On primary ideal decompositions in non-commutative rings

Hisao Tominaga*

*Okayama University

Copyright ©1953 by the authors. Mathematical Journal of Okayama University is produced by The Berkeley Electronic Press (bepress). http://escholarship.lib.okayama-u.ac.jp/mjou
ON PRIMARY IDEAL DECOMPOSITIONS
IN NON-COMMUTATIVE RINGS

HISAO TOMINAGA

It is the purpose of this note to present the condition that every ideal in a (non-commutative) ring is represented as the intersection of a finite number of $s$-primary ideals. Although the fact that the well-known results of E. Noether hold in non-commutative case for those ideals which can be represented as the intersection of a finite number of right primary ideals has been shown under maximum condition \[3\]
, a necessary and sufficient condition that such a representation exists for every ideal is still unknown.

Throughout this note, the term "ideals" will mean "two-sided ideals" and $\mathfrak{N}$ will be a ring considered.

1. The right [left] quotient $a b^{-1} [b^{-1} a]$ of the ideals $a$ and $b$ is defined by $a b^{-1} = \{ x \in \mathfrak{R} \mid x b \subseteq a \} \{ b^{-1} a = \{ x \in \mathfrak{R} \mid b x \subseteq a \} \}$. The following properties of quotients are easily verified:

1) $\left( a b^{-1} \right) c^{-1} = a (c b)^{-1}$,
2) $\left( \bigcap_{\lambda} a_{\lambda} \right) b^{-1} = \bigcap_{\lambda} a_{\lambda} b^{-1}$,
3) $a \left( \sum_{\lambda} b_{\lambda} \right)^{-1} = \bigcap_{\lambda} a b_{\lambda}^{-1}$, $a$, $b$, $a_{\lambda}$ and $b_{\lambda}$ are ideals.

Let $a$, $b$ be ideals, if $a b^{-1} \supseteq a$, we say that $b$ is non-prime to $a$. If, for some positive integer $k$, $a b^{-k} = a b^{-\left(k+1\right)}$, then we say that $a b^{-k}$ is the right limit ideal of $a$ by $b$. Clearly, the right limit ideal $a b^{-k} = \bigcup_{i=1}^{\infty} a b^{-i}$. The left limit ideal is defined in the obvious way. In the case where the right limit ideal of $a$ by $b$ coincides with the left one, we call it the limit ideal of $a$ by $b$.

An ideal $p$ is said to be prime [1] if $a^{-1} p = p a^{-1} = p$ for any ideal $a \nsubseteq p$. As well-known, for every prime divisor $p$ of any ideal $a$, there exists a minimal prime divisor of $a$ which are contained in $p$ [1]. The intersection of all the minimal prime divisors of $a$ is called the radical of $a$ and denoted by $\sqrt{a}$.

An ideal $q$ is called right [left] primary if $q a^{-1} = q [a^{-1} q = q]$ for any ideal $a \nsubseteq q$ [3]. An ideal called primary if it is both right and

1) This note has been completed by the encouragement of Prof. M. Moriya. I express him my hearty thanks.
2) Numbers in brackets refer to the bibliography at the end of the note.
3) $a b^{-k}$ is defined as $(a b^{-\left(k-1\right)}) b^{-1}$ inductively.
left primary, and a [right, left] primary ideal is called s-[right, left] primary if its radical is nilpotent modulo the ideal.

**Theorem 1.** *The radical of an s-right primary ideal q is prime.*

In fact, let \( ab \subseteq q \), then \((ab)^n \subseteq q\) for a sufficiently large \( n \). Let \( n \) be the least with this property. If \( a, b \notin \bar{a} \), then \((ab)^n = (ab)^{n-1}ab \subseteq q\) implies \((ab)^{n-1} \subseteq q\) because q is right primary. \( (\text{We set} \ (ab)^{n-1} = \emptyset \text{ if} \ n = 1) \). But this contradicts with the minimality of \( n \).

If a prime ideal is the radical of an s-primary ideal q, we say that q is an s-primary ideal belonging to the prime ideal. And a prime ideal p is called a prime ideal associated with an ideal \( a \) [2] if there exists an s-primary ideal q belonging to p such that q = ar⁻¹, where r is an ideal not contained in a.

2. In this section, we assume that \( a = q_1 \cap \cdots \cap q_n \), where \( q_i \) are s-primary and the representation is irredundant[1]. As easily verified, a prime ideal p is a minimal prime divisor of \( a \) if and only if p is minimal in the set \( \{ q_i \} \). If \( q_i = q_j \) for all \( i \), then \( a \) is also a primary ideal with the radical \( p' \). In fact, \( p' \) is the unique minimal prime divisor of \( a \) and the rest of the proof is easy.

**Theorem 2.** *Let \( a = q_1 \cap \cdots \cap q_n \), where \( q_i \) are s-primary, then \( \bar{a} \) is nilpotent modulo \( a \).*

In fact, let \( k_i \) be the nilpotency index of \( \bar{q_i} \) modulo \( q_i \), then \((\bar{a})_{k_1^{+} \cdots \cdots + k_n} \subseteq a \).

From the preceding, we can assume, without loss of generality, that \( \bar{a} \) does not coincide with any \( \bar{q}_i \) \( (i \neq j) \) and so that the representation \( a = q_1 \cap \cdots \cap q_n \) is a short representation of \( a \).

If \( n > 1 \), then \( a \) is not primary. In fact, let \( \bar{a}_i \) be minimal in the set \( \{ \bar{q}_i \} \). Then there exist elements \( a_i \) \( (i = 2, \cdots, n) \) such that \( a_i \in \bar{q}_i \setminus \bar{a}_i \), where \( \bar{q}_i \setminus \bar{a}_i \) means the complement of \( \bar{a}_i \) in \( \bar{q}_i \). And so, for some positive integer \( m \), \( (a_i)^m \subseteq q_i \) \( (i = 2, \cdots, n) \), where \( (a_i) \) means the two-sided ideal generated by \( a_i \). As clearly \( q_i \supseteq a \), there exists an element \( q_i \in q_i \setminus a \), and \( (q_i)_{\sum_{i=2}^{n} a_i} \subseteq a \). Suppose now that \( a \) is primary, then \( (q_i) \subseteq a \) implies \( \sum_{i=2}^{n} a_i \subseteq \bar{a} \subseteq \bar{q}_i \), but it is impossible.

In the rest of this section, we assume that \( a = q_1 \cap \cdots \cap q_n \) be

---

1) A representation \( a = q_1 \cap \cdots \cap q_n \) is called irredundant if none of the \( q_i \) contains the intersection of the remainings.

2) The term "short representations" will be used for representations as the intersection of a finite number of s-primary ideals.
ON PRIMARY IDEAL DECOMPOSITIONS IN NON-COMMUTATIVE RINGS

A short representation of $a$. By using the same argument as in p. 35 of [4], we can prove, for every short representation of $a$, the uniqueness of the number of primary components and the radicals of the primary components. We state here the proof of the uniqueness of the isolated components.

If $p \subseteq \mathfrak{R}$ is any prime ideal, then we denote by $a'(p)$, where $a'$ is an ideal, the set of all the elements $b$ of $\mathfrak{R}$ such that $b \subseteq a'$ for some ideal $r \subseteq p$. (If $p = \mathfrak{R}$, $a'(r) = a'$, by definition.) As easily verified, $a'(p)$ is a two-sided ideal containing $a'$.

**Lemma 1.** Let $a = q_1 \cap \cdots \cap q_n$ be a short representation. If $p$ is a prime ideal containing $\overline{q}_1, \ldots, \overline{q}_r$ $(1 \leq r \leq n)$ but not containing $\overline{q}_{r+1}, \ldots, \overline{q}_n$, then $a(p) = q_i \cap \cdots \cap q_r$. If $p$ contains none of the $\overline{q}_i$, then $a(p) = \mathfrak{R}$.

We first assume that $p$ contains $\overline{q}_1, \ldots, \overline{q}_r$. Let $b$ be any element of $a(p)$, then for some ideal $\overline{s} \subseteq p$, $\overline{b} \subseteq \overline{s}$ and so $(\overline{b})\overline{s} \subseteq q_i$ $(i = 1, \ldots, n)$. As $\overline{s} \subseteq \overline{q}_i$ $(i = 1, \ldots, r)$ and $q_i$ are $s$-primary, $(\overline{b})\overline{s} \subseteq q_{r+1}$ $(i = 1, \ldots, r)$. Hence $a(p) \subseteq q_i \cap \cdots \cap q_r$. The converse inclusion is proved as following. If $n = r$, it is trivial. Therefore, we assume that $r < n$ and let $c$ be any element of $q_i \cap \cdots \cap q_r$. For $i = r + 1, \ldots, n$, we choose elements $p_i \in \overline{q}_i \setminus p$, then for some positive integer $k$, $(p_i)^k \subseteq q_i$ $(i = r + 1, \ldots, n)$. If we set $r' = (p_{r+1})^k \cdots (p_n)^k$, then $r' \subseteq p$ and $r' \subseteq q_{r+1} \cap \cdots \cap q_n$. Since $c \subseteq q_i \cap \cdots \cap q_r$, it follows that $cr' \subseteq a$, and hence $c \in a(p)$.

If $p$ contains none of the $\overline{q}_i$, then the last part of the above proof shows that there is an ideal $r' = (p_1)^k \cdots (p_n)^k$ which is not contained in $p$, where $p_i$ is in $\overline{q}_i \setminus p$ and $(p_i)^k \subseteq q_i$. Hence, $y\overline{r}' \subseteq a$ for all elements $y \in \mathfrak{R}$, that is, $a(p) = \mathfrak{R}$.

**Corollary**. Let $a = q_1 \cap \cdots \cap q_n$ be a short representation. Then $a(p)$ is s-primary if $p$ is a minimal prime divisor of $a$.

This is the direct consequence of Lemma 1.

---

1) Let $a = q_1 \cap \cdots \cap q_n$ be a short representation. Consider a subset $S$ of the set \{q_i\} having the property that if $q_i \in S$, then $q_j \subseteq q_i$ implies $q_j \subseteq S$. The intersection of the s-primary components belonging to the prime ideals in $S$ is called an isolated component of $a$.

2) D.C. Murdoch proved that if $a$ is represented as the intersection of a finite number of right primary ideals, then $u(a, p)$ is right primary for every minimal prime divisor $p$ of $a$, where the maximum condition for ideals is assumed (Corollary 2 to Theorem 17 of [3]). In our case, we obtain that $u(a, p) = I(a, p) = a(q)$. Thus our corollary corresponds to the Murdoch's result stated above.
Let \( b = q_{i_1} \cap \cdots \cap q_{i_m} \) be an isolated component of \( a \), then \( a(q_{i_s}) \) \((s = 1, \ldots, m)\) is represented as the intersection of the primary ideals belonging to a subset of \( \{ q_{i_j} \} \( j = 1, \ldots, m\) containing \( q_s \). It follows that \( b = a(q_{i_1}) \cap \cdots \cap a(q_{i_m}) \).

We summarize here the uniqueness theorems.

**Theorem 3** If an ideal \( a \) is represented as the intersection of a finite number of \( s \)-primary ideals, then there exists a short representation of \( a \). And,

1. The number of \( s \)-primary components in every short representation of \( a \) and the radicals of them are uniquely determined.
2. The radicals of primary ideals belonging to an isolated component of \( a \) uniquely determine the isolated component, and so the isolated components of \( a \) coincide in all the short representations.

Let \( b \) be an arbitrary ideal contained in \( q_i \), then \( \mathcal{R} = q_i b^{-x} = q_i b^{-(s+\epsilon)} = b^{-(s+\epsilon)}q_i = b^{-s}q_i \) for a sufficiently large \( k \). On the other hand, if \( b \nsubseteq q_i \), then \( q_i = q_i b^{-k} = b^{-k}q_i \) for every positive integer \( k \). Thus, for any ideal \( c \), there exists a positive integer \( t \) such that \( ac^{-t} = ac^{-t+\epsilon} = c^{-t+\epsilon}a = q_{i_1} \cap \cdots \cap q_{i_m} \), where \( \{ q_{i_1}, \ldots, q_{i_m} \} \) is a subset of \( \{ q_1, \ldots, q_n \} \). This proves the next

**Theorem 4.** Let \( a = q_1 \cap \cdots \cap q_n \) be a short representation. Then, for any ideal \( b \), there exists the limit ideal of \( a \) by \( b \). And the number of ideals which, starting from \( a \), are obtained by repeating the procedures to make limit ideals successively is finite, and is uniquely determined by \( a \).

From the existence of the limit ideal of \( a \), we see readily the following

**Corollary.**\(^1\) \( ab^{-1} \supset a \) if and only if \( b^{-1}a \supset a \), accordingly, \( a \) is primary if and only if \( a \) is right (or left) primary.

Let \( p \) be a minimal prime divisor of \( a (\subset \mathfrak{M}) \) and let \( h \) be a positive integer such that \( p^h \subseteq q_i \) for all \( q_i \) with \( q_i \subseteq a \) containing \( p \), then clearly \( ap^{-h} \supset a \). Hence we have

**Theorem 5.** Let \( q_1 \cap \cdots \cap q_n = a \) be a short representation of \( a \subset \mathfrak{M} \). If \( p \) is a minimal prime divisor of \( a \), then \( p \) is non-prime to \( a \).

**Lemma 2.** If \( q \) is \( s \)-primary, then for any ideal \( x \nsubseteq q \), \( qr^{-1} \) is \( s \)-left primary.

Clearly, \( qr^{-1} = q \). \( uv \subseteq qr^{-1} \) implies \( uv \subseteq q \), where \( u, v \) are ideals. If \( u \nsubseteq qr^{-1} \), then we have \( vr \subseteq q \). Thus \( v \subseteq qr^{-1} \).

---

\(^1\) This corollary is derived from only the fact that there exists the limit ideal of \( a \).
ON PRIMARY IDEAL DECOMPOSITIONS IN NON-COMMUTATIVE RINGS

Theorem 6. If every ideal in \( R \) is represented as the intersection of a finite number of \( s \)-primary ideals, then a prime divisor \( \wp \) of an arbitrary ideal \( \alpha \) is a prime ideal associated with \( \alpha \) if and only if \( \wp \) coincides with one of the radicals \( \bar{\eta}_j \) in a short representation \( \alpha = q_1 \cap \cdots \cap q_n \). And every primary component \( q_j \) \( (j = 1, \cdots, n) \) has the following property: For any ideal \( \eta \subseteq q_j, \not\subseteq \alpha, \alpha \eta^{-1} \) is not an \( s \)-primary ideal belonging to \( \bar{\eta}_j \).

The second part of this theorem follows from \( \alpha \eta^{-1} = q_1 \cap \cdots \cap q_{j-1} \cap q_{j+1} \cap \cdots \cap q_n \eta^{-1} \).

Now we shall prove the first part. As \( \alpha = q_1 \cap \cdots \cap q_n \) is a short representation, \( \cap q_j \not\subseteq q_i \). Clearly \( \alpha(\cap q_j)^{-1} = q_i(\cap q_j)^{-1} \). By Lemma 2 and Corollary to Theorem 4, \( \alpha(\cap q_j)^{-1} \) is an \( s \)-primary ideal belonging to \( \bar{\eta}_j \). (If \( n = 1 \), we set \( \cap q_j = R \).) Conversely, let \( \wp \) be a prime ideal associated with \( \alpha \), that is, \( q = \alpha \wp^{-1}(\wp \not\subseteq \alpha) \) be an \( s \)-primary ideal belonging to \( \wp \). If \( r \subseteq q_1, \cdots, q_r \) but \( \not\subseteq q_{r+1}, \cdots, q_n \), then \( \alpha \wp^{-1} = q_1 \cap \cdots \cap q_r \wp^{-1} \cap \cdots \cap q_n \wp^{-1} \). Again by Lemma 2 and Corollary to Theorem 4, the ideals \( q_{r+1} \wp^{-1}, \cdots, q_n \wp^{-1} \) are \( s \)-primary. By Theorem 3, (1), we have \( \alpha \wp^{-1} = q_i \wp^{-1} \) for some \( i \) \((r + 1 \leq i \leq n)\). Hence \( \wp \) coincides with \( \bar{\eta}_i \).

Summarizing the above-mentioned results, we obtain

Theorem 7. In order that every ideal in \( R \) is represented as the intersection of a finite number of \( s \)-primary ideals, the following conditions are necessary:

(A) The radical of any ideal \( \alpha \) is nilpotent modulo \( \alpha^3 \).

(B) For any ideals \( \alpha, \beta \), there exists the limit ideal of \( \alpha \) by \( \beta \) and there exists a finite number \( n(\alpha) \) of ideals which, starting from \( \alpha \), are obtained by repeating the procedures to make limit ideals successively. The number \( n(\alpha) \) is uniquely determined by \( \alpha \).

(C) Each minimal prime divisor of any ideal \( \alpha \subseteq R \) is non-prime to \( \alpha \).

(D) If \( \wp \) is an arbitrary prime ideal associated with an ideal \( \alpha \), there exists an \( s \)-primary ideal \( \eta \supseteq \alpha \) belonging to \( \wp \) such that, for any ideal \( \beta \subseteq \eta, \not\subseteq \alpha, \alpha \beta^{-1} \) is no primary ideal belonging to \( \wp \).

3. In this section, we assume first the condition (B) in Theorem 7. Let \( \alpha \) be an ideal and let \( \wp \subseteq R \) be a minimal prime divisor of \( \alpha \).

1) In any ring with maximum condition for ideals, the condition (A) is satisfied. See, for example, Theorem 10 of [3].
Then we consider the set \( M \) of the limit ideals of \( \alpha \) by \( \gamma' \), where \( \gamma' \) runs over all ideals not contained in \( \gamma \). Clearly, by (B), \( M \) is a finite set. Therefore, there exists a maximal ideal \( \alpha_0 \) in \( M \) and, for some ideal \( \gamma \in \gamma \), \( \alpha_0 = \alpha \gamma^{-k} = \alpha \gamma^{-(k+1)} = \gamma^{-(k+1)} \alpha = \gamma^{-k} \alpha_0 \), where \( k \) is a sufficiently large positive integer. Being \( \gamma \in \gamma \), \( \gamma \) is obviously a minimal prime divisor of \( \alpha_0 \). By the definition, \( \alpha_0 \subseteq \alpha(\gamma) \). Let now \( b \) be any element in \( \alpha(\gamma) \). Then there exists an ideal \( \delta \subseteq \gamma \) such that \( b \delta \subseteq \gamma \subseteq \alpha_0 \).

Hence, \( b \in \alpha_0 \delta^{-1} = \alpha(\delta \gamma^{-i})^{-1} \subseteq \bigcup_{i=1}^{\infty} \alpha(\delta \gamma^{-i})^{-1} \). Since \( \alpha_0 \subseteq \bigcup_{i=1}^{\infty} \alpha(\delta \gamma^{-i})^{-1} \in M \) and \( \alpha_0 \) is maximal in \( M \), then \( \bigcup_{i=1}^{\infty} \alpha(\delta \gamma^{-i})^{-1} = \alpha_0 \), whence \( b \in \alpha_0 \). Thus, we have \( \alpha(\gamma) = \alpha_0 \) and, furthermore, \( \alpha_0 = \alpha_0 \delta^{-1} \) for every \( \delta \in \gamma \).

Let us assume next the condition (C) in Theorem 7, in addition to (B). If \( \alpha(\gamma) = \gamma \), then there exists a minimal prime divisor \( \gamma' = \gamma \) of \( \alpha(\gamma) \). Since, by (C), \( \gamma' \) is non-prime to \( \alpha(\gamma) \), for some ideal \( \beta \neq \alpha(\gamma) \), \( \beta \gamma' \subseteq \alpha(\gamma) \). But \( \beta \in \alpha(\gamma) \gamma'^{-1} = \alpha(\gamma) \). This contradiction shows \( \alpha(\gamma) = \gamma \). Hence we have proved the following

**Lemma 3.** Let \( R \) satisfy the conditions (B) and (C) in Theorem 7. Then \( \alpha(\gamma) \) is primary, where \( \gamma \in R \) is a minimal prime divisor of \( \alpha \), and there exists an ideal \( \gamma_0 \neq \gamma \) such that \( \alpha(\gamma) = \alpha \gamma_0^{-1} = \gamma_0^{-1} \alpha_0 \).

We prove next the following

**Lemma 4.** If \( R \) satisfies the conditions (A) and (B) in Theorem 7, then the number of prime ideals associated with a non-primary ideal \( \alpha \) is finite.

Let \( \{\gamma_\alpha\} \) be the set of all prime ideals associated with \( \alpha \) and let \( q_\alpha = \alpha \gamma_\alpha^{-1} (\gamma_\alpha \neq \alpha) \) be a primary ideal belonging to \( \gamma_\alpha \). By Lemma 3 and the condition (A), the set \( \{\gamma_\alpha\} \) is not empty.

At first, let \( \gamma_1 \supseteq \gamma_2 \supseteq \cdots \supseteq \gamma_k \) be a chain in \( \{\gamma_\alpha\} \), then \( k \leq n(\alpha) \). If not, we define the ideals \( \gamma_i (i = 1, \ldots, k) \) by setting \( \gamma_i = \gamma_i \gamma_\alpha^{-1} \gamma_\alpha \). Then, for some positive integer \( h \), \( \gamma_1 \supseteq \gamma_2 \cdots \supseteq \gamma_i \gamma_\alpha^{-1} \gamma_\alpha \), where, by (A), we assume \( \gamma_1 \gamma_\alpha^{-1} \gamma_\alpha \subseteq q_m \) for every \( m \) (\( 1 \leq m \leq k \)). As \( q_m \gamma_\alpha \subseteq \alpha \) and \( \gamma_i \gamma_\alpha^{-1} \gamma_\alpha \subseteq q_m \), we have \( \gamma_m \subseteq \gamma_i \). But, if \( t > j \), \( \gamma_t \neq \gamma_j \), because \( \gamma_1 \supseteq \gamma_2 \cdots \supseteq \gamma_i \gamma_\alpha^{-1} \gamma_\alpha \) implies \( \gamma_t \gamma_\alpha^{-1} \gamma_\alpha \subseteq \gamma_\alpha \subseteq \gamma_t \subseteq \gamma_i \gamma_\alpha^{-1} \gamma_\alpha \) in which each term is the limit ideal of the preceding one and whose length is \( k \). But it contradicts with the condition (B). From this, we can easily see that there is a maximal one in \( \{\gamma_\alpha\} \).

Next, let \( \{\gamma_1, \ldots, \gamma_k\} \) be a finite subset of \( \{\gamma_\alpha\} \) and let every \( \gamma_i \) be not contained in any remaining one. Then \( k \leq n(\alpha) \). If not, by
using the above argument, we can construct an ascending chain 
\( a \subseteq r_1 \subseteq \ldots \subseteq r_k \) in which each term is the limit ideal of the preceding one and whose length is \( k \).

By the facts proved above, we see that there exists a finite number of maximal elements in \( \{ v_i \} \). Now, we omit from the set 
\( \{ v_i \} = M \) all the maximal elements \( v_{1,1}, \ldots, v_{1,n} \), and denote by \( M \) the set of the remaining ideals. Since \( M \) has obviously a finite number of maximal elements \( v_{2,1}, \ldots, v_{n,2} \), we obtain the set \( M \) by omitting \( v_{2,1}, \ldots, v_{n,2} \) from \( M \). Clearly, each ideal of \( v_{2,1}, \ldots, v_{n,2} \) is contained in some of the \( v_{1,i} \)'s. Repeating this procedure, we obtain a descending chain \( M \supseteq M \supseteq \ldots \), but \( M + 1 \) is the empty set, because, otherwise, there exists a descending chain of prime ideals from \( \{ v_i \} \) whose length exceeds \( n - 1 \). q.e.d.

We assume here the condition (D) besides (A), (B) and (C) in Theorem 7.

Let \( v_1, \ldots, v_n \) be all the prime ideals associated with an non primary ideal \( a \) (by Lemma 4), and let \( q_1, \ldots, q_n \) be the primary divisors of \( a \) belonging to \( v_1, \ldots, v_n \) respectively which possess the property in (D). We set \( b = q_1 \cap \ldots \cap q_n \). By Lemma 3, every minimal prime divisor of \( a \) is a prime ideal associated with \( a \). Hence \( b \subseteq a \). As \( a \) is nilpotent modulo \( a \) (by (A)), \( ab^{-1} \supseteq a \). We suppose now that \( b \supseteq a \). If \( ab^{-1} \) is non primary, by Lemma 3, we have, for some ideal \( r_1 \subseteq ab^{-1} \), a primary ideal \( ab^{-1} \subseteq R \) and set \( r = r_1 b \). On the other hand, if \( ab^{-1} \) is primary, we set \( b = r \). Hence, in either case, we have a primary ideal \( q = ar^{-1} \), where \( r \subseteq a \), \( r \subseteq b \), and \( q \) is a prime ideal associated with \( a \). Therefore, for some \( i \), \( q = v_i \). Since \( r \subseteq b \subseteq q_i \), the ideal \( q = ar^{-1} \) is not primary (by (D)), but this is a contradiction. Hence, we have \( a = b \). This proves the sufficiency part of the next principal theorem.

Theorem 8. Every ideal in \( R \) is represented as the intersection of a finite number of \( s \)-primary ideals if and only if the conditions (A), (B), (C) and (D) are satisfied.

Bibliography


Department of Mathematics,
Okayama University

(Received July 20, 1953)