1 = \alpha_1 \circ \eta_3 \circ \nu_4$ and $\alpha_1 = \alpha_1 \circ \eta_3 \circ \nu_4 \circ \gamma_3$. $\pi_3(R_3)$ $= 12 + 12 = \{\alpha_1\} + \{\beta_2\}$, $\pi_3(R_4) = 2 + 2 = \{\alpha_1\} + \{\beta_2\}$, and $\pi_3(R_3) = 2 + 2 = \{\alpha_1\} + \{\beta_2\}$, where $\beta_2 = \beta_2 \circ \mu_3$, $\beta_2 = \beta_2 \circ \eta_3 \circ \nu_4$ and $\beta_2 = \beta_2 \circ \eta_3 \circ \nu_4 \circ \gamma_3$.

2. The groups $\pi_r(R_3)$

2.1. To determine $\pi_r(R_3)$, we must calculate the kernel of $i_* : \pi_1(R_3) \to \pi_1(R_3)$. For this purpose, we first consider a principal bundle $\mathfrak{B} = \{B, g, S^n, G, \} \langle \gamma \rangle$ over $S^n$. Let $S^{n-1}$ be a great $(n-1)$-sphere on $S^n$ determined by setting the last real coordinate to zero, and $E^n_+$, $E^n_-$ the closed hemispheres of $S^n$ determined by $S^{n-1}$. Moreover, let $a_i$ in $E^n_+$ and $a_i$ in $E^n_-$ be the poles of $S^{n-1}$, and $V_i$ and $V_i$ be open cells on $S^n$ bounded by $(n-1)$-spheres parallel to $S^{n-1}$ and containing $E^n_+$ and $E^n_-$ respectively. If the bundle $\mathfrak{B}$ is in normal form, that is, its coordinate neighborhoods are $V_i$ and $V_i$, and $g_{a_i}(a_i) = e$ the identity of $G$, where $a_i$ is the reference point on $S^{n-1}$ and $g_{a_i} : V_i \cap V_i \to G$ is the coordinate transformation, then the map $T = g_{a_i} : S^{n-1} : S^{n-1} \to G$ is known as the characteristic map of $\mathfrak{B}$; and, if $r \leq 2n - 3$, the image of the homomorphism $d : \pi_{r+1}(S^n) \to \pi_r(G_1)$ is the group $\xi_r T_*$ where $G_1$ is the fibre over $a_i$ and $\xi = \phi_1, a_i = \phi_i | a_i \times G$ and $\phi_i : V_i \times G \to S^{n-1}(V_i)$ is the coordinate function.

To prove this property, we consider the diagram:

$$
\begin{array}{ccc}
\pi_{r+1}(S^n) & \xrightarrow{p_*} & \pi_{r+1}(B, G_1) \\
\downarrow k_* & & \uparrow h_* \\
\pi_{r+1}(S^n, E^n_+) & \xrightarrow{t_*} & \pi_{r-1}(S^{n-1}) \\
\end{array}
$$

where $k$ and $l$ are inclusion maps, $\partial$ and $\partial'$ the boundary homomorphisms and $h : (E^n_+, S^{n-1}) \to (S^n, a_i)$ the map such that, for $x \in E^n_+$, $h(x)$ lies in the great circle arc $C(x) = a_2 xa_1$, and its arc length from $a_2$ is twice that of $x$; and, finally, if $k(x)$ is the point $C(x) \cap S^{n-1}$, $h' : (E^n_+, S^{n-1}) \to (B, G_1)$ is the map defined by

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1) Cf. [7], 23.2. Theorem.
ON THE HOMOTOPY GROUPS OF ROTATION GROUPS

Then, $k_*, \beta_*$ and $\partial'$ are isomorphic onto and commutative relations hold in the square and triangles. Moreover, as the composed map $k_*^{-1} l_* \theta^{-1} : \pi_r(S^{n-1}) \to \pi_{r+1}(S^n)$ is the suspension $E$, $\xi_* T_* \pi_r(S^{n-1}) = \partial \beta_*^1 E \pi_r(S^{n-1}) = \Delta E \pi_r(S^{n-1})$. If $r < 2n - 3$, this shows the above property, as $E : \pi_r(S^{n-1}) \to \pi_{r+1}(S^n)$ is onto. The last property is stated as follows.

**Theorem.** In the above notations, the image group $\xi_* T_* \pi_r(S^{n-1})$ is the subgroup $\Delta E \pi_r(S^{n-1})$ of $\pi_r(G)$, and hence, if $E : \pi_r(S^{n-1}) \to \pi_{r+1}(S^n)$ is onto, the group $\xi_* T_* \pi_r(S^{n-1})$ is equal to the image of $\Delta : \pi_{r+1}(S^n) \to \pi_r(G)$.

2.2. Now, we consider the principal bundle $\{ R_{n+1}, \pi, S^n, R_n, R_0 \}$. If $\alpha$ is an element of $\pi_{r+1}(S^n)$ such that $\Delta \alpha = 0$, then there is an element $\beta \in \pi_{r+1}(R_{n+1})$ such that $\beta_* \beta = \alpha$, by exactness of the homotopy sequence stated in 1.1. Let $f : S^{r+1} \to R_{n+1}$ be a map representing $\beta$, and $\tilde{f} : S^{r+1} \times S^n \to S^n$ be the map determined by the formula: $\tilde{f}(x, y) = f(x) \cdot y$, where $x \in S^{r+1}$ and $y \in S^n$. Then $\tilde{f} | S^{r+1} \times y_0$ represents $\beta_* \beta = \alpha$ and $\tilde{f} | x_0 \times S^n$ represents $\iota_0$ for base points $x_0, y_0 \in S^{r+1}$ and $y_0 \in S^n$, and hence $f$ has type $(\alpha, \iota_0)$. It is known that the existence of a map of type $(\alpha, \beta)$ is equivalent to $[\alpha, \beta] = 0$, where $[\alpha, \beta]$ is the Whitehead product of $\alpha$ and $\beta$ [11, Corollary (3.5)], and therefore we have

**Theorem.** If $\alpha$ is a element of $\pi_{r+1}(S^n)$ such that $\Delta \alpha = 0$, then $[\alpha, \iota_0] = 0$.

2.3. By using above two theorems, we can prove

**Lemma.** The map $\Delta : \pi_r(S^n) \to \pi_r(R_0)$ is onto and its kernel is the infinite cyclic subgroup of $\pi_r(S^n)$ generated by $12 \nu_3$.

As the subgroup $\{ \mu_3 \} \subset \pi_r(S^n)$ is equal to $E \pi_0(S^n)$, by Theorem of 2.1, $\Delta \{ \mu_3 \} = T_{\mu_3} \pi_r(S^n)$ and its generator is

\[
T_{\mu_3} \mu_3 = (\alpha_3 + 2 \beta_3) \circ \mu_3
\]

\[
= (\alpha_3) \circ \mu_3 + (2 \beta_3) \circ \mu_3 + (\alpha_3 + 2 \beta_3) \circ H(\mu_3)
\]

1) In [12], Theorem 5.15, G.W. Whitehead proved that, if $\alpha \in \pi_r(S^n)$ and $\beta_1, \beta_2 \in \pi_n(X)$, and if $n < 3r - 3$, then $(\beta_1 + \beta_2) \alpha = \beta_1 \alpha + \beta_2 \alpha + [\beta_1, \beta_2] \circ H(\alpha)$, where $H(\alpha) \circ \pi_r(S^{2n-1})$ is the generalized Hopf invariant; and Blakers and Massey [2, (5.5)] showed that the definition of the generalized Hopf invariant $H(\alpha)$ can be extended by one dimension, and the above relation also holds for the case $n = 3r - 3$. 

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\[
= -\alpha_5 \circ \mu_5 + 2\beta_5 \circ \mu_5 + [\alpha_5, \alpha_5] \circ H(\mu_5) + [\beta_5, \beta_5] \circ H(\mu_5) - [\alpha_5, \beta_5] \circ 2H(\mu_5) \\
= -\alpha_6 + 2\beta_6.
\]

The last equality follows from the fact that

\[
[\alpha_5, \alpha_5] = [\rho_5, \epsilon_5, \rho_5, \epsilon_5] = \rho_5[\epsilon_5, \epsilon_5] = 0, \\
[\beta_5, \beta_5] = [\sigma_5, \epsilon_5, \sigma_5, \epsilon_5] = \sigma_5[\epsilon_5, \epsilon_5] = 0,
\]

because \(E([\epsilon_5, \epsilon_5]) = 0\) [11, Theorem 3.11] and the fact that \(E: \pi_5(S^3) \rightarrow \pi_6(S^4)\) is isomorphic onto imply \([\epsilon_5, \epsilon_5] = 0\). The above calculation shows that \(d\{\mu_5\} = \{-\alpha_6 + 2\beta_6\} = 12\).

On the other hand, it is known that \([\nu_5, \epsilon_5] = 2\nu_5 \circ \nu_7\) and \(\nu_5 \circ \nu_7\) has order \(24^2\); and hence, by Theorem of 2.2, \(d\{\nu_5\}\) is cyclic subgroup of \(\pi_6(R_4)\) of order 12. Set \(d\nu_5 = \alpha\) and \(d\mu_5 = \beta\), then \(\{\alpha\}\) and \(\{\beta\}\) are both cyclic subgroups of order 12. If \(m\alpha = n\beta\) for some integers \(m\) and \(n\) such that \(0 < m, n < 12\), then \(d(m\nu_5 - n\mu_5) = 0\), and so \([m\nu_5 - n\mu_5, \epsilon_5]\) = 0 by Theorem of 2.2. As \([\nu_5, \epsilon_5] = [\sigma_5 \circ E\mu_5, \epsilon_5 \circ E\epsilon_5] = [\epsilon_5 \circ \epsilon_5] \circ (\mu_5 \circ \epsilon_5) = 2\nu_5 \circ E\epsilon_5 \circ \mu_5 = 2\nu_5 \circ E\epsilon_5 \circ \epsilon_5 = 2\nu_5 \circ \nu_7 = 4\nu_5 - 2\mu_5 \circ \nu_7\), it follows that \((2m - 4n)\nu_5 - 2n\mu_5 \circ \nu_7 = 0\), and hence \(2m - 4n = 0\) mod \(24^2\). This shows that \(m - 2n = 0\) or \(-12\), and so \(m + 12a = 2n\), where \(a = 0\) or 1. As \(\alpha\) has order 12, this shows that \(2n\alpha = (m + 12a)\alpha = m\alpha\) and hence \(2m\alpha = n\beta\). If \(n = 0\), this relation contradicts the fact that both \(\alpha\) and \(\beta\) have order 12; and therefore \(n = 0\), and \(m = 0\). Thus the intersection of \(\{\alpha\}\) and \(\{\beta\}\) contains the zero element only, and hence \(\{\alpha\} + \{\beta\} = \pi_6(R_4)\), as \(\pi_6(R_4) = 12 + 12\) has 12 \(\times\) 12 elements.

The above results show that \(d\pi_7(S^4) = d\{\mu_5\} + \{\nu_5\}\) = \(\pi_6(R_4)\), and the above lemma is completed.

2.4. To determine the image of \(d: \pi_{r+1}(S^4) \rightarrow \pi_r(R_4)\) for \(r > 7\) and \(8\), we consider the homomorphism \(Y_r: \pi_r(X) \rightarrow \pi_{r+1}(X)\), for \(r > 2\), defined by

\[Y_r(\alpha) = \alpha \circ \eta_r, \quad \alpha \in \pi_r(X).\]

1) This is a consequence of the fact that \([\iota_5, \iota_5] = 2\nu_5 \circ \eta_7 [9, Lemma (4.6)], and \([\alpha, \beta] = ( -1 )^{[2r]} [\alpha, \beta] \) for \(\alpha \in \pi_r(X)\) and \(\beta \in \pi_r(X) [13, (3.3)]\).

2) If we apply to \(\nu_{14} \phi_{25}\), the Hopf homomorphism \(H_6: \pi_r(S^3) \rightarrow \pi_{r+1}(S^{2n})\) of [9, (3.2)] for \(r = 10\) and \(n = 4\), and use [9, (3.4)] or [8, (2.7)], then \(H_6(\nu_{14} \phi_{25}) = H_6(\nu_{14}) \circ E\nu_{14} \phi_{25} = \iota_5 \circ \nu_8 \phi_{25}\), and so \(\nu_{14} \phi_{25}\) has order 24.

3) Cf. [12], (3.58), where \(\iota_5 \circ \iota_5\) is the join of \(\iota_5\) and \(\iota_5\).

4) \([\iota_6, \iota_6] = 2\iota_6 \circ -\mu_5\) and \(E\iota_6 \circ \mu_6 = 2\iota_6\), cf. [9], Lemma (4.3), (ii).

5) As in the footnote 2), \(H_6(\iota_5 \circ (\nu_{14} \circ \iota_4)) = H_6(\nu_{14} \phi_{25}) \circ H_6(E\nu_{14} \phi_{25}) = \iota_5 \circ \nu_8\), by [9, (3.3)] or [8, (2.6)], and hence, if \((\alpha \iota_5 \circ (\nu_{14} \circ \iota_4) \circ \nu_8 = 0\), \(\alpha\) is a multiple of 24.
ON THE HOMOTOPY GROUPS OF Rotation Groups

As \( \eta_r \) is the suspension of \( \eta_{r-1} \), \( Y_r \) is clearly a homomorphism. Let \( \mathcal{B} = \{B, p, X, Y, G\} \) be a fibre bundle, \( Y_0 \) the fibre over \( x_0 \in X \) and \( \delta_r : \pi_{r+1}(X) \to \pi_r(Y_0) \) the boundary homomorphism of the homotopy sequence of this fibre bundle, then we have

**Lemma.** \( \delta_r(Y_r) = Y_{r-1} \delta_r(Y_{r-1}) \).

Let \( E^*_r \) and \( E^*_r \) be as in 2.1, and \( h : (E^*_r, S^{r-1}) \to (S^r, a_0) \) be a map representing \( \eta_r \in \pi_r(S^r) \), then \( h_r : \pi_{r+1}(E^*_r, S^{r-1}) \to \pi_{r+1}(S^r) \) is isomorphic onto \( \pi_r(S^r) \). As \( \delta_r : \pi_{r+1}(E^*_r, S^{r-1}) \to \pi_r(S^r) \) is isomorphic onto, there is a map \( g : (E^*_r, S^r) \to (E^*_r, S^{r-1}) \) such that \( g | S^r : S^r \to S^{r-1} \) represents \( \eta_{r-1} \in \pi_r(S^{r-1}) \), and \( g \) represents a generator of \( \pi_{r+1}(E^*_r, S^{r-1}) \), and therefore \( hg \) represents the generator \( \eta_r \) of \( \pi_{r+1}(S^r) \).

We consider the diagram

\[
\begin{array}{ccc}
\pi_r(B, Y_0) & \xrightarrow{\delta_r(Y_r)} & \pi_r(B, Y_0) \\
\pi_r(X) & \xrightarrow{h_r} & \pi_{r+1}(X) & \xleftarrow{\delta_r} & \pi_{r+1}(B, Y_0) \\
\pi_{r-1}(Y_0) & \xrightarrow{\delta_r(Y_{r-1})} & \pi_{r-1}(Y_0) & \xleftarrow{\delta_r} & \pi_r(Y_0)
\end{array}
\]

Let \( f : (S^r, a_0) \to (X, x_0) \) and \( f' : (E^*_r, S^{r-1}) \to (B, Y_0) \) be a representative of \( \alpha \in \pi_r(X) \) and \( \delta_r(Y_r) \in \pi_r(B, Y_0) \) respectively, then both \( pf' \) and \( fh \) represent \( \alpha \) and so \( pf' \) is homotopic to \( fh \). Hence, \( pf'g \) is homotopic to \( fhg \), and, as the latter represents \( \alpha \eta_r = Y_r(\alpha) \), \( f'g \) represents \( \delta_r^{-1}Y_r(\alpha) \). Thus \( f'g | S^r : S^r \to S^r \) represents \( \delta_r^{-1}Y_r(\alpha) = \delta_r(\alpha) \). On the other hand, as \( f'g | S^r \) is the composition of \( g | S^r \) and \( f' | S^{r-1} \) and these maps represent \( \eta_{r-1} \) and \( \delta_r^{-1} \alpha \) respectively, \( f'g | S^r \) represents \( (\delta_{r-1}\alpha) \eta_{r-1} = Y_{r-1}\delta_{r-1}\alpha \). Thus we have \( \delta_r Y_r \alpha = Y_{r-1}\delta_{r-1}\alpha \).

**2.5. Lemma.** The map \( \delta_r : \pi_{r+1}(S^r) \to \pi_r(R) \) is isomorphic onto for \( r = 7, 8 \).

For the case \( \mathcal{B} = \{R, p, S^r, R, R\} \), \( Y_r : \pi_r(S^r) \to \pi_0(R) \) is onto by 1.2, 1.4 and 2.3 respectively, and therefore, by the lemma of 2.4, \( \delta_r \) is onto. Similarly, \( \delta_s \) is also onto. Finally, as \( \pi_{r+1}(S^r) \) and \( \pi_r(R) \) are the same type \( 2 + 2 \), for \( r = 7 \) and \( 8 \), isomorphic properties are followed from ontoness.

**2.6. Now we can determine \( \pi_r(R) \).**

**Proposition.** \( \pi_r(R) = 0 \), \( \pi_1(R) = 0 \), \( \pi_0(S^r) = \infty \), and \( \pi_0(R) = 0 \), where \( \eta_r \) satisfies \( p_r \eta_r = 12 \nu_1 \in \pi_2(S^r) \).
Consider the exact homotopy sequence of the bundle \( \{ R_s, \bar{p}, S^s, R_s, R_s \} \):

\[
\pi_r(S^s) \xrightarrow{d_r} \pi_r(R_s) \rightarrow \pi_r((R_s, R_s)) \xrightarrow{\bar{p}_*} \pi_r(S^s) \xrightarrow{\partial_{r-1}} \pi_{r-1}(R_s) \xrightarrow{i_*} \pi_{r-1}(R_s).
\]

For \( r = 6, 7 \) and 8, \( \partial_r \) is onto by 2.3 and 2.5, and hence \( \bar{p}_* \) is isomorphic into by exactness. For the case \( r = 6 \), kernel \( i_* = 2 \) by 1.3, and \( \pi_6(S^s) = 2 \), and so \( \partial_6 \) is isomorphic onto. This shows that \( \pi_6(R_s) = 0 \). For \( r = 7 \), kernel \( \partial_7 = \{ 12 \nu_7 \} \) by 2.3, and hence \( \pi_7(R_s) = \infty \). Finally, for \( r = 8 \), as \( \partial_8 \) is isomorphic onto, \( \pi_8(R_s) = 0 \).

3. The groups \( \pi_n(R_s) \) for \( n \geq 6 \)

3.1. Proposition. \( \pi_n(R_s) = 0 \) for \( n \geq 6 \).

In the homotopy sequence \( \pi_n(R_s) \xrightarrow{i_*} \pi_n(R_s) \xrightarrow{\bar{p}_*} \pi_n(S^s) \xrightarrow{\partial} \pi_n((R_s, R_s)) \xrightarrow{i_*} \pi_n(R_s) \), image \( \partial = 2 \) and \( \pi_6(S^s) = 2 \) imply the onto-ness of \( i_* \), and hence \( \pi_6(R_s) = 0 \), because \( \pi_6(R_s) = 0 \). By 1.1, \( \pi_6(R_s) \rightarrow \pi_6(R_{n+1}) \) is onto for \( n \geq 6 \), and therefore we have 3.1.

3.2. Proposition. \( \pi_7(R_s) \) is equal to i) \( \infty + 2 = \{ r_7 \} \) + \{ \nu_7 \} or ii) \( 2 \mu_7 = r_7 \cdot \nu_7 \).

\( \pi_7(R_s) = 24 = \{ \delta_7 \} \), where \( \bar{p}_* \delta_7 = \pi_7(S^s) \) and, in the case ii), \( 2 \mu_7 = r_7 \cdot \nu_7 \).

3.3. Now, we consider some maps. Representing \( S^7 \) by Cayley numbers of absolute value 1, and taking a map \( \bar{p} : S^7 \rightarrow R_s \) defined by \( \bar{p}(c') = cc'c^{-1} \), where \( c \in S^7 \) and \( c' \in S^6 \) and the real part of \( c \) is zero. Then \( \bar{p} \) is a continuous map and it is known that \( \bar{p} : S^7 \rightarrow S^6 \) represents a non-zero element of \( \pi_7(S^6) \).

It is known that the bundle \( R_s \) is equivalent to the product bundle \( S^7 \times R_s \) and the map \( \bar{p} : S^7 \rightarrow R_s \), defined by \( \bar{p}(c \cdot c') = cc' \), where \( c, c' \in S^7 \), is clearly a cross-section of this product bundle, and so, in the direct sum decomposition \( \pi_7(R^s) \approx \pi_7(S^7) + \pi_7(R_s) \), the iso-group of \( \pi_7(S^7) \) into \( \pi_7(R_s) \) is given by \( \bar{p} \).

3.4. Proposition. i) \( \pi_7(R_s) = \infty + 4 = \{ r_7 \} + \{ \epsilon_1 \} \) and \( \pi_7(R_s) = \)

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1) \( \bar{p} \) is equivalent to \( \bar{f} \) of \([12], (8.12)\), which has the property that \( \bar{pf} \) represents \( \pi_7 \).

2) Cf. \([7], (8.5), (8.6)\) and \((17.9)\).
ON THE HOMOTOPY GROUPS OF ROTATION GROUPS

\[ \infty + 4 + \infty = \{\tau_1\} + \{v_1\} + \{\xi_1\}, \text{ if } \pi_1(R_3) \text{ is the case i) of 3.2; or ii) } \pi_1(R_3) = \infty = \{\tau_1\} \text{ and } \pi_1(R_3) = \infty + \infty = \{\tau_1\} + \{\xi_1\}, \text{ if } \pi_1(R_3) \text{ is the case ii) of 3.2; and the relation } 2\xi_1 = \delta_1 \text{ holds in } \pi_1(R_3). \]

In the sequence \( \pi_1(R_3) \overset{i_3}{\rightarrow} \pi_1(R_3) \overset{p_3}{\rightarrow} \pi_1(S^3) \overset{d_3}{\rightarrow} \pi_1(R_3) \), because \( \delta_3 \circ \eta_1 \epsilon \pi_1(R_3) = 0 \), kernel \( i_3 \) is \( T_{\infty} \pi_1(S^3) = \{ \delta_3 \circ \eta_1 \circ \eta_3 \} \), and hence \( i_3 \) is isomorphic into. As \( \pi_1(R_3) = 0 \), \( \pi_1(R_3) / \text{kernel } p_3 \approx \pi_1(S^3) = 2 \). On the other hand, in the homotopy sequence of \( (R_1, R_3) : \pi_1(R_3) \overset{i_3}{\rightarrow} \pi_1(R_3) \overset{i_3}{\rightarrow} \pi_1(R_1, R_3) \), \( i_3 \) is isomorphic and \( \pi_1(R_1, R_3) = 0 \), and therefore \( \pi_1(R_3) \), \pi_1(R_3) / \{ \gamma_1 \} \approx \pi_1(R_1, R_3) = 4^2 \). These relations imply the above proposition for \( \pi_1(R_3) \). \( \pi_1(R_3) \) follows from 3.3.

3.5. Proposition. \( \pi_1(R_3) = 2 + 2 = \{\tilde{\delta}_1\} + \{\varepsilon_1\} \) and \( \pi_1(R_3) = 2 + 2 + 2 = \{\tilde{\delta}_3\} + \{\varepsilon_3\} + \{\zeta_3\} \), where \( \tilde{\delta}_3 = i_3 \delta_3, \varepsilon_3 = \varepsilon_1 \circ \eta_1 \) and \( \zeta_3 = \zeta_1 \circ \eta_1 \).

In the sequence: \( \pi_1(R_3) \overset{i_3}{\rightarrow} \pi_1(R_3) \overset{p_3}{\rightarrow} \pi_1(S^3) \overset{d_3}{\rightarrow} \pi_1(R_3) \) \( i_3 \) is isomorphic, and hence \( p_3 \) is onto. The kernel of \( i_3 \) is equal to \( T_{\infty} \pi_1(S^3) \) and its generator is \( T_{\infty} \nu_2 = \delta_3 \circ \nu_2 = 2 \delta_3 \), as \( p_3(\delta_3 \circ \nu_3) = p_3(\delta_3) \circ \nu_3 = 2 \delta_3 \). Thus image \( i_3 \approx \{\tilde{\delta}_3\} / \{2 \delta_3\} = 2 \). On the other hand, as \( p_3(\varepsilon_1 \circ \eta_1) = p_3(\varepsilon_1) \circ \eta_1 = \eta_3 \circ \eta_1 = 0 \) in \( \pi_1(S^3) \), the element \( \varepsilon_3 = \varepsilon_1 \circ \eta_1 \) of \( \pi_1(R_3) \) does not belong to image \( i_3 \) and clearly has order 2. Thus we have \( \pi_1(R_3) = 2 + 2 \) and the above proposition.

3.6. Proposition. For \( n \gg 9 \), corresponding to the case i) or ii) of 3.2, i) \( \pi_1(R_3) = \infty + 8 = \{\tau_1\} + \{\zeta_1\} \), or ii) \( \pi_1(R_3) = \infty = \{\zeta_1\} \), where \( \zeta_1 = i_3 \zeta_1 \), and the relation \( 2\zeta_1 = \varepsilon_1 \) holds. \( \pi_1(R_3) = 2 + 2 = \{\tilde{\delta}_3\} + \{\zeta_3\} \), and \( \pi_1(R_3) = 2 = \{\tilde{\delta}_3\} \) for \( n \gg 10 \).

The groups \( \pi_1(R_3) \) and \( \pi_1(R_3) \) are the immediate consequence of the property that \( T_{\infty} : S^1 \rightarrow R_{\infty} \), the characteristic map of the principal bundle \( \{R_0, \rho, S^1, R_1, R_3\} \), represents the element \( -\varepsilon_1 + 2 \xi_1 \) of \( \pi_3(R_3) \), which can be proved by the same proofs of the fact that \( T_{\infty} \) represents \( -\alpha_s + 2\beta \) by using Cayley numbers instead of quaternions.

The characteristic map \( T_{\infty} : S^3 \rightarrow R_3 \) is homotopic to the characteristic map \( T_1 : S^3 \rightarrow R_3 \) of the unitary bundle\(^1\). Because \( T_1 : S^3 \rightarrow S^1 \) is essential\(^2\), \( T_1 \) represents \( a\tilde{\delta}_3 + b\varepsilon_3 + \zeta_3 \) of \( \pi_1(R_3) \), where \( a, b = 0 \) or 1. These properties show that \( T_{\infty} \) represents the image of \( a \tilde{\delta}_3 + b\varepsilon_3 + \zeta_3 \) under the map \( i_3 : \pi_1(R_3) \rightarrow \pi_1(R_3) \) and the latter is \( a\tilde{\delta}_3 + \zeta_3 \), where \( a = 0 \) or 1. Thus we have 3.6.

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1) \( T_{\infty} \) represents \( \delta_3 \in \pi_1(R_3) \), cf. proofs of the proposition of 1.3.
2) Cf. [1], Theorem 1.2.
3) Cf. [7], 23.6. Theorem, and 24.2 - 24.5.
By the results of §§2-3, we obtain Theorem 1 completely.

4. Some remarks on \( \pi_r(S^n) \)

4.1. It was proved by G. W. Whitehead that, if \( \alpha \in \pi_r(R) \) and \( p_*(\alpha) \in \pi_r(S^{n-1}) \) is not zero, then \( J(\alpha) \in \pi_{r+n}(S^n) \) is a non-zero element for \( r < 2n - 3 \); and A. L. Blakers and W. S. Massey generalized it for \( n \leq 2n - 3 \) that, if \( \alpha \in \pi_r(R) \) and the suspension \( E\nu_*(\alpha) \) is not zero, then \( J(\alpha) \neq 0 \). By the analogous process and making use of the Hopf homomorphism \( H_0 : \pi_r(S^n) \rightarrow \pi_{r+1}(S^{n+1}) \) of [9], we can prove more generally

4.2. Theorem. If \( \alpha \in \pi_r(R) \) and the \( m \)-hold suspension \( E^m\nu_*(\alpha) \) of \( p_*(\alpha) \in \pi_r(S^{n-1}) \) is not zero, then \( J(\alpha) \) is a non-zero element of \( \pi_{r+m}(S^n) \), where \( m \) is the minimum value of \( n+1 \) and \( r - 2n + 4 \).

If \( \alpha \in \pi_r(R_r) \), then \( J(\alpha) \) is represented by the Hopf construction of the mapping \( S^r \times S^{n-1} \rightarrow S^{n-1} \) of type \( (\nu_*(\alpha), \nu_{n-1}) \), and therefore \( H_0(J(\alpha)) = (1)^{r(n-1)}E(\nu_*(\alpha) * \nu_{n-1}) \). If \( E^m\nu_*(\alpha) \in \pi_{r+m}(S^{n+m}) \) is not zero, \( 5.2 \) implies \( J(\alpha) \) is a non-zero element of \( \pi_{r+m}(S^n) \).

For the case \( \pi_0(S^n) \), \( E^m\nu_*(\alpha) \) is not zero, because \( E^m(12\nu_4) = 12\nu_{m+4} \in \pi_{m+1}(S^{m+n+4}) \) is not zero. \( 5.2 \) implies \( J(\alpha) \) is a non-zero element of \( \pi_0(S^n) \).

For the case \( \pi_1(S^n) \), \( E^m\nu_*(\alpha) \) is not zero, because \( aJ(\delta) = J(a\delta) \) is not zero for \( a = 1, 2, \ldots, 23 \). Hence

4.4. Proposition. \( \pi_1(S^n) \) contains a cyclic subgroup whose order is a multiple of 24.

\( \pi_0(S^1) \) contains a non-zero element \( J(\varepsilon_0) \), and hence, by [4, Theorem 4], \( \nu_1 \) \( J\varepsilon_0 \) is a non-zero element of \( \pi_0(S^n) \).

Now, we consider the homotopy sequence of the bundle \( \{ R_r, p, S^r, R_0, R_1, R_2 \} : \)

\[ \pi_0(R_0) \xrightarrow{p_*} \pi_0(S^n) \xrightarrow{d} \pi_0(R_1) \xrightarrow{i_*} \pi_0(R_1). \]

As kernel \( i_* = 12 \) by 3.5 and \( \pi_0(S^1) = 24 \), image \( p_* = \text{kernel } d = \{ 12\nu_4 \} = 2 \). Hence \( \pi_0(R_1) \) contains a element \( \varepsilon_0 \) such that \( p_* \varepsilon_0 = 12\nu_4 \).

1) Cf. [12], Corollary 5.14, for the definition of \( J : \pi_r(R_0) \rightarrow \pi_{r+n}(S^n) \) and these results.
2) Cf. [2], (5.5).
3) Cf. [9], Corollary (3.6).
ON THE HOMOTOPY GROUPS OF ROTATION GROUPS

Therefore \( \pi_{16}(S^7) \) contains a non-zero element \( J(e_8) \), and consequently \( \pi_{16}(S^8) \) contains a non-zero element \( \nu_5 J(e_8) \).

By the same manner as above, considering the sequence of the bundle \( \{ R_n, p, S^{n-1}, R_{n-1}, R_{n-1} \} \) and using 4.2, it follows immediately that

4.5. Corollary. If the homomorphism \( i_* : \pi_{r-1}(R_{n-1}) \to \pi_{r-1}(R_n) \) is isomorphic and the image of the suspension \( E^n : \pi_r(S^{n-1}) \to \pi_{r+m}(S^{n-1+m}) \) contains a cyclic subgroup of order \( p \), where \( m = \min (n + 1, r - 2n + 4) \), then \( \pi_{r+n}(S^n) \) contains a cyclic subgroup whose order is a multiple of \( p \).

As \( i_* : \pi_{r-1}(R_1) \to \pi_{r-1}(R_3) \) is isomorphic, it follows immediately from this property that \( \pi_{16}(S^8) \) and \( \pi_{16}(S^8) \) is not zero.

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1) \( E : \pi_{16}(S^7) \to \pi_{16}(S^8) \) is isomorphic, cf. [9], Appendix 2, viii).