A construction of closed surfaces of negative curvature in E4

Tominosuke Otsuki*

*Mathayama University

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A CONSTRUCTION OF CLOSED SURFACES 
OF NEGATIVE CURVATURE IN $E^4$

TOMINOSUKE ŌTSUKI

Introduction. The author recently proved a theorem\(^1\) as follows.

Theorem. Let

$$A_{\alpha i j} x^i x^j = 0^\alpha, \quad A_{\alpha i j} = A_{\alpha j i},$$

$$i, j = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, N$$

be a system of quadratic equations in $x^i$. If

$$\sum_\alpha (A_{\alpha i k} A_{\alpha j k} - A_{\alpha i k} A_{\alpha j k}) x^i y^j x^k y^k < 0$$

for any $x^i, y^i$, it has a non-trivial real solution in $x^i$, when $N < n$.

This gives the corollary\(^2\):

Corollary. Let

$$\Psi_\alpha (x, x) \equiv A_{\alpha i j} x^i x^j, \quad A_{\alpha i j} = A_{\alpha j i},$$

$$i, j = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, N$$

be $N$ quadratic forms of $x^i$. If

$$\sum_\alpha (A_{\alpha i k} A_{\alpha j k} - A_{\alpha i k} A_{\alpha j k}) x^i y^j x^k y^k < 0$$

for any two linearly independent vectors $x = (x^i), \ y = (y^i)$, then $N > n - 1$.

Proof. Let us suppose that $N \leq n - 2$. By virtue of the theorem above, there exists a vector $x = (x^i)$ such that $x \neq 0$ and

$$\Psi_\alpha (x, x) = 0, \quad \alpha = 1, 2, \ldots, N.$$ 

For a system of linear equations in $y^i$ such that

$$x^i y^i = 0,$$

$$\Psi_\alpha (x, y) = 0, \quad \alpha = 1, 2, \ldots, N$$

there exists a non-trivial solution $y = (y^i)$. For the $x^i, y^i$, we have

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2) The summation convention of tensor analysis is used throughout.

3) The author was informed by Prof. S.S. Chern of this corollary.
\[
\sum_a (A_{a_{ik}} A_{a_{jk}} - A_{a_{ik}} A_{a_{jk}}) x^i y^j x^k y^k = 0.
\]

this contradicts to the assumption.

As a geometrical application of the corollary, we have a theorem as follows.

**Theorem.** An n-dimensional Riemann manifold with the property that at least at a point, the sectional curvatures for all plane elements are negative cannot be isometrically imbedded in an Euclidean space of dimension \(2n - 2\).

**Proof.** Let \(V_n\) be a Riemann manifold with the property stated in the theorem. Assume that \(V_n\) can be isometrically imbedded in an Euclidean space of dimension \(n + N\). Let the line element of \(V_n\) be given by \(ds^2 = g_{ij}(u) du^i du^j\) in a coordinate neighborhood \((u^i)\) of the point. Let

\[
R_{ij}^{hk} = \frac{\partial \Gamma_{ij}^h}{\partial u^k} - \frac{\partial \Gamma_{ij}^k}{\partial u^h} - \Gamma_{j}^{m} \Gamma_{i}^{h} \Gamma_{m}^{k} + \Gamma_{i}^{m} \Gamma_{j}^{h} \Gamma_{m}^{k}
\]

be the components of the Riemann-Christoffel tensor of \(V_n\), where \(\Gamma_{ij}^k\) denote the Christoffel symbols of the second kind made by \(g_{ij}\).

Let \(\varphi_{a} = A_{a_{ij}} du^i du^j\), \(A_{a_{ij}} = A_{a_{jk}}\), \(a = 1, 2, \ldots, N\), be the second fundamental forms of \(V_n\). As is well known, we have the equations

\[
R_{ij}^{hk} = -\sum_a (A_{a_{ik}} A_{a_{jk}} - A_{a_{ik}} A_{a_{jk}}).
\]

By the assumption, it follows that

\[
\sum_a (A_{a_{ik}} A_{a_{jk}} - A_{a_{ik}} A_{a_{jk}}) x^i y^j x^k y^k < 0
\]

for any two linearly independent tangent vectors \(x^i, y^i\) to \(V_n\) at the point. Hence, by means of the corollary above, it must be \(N \geq n - 1\).

Furthermore, we have the theorem:

**Theorem.** A compact n-dimensional Riemann manifold with the property that at any point, the sectional curvatures for all plane elements are non-positive cannot be isometrically imbedded in an Euclidean space of dimension \(2n - 1\).

**Proof.** Let \(V_n\) be a Riemann manifold with the property stated in the theorem. Let us suppose that \(V_n\) can be isometrically imbedded in an Euclidean space of dimension \(2n - 1\). Let \(O\) be a fixed point of the space and let \(P\) be a point of \(V_n\) whose distance from
O is maximum. We may consider that the second fundamental form \( \nu = A_{ij} du^i du^j \) is correspond to the normal direction \( \overrightarrow{OP} \) to \( V_n \) at \( P \). Since \( \text{dist} (O, P) \) is maximum on \( V_n \), we have \( \overrightarrow{OP} \cdot dP = 0, \) \( dP \cdot dP + \overrightarrow{OP} \cdot d^2P \leq 0 \) at \( P \), hence \( e_{n+1} d^2P = -A_{11j} du^i du^j < 0 \), where \( a \) dot indicates the inner product and \( e_{n+1} \) is the unit vector of the direction \( \overrightarrow{OP} \). Accordingly, \( \nu \) is positive definite at \( P \). We must have the following relation

\[
(A_{1j} A_{1jk} - A_{1jk} A_{1j}) x^i y^j x^h y^k > 0
\]

for any two linearly independent tangent vectors \( x^i, y^j \) to \( V_n \) at \( P \). Hence we obtain from the assumption

\[
\sum_{\beta = 2}^{n} (A_{\beta 1k} A_{\beta 1k} - A_{\beta k} A_{\beta k}) x^i y^j x^h y^k < 0.
\]

By virtue of the corollary, we must have \( N = n - 2 \geq n - 1 \), which is a contradiction.

It is well known that there exists no closed surface with non-positive curvature at every point in a 3-dimensional Euclidean space. According to the argument above, for any compact \( n \)-dimensional Riemann manifold \( V_n \) with the property that at any point, the sectional curvatures for all plane elements are non-positive, the minimum of the dimensions of the Euclidean spaces in which \( V_n \) can be isometrically imbedded is not smaller than \( 2n \). Can such \( V_n \) be always isometrically imbedded in an Euclidean space of dimension \( 2n \)? As is well known, any \( n \)-dimensional Riemann manifold can be locally and isometrically imbedded in an Euclidean space of dimension \( n(n+1)/2 \). For \( n = 2, 3 \), we have \( n(n+1)/2 \leq 2n \). Especially, can a compact 2-dimensional \( V_2 \) everywhere with non positive curvature be always isometrically imbedded in an Euclidean space of dimension \( 4 \)? The author don't know any literatures on this question. In the present paper, we shall show that there exist closed orientable surfaces everywhere with negative curvature in an Euclidean space of dimension \( 4 \). This will show that the question above is reasonable.

Let \( V_2 \) be a compact, orientable Riemann manifold of dimension 2. Let \( ds^2 = \omega_1 \omega_1 + \omega_2 \omega_3 \) be the line element of \( V_2 \). Let \( K \) be the Gaussian total curvature of \( V_2 \). Then, as is well known, we have the equation

\[
\frac{1}{2\pi} \int_{V_2} K \omega_1 \wedge \omega_2 = \chi(V_2)
\]
where $\chi(V_2)$ denotes the Euler-Poincaré characteristic of $V_2$. If we put the genus of $V_2 = \rho$, we have

$$\frac{1}{2\pi} \int_{V_2} K \omega_1 \wedge \omega_2 = 2(1 - \rho).$$

Hence, if $K < 0$, it must be $1 < \rho$ and if $K = 0$, $K = 0$, it must be $2 < \rho$. This shows that the only compact, orientable and flat Riemann manifold of dimension 2 is a topological torus.

1. Tubiform surfaces in an Euclidean space of dimension 4.

Let $\mathcal{C}$ be a curve of class $C^{r_1}$ in a 3-dimensional hyperplane of an Euclidean space $E$ of dimension 4. For any point $\bar{P} \in \mathcal{C}$, let $\{\bar{P}, e_i\}$, $i = 1, 2, 3, 4$, be the Frenet frame of $\mathcal{C}$, that is

$$\begin{align*}
\frac{d\bar{P}}{du} &= du e_1, \\
de_1 &= \sigma(u) du e_2, \\
de_2 &= -\sigma(u) du e_1 + \tau(u) du e_3, \\
de_3 &= -\tau(u) du e_2, \\
de_4 &= 0
\end{align*}$$

(1)

where $u$ denotes the parameter which represents the arc length measured from a fixed point of $\mathcal{C}$ to $\bar{P}(u)$ on $\mathcal{C}$, $\sigma(u)$ and $\tau(u)$ denote the first and second curvatures of the curve respectively.

Making use of a positive function $R(u)$, let us construct a tubiform surface $M$ as

$$\begin{align*}
\bar{P}(u, \theta) &= \bar{P}(u) + R(u)(\cos \theta e_3 + \sin \theta e_4)^n.
\end{align*}$$

(2)

If we take a suitable $R(u)^n$, $M$ is a regular submanifold in $E$. Since we have

$$\begin{align*}
P_u &= \frac{\partial \bar{P}}{\partial u} = e_1 + \dot{R}(\cos \theta e_3 + \sin \theta e_4) - R\tau \cos \theta e_2, \\
P_\theta &= \frac{\partial \bar{P}}{\partial \theta} = R(-\sin \theta e_3 + \cos \theta e_4),
\end{align*}$$

1) As for the differentiability assumptions we suppose that the class $r$ of $\mathcal{C} \geq 4$ and the one of $R(u) \geq 2.
2) P, \bar{P}$ denote also their position vectors.
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(3) \[
\begin{align*}
E &= P_u \cdot P_u = 1 + \dot{R}^2 + R^2 \tau^2 \cos^2 \theta, \\
F &= P_u \cdot P_\theta = 0, \\
G &= P_\theta \cdot P_\theta = R^2
\end{align*}
\]

where the dots indicate derivatives with respect to \( u \), the line element of \( M \) is written as

(4) \[
ds^2 = (1 + \dot{R}^2 + R^2 \tau^2 \cos^2 \theta) du^2 + R^2 d\theta^2.
\]

Let us put

\[
W^2 = EG - F^2 = R^2(1 + \dot{R}^2 + R^2 \tau^2 \cos^2 \theta),
\]

then, according to the well-known formula on the Gaussian total curvature:

\[
K = -\frac{1}{4W^4} \left| \begin{array}{ccc}
E & E_u & E_\theta \\
F & F_u & F_\theta \\
G & G_u & G_\theta
\end{array} \right| - \frac{1}{2W} \left\{ \frac{\partial}{\partial \theta} \left( \frac{E_\theta - F_u}{W} \right) - \frac{\partial}{\partial u} \left( \frac{F_\theta - G_u}{W} \right) \right\},
\]

we have

\[
K = -\frac{1}{2W} \left\{ \frac{\partial}{\partial \theta} \left( \frac{E_\theta}{W} \right) + \frac{\partial}{\partial u} \left( \frac{G_u}{W} \right) \right\}
\]

\[
= -\frac{1}{2W^4} \left\{ W^2 E_{\theta \theta} - \frac{1}{2} (W^2)_\theta E_\theta + W^2 G_{u u} - \frac{1}{2} (W^2)_u G_u \right\}.
\]

Since we have \( E_\theta = -2R^2 \tau^2 \cos \theta \sin \theta \), \( E_{\theta \theta} = -2R^2 \tau^2 (\cos^2 \theta - \sin^2 \theta) \), we get

\[
2W^4 K = -\left\{ GEE_{\theta \theta} - \frac{1}{2} GE_\theta E_\theta + GG_{u u} E - \frac{1}{2} (GE_\theta + G_u E) G_u \right\}
\]

\[
= 2R^4 \tau^2 (1 + \dot{R}^2 + R^2 \tau^2 \cos^2 \theta) (\cos^2 \theta - \sin^2 \theta)
\]

\[
+ 2R^6 \tau^4 \cos^2 \theta \sin^2 \theta
\]

\[
- 2R^2 (R \dot{R} + \dot{R}^2) (1 + \dot{R}^2 + R^2 \tau^2 \cos^2 \theta)
\]

\[
+ 2R \dot{R} \{R \dot{R}(1 + \dot{R}^2) + R^2 \dot{R} \dot{R} + (2R^3 \dot{R} \tau^2 + R^4 \tau \dot{\tau}) \cos^2 \theta \}
\]

\[
= -2R^3 \{\dot{R} + R^2 \tau (1 + \dot{R}^2) \}
\]

\[
+ 2R^4 \{ \tau^2 (2 + 3 \dot{R}^2) + R \dot{R} \tau \dot{\tau} - \tau^2 R \dot{R} \} \cos^2 \theta + 2R^4 \tau \cos^2 \theta,
\]

that is

\[
R^4 (1 + \dot{R}^2 + R^2 \tau^2 \cos^2 \theta)^2 K
\]

(5) \[
= -2R^3 \{ \dot{R} + R^2 \tau (1 + \dot{R}^2) \}
\]

\[
+ 2R^4 \{ \tau^2 (2 + 3 \dot{R}^2) + R \dot{R} \tau \dot{\tau} - \tau^2 R \dot{R} \} \cos^2 \theta + 2R^4 \tau \cos^2 \theta.
\]
When \( r = 0 \) and \( \dot{R} = 0 \), we have \( K = 0 \) by the equation above. Especially, if \( \mathcal{C} \) is a closed curve, it must be \( R = \text{constant} \). Thus we have the following theorem.

**Theorem.** Let \( \mathcal{C} \) be a differentiable closed Jordan curve of class \( C^r \) on a plane \( E^2 \) in \( E^4 \). Let \( M \) be a tubiform surface which is the locus of circles with constant radius, lying in the normal planes to \( E^2 \) through all the points of \( \mathcal{C} \) and whose centers are these points. Then \( M \) has zero Gaussian total curvature.

We can easily see that any locally Euclidean, closed, orientable 2-dimensional Riemann manifold can be isometrically imbedded in \( E^4 \).

Now, since the right hand side of (5) is a quadratic form with respect to \( \cos^2 \theta \), in order that \( K \leq 0 \) for \( 0 \leq \theta \leq 2\pi \), it is necessary and sufficient that

\[
\dot{R} + R \tau^2 (1 + \dot{R}^2) \geq 0,
\]

and

\[
-2R^3 \{\dot{R} + R \tau^2 (1 + \dot{R}^2)\} + 2R^4 \{\tau^2 (2 + 3 \dot{R}^2) + R \ddot{R} \tau^2 - \tau^2 \dot{R} \ddot{R}\} + 2R^6 \tau^4
\]

\[
= -2R^3 (1 + R^2 \tau^2) \dddot{R} + 2R^4 \tau^2 (1 + 2 \dot{R}^2) + 2R^5 \dddot{R} \tau^2 + 2R^6 \tau^4 \leq 0,
\]

that is

\[
R \geq \frac{1}{1 + \tau^2 R^2} \{\tau^2 R (1 + 2 \dot{R}^2) + \tau^2 R^2 \dot{R} + \tau^4 R^3\},
\]

putting \( \cos \theta = 0, 1 \) respectively in the right hand side of (5). Especially, in order that \( K < 0 \) for \( 0 \leq \theta \leq 2\pi \), it is necessary and sufficient to take only the signs of inequality in (6) and (7).

2. A construction of surfaces of negative curvature.

The genus of a closed orientable surface everywhere with negative Gaussian total curvature \( \geq 2 \). Making use of argument stated in the last section, we will construct such a surface of genus 7 in \( E^4 \) as follows.

We take a fixed rectangular coordinate system in \( E^4 \) and let \( E^3 \) be the 3-dimensional subspace spanned by the first three axises of the coordinate system.

Firstly we shall constitute a figure composed of curves as indicated in Fig. 1 which has the properties as follows:
1) The part indicated by full lines lies in $E^3$ and is symmetric with respect to the point $O$. 
2) The line segments $\overline{I_iA_i}$, $\overline{I_iD_i}$, the arc $\overline{I_iB_i}$ lie on a plane, $\overline{OE_i}$ is tangent to the arc $\overline{I_iB_i}$ at $B_i$ and $\angle O\overline{I_iD_i} < \pi/2$. $\overline{I_iD_i}$, $\overline{I_iB_i}$; $\overline{I_iD_i'}$, $\overline{I_iB_i''}$ are obtained from $\overline{I_iD_i}$, $\overline{I_iB_i}$ by rotating these through angles of $2\pi/3$, $4\pi/3$ around the axis $OA_i$ respectively in $E'$.

Fig. 2.

3) The curve $\overline{A_1A_2A_3}$ lies on a plane which contains the line $A_1A_2$ and is parallel to the fourth axis of the coordinate system. The curve $\overline{A_1A_2A_3}$ osculates of a suitable order with the straight line $A_1A_2$ at $A_1$ and $A_2$. 
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4) The curves $\overline{B_iB_3B_4}$, $\overline{B_i'B_3'B_4'}$, $\overline{B_i''B_3''B_4''}$ are related to the straight lines $B_iB_3$, $B_i'B_3'$, $B_i''B_3''$ in the same manner as $\overline{A_iA_3A_2}$ is related to the straight line $A_iA_3$ as stated above.

5) The curve $\overline{D_iD_3D_4}$ lies in the 3-dimensional subspace which contains the plane $(D_iD_3I_iI_3)$ and is parallel to the fourth axis of the coordinate system. The curve $\overline{D_iD_3D_4}$ is spiral and osculates of a suitable order with the lines $I_iD_i$, $I_3D_3$ at $D_i$, $D_3$ respectively. $\angle OD_iI_i$ is an obtuse angle.

6) The curves $\overline{D_i'D_3'D_4'}$, $\overline{D_i''D_3''D_4''}$ have the same property as $\overline{D_iD_3D_4}$.

7) These curves does not intersect each other save for the points $I_i$, $I_3$.

In the next place, we shall constitute surfaces $\overline{\mathcal{F}_i}$, $\overline{\mathcal{F}_3}$ around the points $I_i$, $I_3$ in $E^3$ as indicated in Fig. 2 which have the properties as follow:

1) They have everywhere negative curvature.

2) The sections of the surfaces by the normal planes in $E^3$ to the curves at $A_i$, $B_i$, $B_i'$, $B_i''$, $C_i$, $D_i'$, $D_i''$, $i = 1, 2$, are circles with centers at these points. Along these circles the surfaces are approximately surfaces of revolution.

Lastly, we shall constitute tubiform surfaces everywhere with negative curvature around the curves $\overline{A_iA_3A_2}$, $\overline{B_iB_3B_4}$, $\overline{B_i'B_3'B_4'}$, $\overline{B_i''B_3''B_4''}$, $\overline{D_iD_3D_4}$, $\overline{D_i'D_3'D_4'}$, $\overline{D_i''D_3''D_4''}$ and $\overline{I_iI_3}$ by the manner stated in Section 1.

We shall show in the following section that we can, in fact, construct a surface as stated above.

3. Spiral curves and inequalities (6), (7).

We can join $D_i$, $D_3$ by a suitable helix such that the straight lines $I_iD_i$, $I_3D_3$ are tangent to it at $D_i$, $D_3$ respectively. While the torsion of a helix is a constant $\neq 0$ and the one of a straight line is zero. Accordingly, making use of the curve, the metric of tubiform surface constituted as in Sections 1, 2 around it becomes discontinuous at each point of the circles corresponding to $D_i$, $D_3$ by (4). Hence it is necessary to compensate it so that the continuity of metric and also Gaussian total curvature is preserved at the joints.

Let us consider a spiral curve $\mathcal{C}$ in $E^3$ as follows:
(8) \( x = a \rho(\theta) \cos \theta, \ y = a \rho(\theta) \sin \theta, \ z = a \tan \beta \int_{\theta_0}^{\theta} \sqrt{\rho^2 + \rho'^2} \, d\theta, \)

where the dash denotes the derivative with respect to \( \theta \), \( \beta = \angle OD_1I \), \( a > 0 \), \( a \tan \beta \int_{\theta_0}^{\theta} \sqrt{\rho^2 + \rho'^2} \, d\theta = h = D_1D_2 \) and \( \rho(\theta) \) is a function of class \( C^{r+1} \) as follows:

For small positive constants \( \varepsilon_0, \theta_0, \)

\[
\begin{aligned}
\rho(\theta) &= 0, \\
\rho(\theta) &= \frac{1 + \varepsilon_0}{\cos \theta} \quad \text{for} \quad -\frac{1}{2} \pi < \theta < 0, \ 2\pi < \theta < \frac{5}{2} \pi, \\
\rho(\theta) &= 1 \quad \text{for} \quad \theta_0 < \theta < 2\pi - \theta_0.
\end{aligned}
\]

By the method of construction of the figure, we can take any positive angle near \( 0 \) for \( \beta \).

Let us put

\[
\max_{\theta_0 < \theta < 2\pi} \sqrt{\rho(\theta)^2 + \rho'(\theta)^2} = M.
\]

Then, for the length \( 2l \) of the subarc \( (0 < \theta < 2\pi) \) of \( \mathcal{C} \), we have easily

\[
\begin{aligned}
l &< \pi a M \sec \beta, \\
h &= l \sin \beta < \pi a M \tan \beta.
\end{aligned}
\]

The torsion of the curve \( \mathcal{C} \) is given by the equation

\[
\tau = \frac{|x'y''z' - x''yz'|}{\{(y'z'' - y''z')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2\}^{1/2}}
\]

as is well known. Putting \( \tau = \varphi(\theta, \beta)/a, \varphi(\theta, \beta) \) is of class \( C^1 \) with respect to \( \theta, \beta \) and especially analytic with respect to \( \beta \). Furthermore it has the properties:

\[
\varphi(\theta, \beta) = 0 \quad \text{for} \quad \theta < 0, \ \theta > 2\pi, \\
\varphi(\theta, \beta) = \cos \beta \sin \beta \quad \text{for} \quad \theta_0 < \theta < 2\pi - \theta_0.
\]

Since \( \varphi(\theta, 0) = 0 \), there exists a positive constant \( N_0 \) for \( \beta_0 > 0 \) such that

\[
|\varphi(\theta, \beta)| < N_0\beta \quad \text{for} \quad 0 < \beta < \beta_0.
\]

On the other hand, for the arclength \( u \) of the curve \( \mathcal{C} \) we have
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\[ du = a \sec \beta \sqrt{\rho^2 + \rho'^2} \, d\theta, \]

hence

\[ \frac{d\tau}{du} = \dot{\tau} = \frac{\cos \beta}{a^3} \sqrt{\rho^2 + \rho'^2} \frac{1}{\partial \varphi(\theta, \beta)} \frac{\partial \varphi(\theta, \beta)}{\partial \theta}. \]

Accordingly, there exists a positive constant \( N_i \) such that

\[ |\tau| \leq \frac{N_i \beta}{a}, \quad |\dot{\tau}| \leq \frac{N_i \beta}{a^3} \quad \text{for} \quad 0 \leq \beta \leq \beta_o. \]

Now, by means of (6), (7), it is sufficient to verify that we can choose a function \( R(u) > 0 \) such that

\[ (6') \quad \ddot{R} > 0, \]

\[ (7) \quad (1 + R^2 \tau^2) \ddot{R} > \tau^2 R(1 + 2 \dot{R}^2) + \tau \dot{R} \dot{R} + \tau^4 R^3 \]

for \( 0 \leq u \leq l, \, \ddot{R}(0) = 0 \) and \( \ddot{R}(l) \) is a suitably small positive number. Furthermore, since we have \( \dot{R}(u) > 0 \) for \( 0 < u < l \), making use of (11), (12), it is more sufficient to replace (7) by the inequality

\[ (13) \quad \ddot{R} > \frac{N_i^2 \beta^2}{a^2} R(1 + 2 \dot{R}^2) + \frac{N_i \beta^4}{a^4} R^3 + \frac{N_i N_i \beta^2}{a^6} R^2 \ddot{R}. \]

Now, putting \( R = b_o + b_i u^2, \, b_o, \, b_i > 0 \), we shall verify that we can choose the constants \( b_o, \, b_i \) such that \( R \) satisfies the condition (13) and the requirements stated above are fulfilled.

Substituting \( R = b_o + b_i u^2, \, \ddot{R} = 2b_i u, \, \dot{R} = 2b_i \) in (13), we get

\[ 2b_i > \frac{N_i^2 \beta^2}{a^2} (b_o + b_i u^2) (1 + 8b_i u^2) + \frac{N_i \beta^4}{a^4} (b_o + b_i u^2)^3 \]

\[ + \frac{2N_i N_i \beta^2}{a^2} (b_o + b_i u^2)^2 b_i. \]

By means of (10), it is sufficient for our purpose if we can determine \( b_o, \, b_i \) so that

\[ 2b_i > \frac{N_i^2 \beta^2}{a^2} (b_o + b_i u^2) (1 + 8b_i u^2) + \frac{N_i \beta^4}{a^4} (b_o + b_i u^2)^3 \]

\[ + \frac{2\pi N_i N_i M \beta^2 \sec \beta}{a^3} (b_o + b_i u^2)^2 b_i. \]

We write the inequality above as

\[ B_o - B_i u^2 - B_i u^4 - B_i u^6 > 0 \]

where
To simplify the evaluation, we shall make the function
\[ f(v) = B_0 - B_1 v - B_2 v^2 - B_3 v^3 \]
has no extreme value. The condition is evidently
\[ B_2^2 - 3B_1 B_3 < 0. \]
If we can do so, \( f(v) \) is monotone decreasing, hence \( f(v) > 0 \) for \( 0 < v < l^2 \) when \( f(l^2) > 0 \).
Now, from (14) we have
\[
B_2^2 - 3B_1 B_3 = \frac{N^2_0 \beta^4}{a^2} \left\{ - \frac{3N^4_0 \beta^2}{a^2} + \frac{2AN^2_0 \beta^4}{a^2} b_0 b_1 + 4(4N_0 + \pi N_1 M \sec \beta) b_0^2 \right\} b_1^4.
\]
Since \( b_1 , N_0 , \beta \neq 0 \), if
\[
- \frac{3N^4_0 \beta^2}{a^2} + \frac{2AN^2_0 \beta^4}{a^2} b_0 b_1 + 4(4N_0 + \pi N_1 M \sec \beta) b_0^2 < 0,
\]
\( f(v) \) becomes a monotone decreasing function.
On the other hand, from the circumstance which is described in Sections 1, 2 we may put \( \beta_0 = \pi / 3 \), that is sec \( \beta \leq 2 \). Inequality (15) is satisfied by \( b_0 , b_1 \) such that
\[
\begin{align*}
4(4N_0 + 2\pi N_1 M) b_0^2 < & \frac{3N^4_0 \beta^2}{a^2}, \\
\frac{2AN^2_0 \beta^2}{a^2} b_0 b_1 < & \frac{3N^4_0 \beta^2}{a^2} - 4(4N_0 + 2\pi N_1 M) b_1^2,
\end{align*}
\]
of which the first gives
\[ b_1 < \frac{\sqrt{3} N_0 \beta}{4(2N_0 + \pi N_1 M) a}. \]
If \( f(v) \) is monotonically decreasing, \( f(l^2) > 0 \) holds good when 

\[ f(4\pi^2 a^2 M) > 0 \]

since \( 1 < \pi a M \sec \beta \leq 2\pi a M \). The last inequality is fulfilled furthermore if we have

\[
2b_i - \frac{N_0^2}{a^2} b_o - \frac{N_0}{a^2} b_o^3 - \frac{4N_0 N_1 M}{a^2} b_o b_i^2
\]

\[
- \left\{ \frac{N_0^2}{a^2} b_o + \frac{3N_0}{a^2} b_o^2 \left[ 8\left( \frac{N_0}{a^2} \sec^2 \beta + \frac{\pi N_0 N_1 M}{a^2} \right) b_o b_i^2 \right] (2\pi a M)^2 \\
- \left\{ \frac{N_0^2}{a^2} b_o + 4\left( \frac{2N_0}{a^2} \sec^2 \beta + \frac{\pi N_0 N_1 M}{a^2} \right) b_i^2 \right\} b_o (2\pi a M)^4 \\
- \frac{N_0}{a^2} b_i^2 (2\pi a M)^6 \right\} > 0.
\]

Thus, we see that the conditions (6'), (7) are satisfied by \( R = b_o + b_i u^2 \) such that \( b_i, b_o \) satisfy the inequalities (16), (17), (18).

For a constant \( b_i \) satisfying (17), a sufficiently small positive constant \( b_o \) satisfies (16). If we put \( b_o = 0 \) in (18), we have

\[
2b_i - 4\pi^2 N_0 M^2 \beta^2 b_i - 64\pi^4 a^2 M^4 N_0 \beta^4 (2N_0 + \pi N_1 M) b_i^2
\]

\[
- 64\pi^4 a^2 N_0 M^2 \beta^4 b_i^2 > 2b_i \left[ 1 - 2\pi^2 N_0^2 M^2 \beta^2 \\
- \frac{3N_0}{2(2N_0 + \pi N_1 M)} (2\pi^2 N_0^2 M^2 \beta^2)^3 - \frac{3N_0^3}{4(2N_0 + \pi N_1 M)^3 (2\pi^2 N_0^2 M^2 \beta^2)^3} \right],
\]

making use of (17).

This relation shows that the left hand side of (18) in which we put \( b_o = 0 \) is positive for sufficiently small \( \beta > 0 \). In fact, the quantity inclosed in the brackets in the right hand side of the inequality above is rewritten as

\[
1 + \frac{2(2N_0 + \pi N_1 M)}{9N_0}
\]

\[
- \frac{2(2N_0 + \pi N_1 M)}{9N_0} \left\{ 1 + \frac{3N_0}{2(2N_0 + \pi N_1 M)} (2\pi^2 N_0^2 M^2 \beta^2)^3 \right\}.
\]

Hence, this is positive for \( \beta \) such that

\[
\beta < \left\{ \frac{2N_0 + \pi N_1 M}{3\pi^2 N_0^2 M^2} \left( \left( 1 + \frac{9N_0}{2(2N_0 + \pi N_1 M)} \right)^{\frac{1}{3}} - 1 \right) \right\}^{\frac{1}{3}}.
\]

In conclusion, if we take \( \beta \) satisfying (19), in the next place \( b_i \) satisfying (17) and lastly sufficiently small \( b_o \) so that (16), (18) are fulfilled, then \( R(u) = b_o + b_i u^2 \) satisfies (6), (7).
On the other hand, from the manner in which the surface is constructed, it may be desirable that \( R(l) \) is small. Let \( r, m \) be given positive numbers. We may here require the following conditions for \( R(u) \):

\[
R(l) < r, \quad \dot{R}(l) < m.
\]

They are sufficiently satisfied if we have

\[
b_0 + 4\pi^2a^2M^2b^2_i < r, \\
4\pi a M b^i < m.
\]

Hence the conditions above are satisfied by \( b_0, b_i \) such that

\[
b_1 < \min\left(\frac{m}{4\pi a M}, \frac{\sqrt{r}}{2\pi a M}\right),
\]

\[
b_0 < r - 4\pi^2a^2M^2b^2_i
\]

which clearly do not contradict to the circumstance under which \( b_0, b_i \) have been determined so that (16), (17), (18), (19) hold good. Now, since \( a \) does not appear in (19), we may consider in our construction of the surface that

\[
OD_i = h = l \sin \beta = a \tan \beta \int_0^\pi \sqrt{\rho^2 + \rho'^2} \, d\theta
\]

can take any positive value.

Finally we call attention to the fact that when we take a curve \( \overline{D_1D_2D_3} \) in Fig. 1 such that it is represented as (8), the direction of the second vector of the Frenet frame at \( D_i \) is the one of the fourth axis of the coordinate system, which can be easily verified. This shows that the surfaces \( \overline{\Gamma_1}, \overline{\Gamma_2}, \overline{\Gamma_3} \) and the tubiform surface constructed as above around the curve \( \overline{D_1D_2D_3} \) can be joined smoothly with one another.

From the argument above we obtain the theorem:

**Theorem.** There exist closed surfaces everywhere with negative curvature and of genus 7 in Euclidean space of dimension 4.

**Department of Mathematics,**

**Okayama University**

*(Received January 5, 1954)*