Some remarks on radical ideals

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SOME REMARKS ON RADICAL IDEALS

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Let $R$ be a (non-commutative) ring. As is well known, every ideal $A$ in $R$ determines a uniquely determined ideal $\bar{A}$ called the radical of $A$, which is defined as the intersection of all (minimal) prime divisors of $A$. Clearly, the operation $A \mapsto \bar{A}$ defined in the set $\mathcal{F}$ consisting of all ideals in $R$ possesses the following properties:

1) $A \subseteq \bar{A}$,
2) $\bar{A} = \bar{A}$,
3) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$, where $A, B \in \mathcal{F}$.

In this note, we consider several properties of ideals $A$ with $\bar{A} = A$, which are called radical ideals by N. H. McCoy [3]¹.

Theorem 1. Let $C$ be an ideal in $R$. Then the following conditions are equivalent to each other:

a) $C$ is a radical ideal.

b) If an ideal $A$ is nilpotent modulo $C$, then $A$ is contained in $C$.

c) $AB \subseteq C$ implies $A \cap B \subseteq C$, where $A, B$ are ideals.

d) $C$ is an intersection of some prime ideals.

Proof. Cor. 4 to Th. 2 of [2] shows that the intersection of all the prime ideals of a ring is $\{0\}$ if and only if the zero ideal is the only nilpotent ideal of the ring. Hence, the equivalence between a) and b) may be easily seen.

Let $C$ be a radical ideal. Then, for each prime divisor $P$ of $C$, $AB \subseteq C$ implies $P \supseteq A$ or $B$. Hence $A \cap B \subseteq P$, accordingly, $A \cap B \subseteq C$. This shows that a) implies c). Conversely, $C$ be not a radical ideal. Then, by b), there exists an ideal $A \not\subseteq C$ such that $A^p \subseteq C$. Thus, c) does not hold.

The equivalence of d) to a) is also easy.

In general, let $A \mapsto A^*$ be a operation defined in $\mathcal{F}$ which satisfies the following axioms:

1') $A \subseteq A^*$,
2') $(A^*)^* = A^*$,
3') $B \subseteq A$ implies $B^* \subseteq A^*$, where $A, B$ are in $\mathcal{F}$.

For arbitrary ideals $A, B$ in $R$ we define the join $A \cup B$ as the ideal

¹) Numbers in brackets refer to the references cited at the end of this note. The term "ideal" will mean a two-sided ideal.


\[(A + B)^*\), and the meet \(A \cap B\) as the intersection of \(A\) and \(B\). We set \(\mathcal{C} = \{A \in \mathfrak{A} \mid A^* = A\}\).

**Lemma.** \(\mathcal{C}\) forms a distributive lattice with respect to the above-defined join and meet if one of the following (trivially equivalent) conditions is satisfied:

a) For any \(C \in \mathcal{C}\), \((AB)^* \subseteq C\) implies \(A^* \cap B^* \subseteq C\), where \(A, B \in \mathfrak{A}\).

b) \((AB)^* = A^* \cap B^*\), where \(A, B \in \mathfrak{A}\).

**Proof.** As our operation \(*\) satisfies the axioms 1'), 2') and 3') we have \((A^* + B^*)^* = (A + B)^*\). Let \(A, B\) and \(C\) be in \(\mathcal{C}\), then \((A \cap B) \cup (A \cap C) = ((A \cap B) + (A \cap C))^* = ((AB)^* + (AC)^*)^* = (AB + AC)^*\), where \(AB + AC\) is a radical ideal. This fact shows that our operation \(A \rightarrow \bar{A}\) satisfies the condition a) of the lemma.

Next, we prove the following theorem which has been proved in the commutative case by S. Mori [4, Satz 1]:

**Theorem 3.** The maximum condition is satisfied for radical ideals in \(R\) if and only if the following conditions are satisfied:

1) The maximum condition is satisfied for prime ideals in \(R\).
2) Every radical ideal is represented as the intersection of a finite number of prime ideals in \(R\).

**Proof.** Necessity: As 1) is a special case of our assumption it is desired only to prove 2).

Let \(C = n P_\sigma\) be a radical ideal, where \(P_\sigma\) is a minimal prime divisor of \(C\). Here, we may assume that \(C\) is not a prime ideal. Then, there exist two ideals \(A, B\) not contained in \(C\) such that \(AB \subseteq C\). We can easily see that \(R \supseteq \mathcal{C}_1 = CB^{-1}\), where \(\{P_\sigma\}\) is a subset of \(\{P_\sigma\}\). Clearly \(C_1\) is a radical ideal.

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1) Theorem 2 has been proved for commutative rings by J.-C. Herz [1].
2) \(CB^{-1}\) is defined as the totality of elements \(x\) such that \(xB \subseteq C[Bx \subseteq C]\).

The following properties of quotients are easily verified (see [5]):

(1) \((AB^{-1})^{-1} = A(BC)^{-1}\).
(2) \((A \cap A)^{-1} = \cap A^{-1}\).
(3) For any prime ideal \(P, PA^{-1} = P\) or \(R\) itself in accordance with \(A \notin P\) or \(A \subseteq P\) respectively.
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If \( C_i \) is not prime, then we repeat the above procedure for \( C_i \) instead of \( C \) and obtain \( C_i = CB_{n-1}^{-1} \), and so on. From our assumption, our procedures must terminate after a finite number of steps. Hence we have a prime ideal \( (R \ni) C_n = CB_{n-1}^{-1} \), where \( B_{n-1} \) is not contained in \( C_n \) as \( C \) is a radical ideal. We now consider all different prime ideals \( P'_\mu (\subseteq R) \) of the form \( P'_\mu = CU'_\mu^{-1} \). As \( C \) is a radical ideal \( U'_\mu \) is not contained in \( P'_\mu \), and \( P'_\mu \) is a minimal prime divisor of \( C \). If \( \mu \equiv \lambda \), then \( U'_\mu \subseteq P'_\lambda \). For, if not, \( U'_\mu \not\subseteq P'_\lambda \) and \( P'_\mu U'_\mu \subseteq C \subseteq P'_\lambda \) imply that \( P'_\mu \subseteq P'_\lambda \). As \( P'_\lambda \) is a minimal prime divisor of \( C \), \( P'_\mu = P'_\lambda \), contradicting with our assumption. The fact proved now shows that the representation \( D = \cap P'_\mu \) is irredundant. The maximum condition for radical ideals implies that the set \( \{P'_\mu\} \) is finite.

By the same manner as in the above, we can see that the set of all different prime ideals \( P''_\mu = U''_\mu^{-1} C \) with \( U''_\mu \not\subseteq P''_\mu \) is finite, and we set \( D_2 = \cap P''_\mu \).

Clearly, \( D = D_1 \cap D_2 \) contains \( C \). We shall prove now that \( D = C \).

If \( D \ni C \), as \( U''_\mu D \subseteq C \) and \( U''_\mu \not\subseteq C \), we can construct a prime ideal \( (R \ni) P' = CU' \cdot^{-1} \) with \( U' \not\subseteq P' \), \( \subseteq D \) by using the same argument as in the first part of the proof. Hence, for some \( \mu \), \( P' = P'_\mu \). On the other hand, \( U' \subseteq D \) implies that \( U' \subseteq P'_\mu = P' \). This is a contradiction.

Sufficiency: We assume now the conditions 1) and 2). Let \( C_1 \subseteq C_2 \subseteq \cdots \) be an infinite ascending chain of radical ideals, where \( C_i \) has a short representation \( P_{i_1} \cap \cdots \cap P_{i_n} \) with its minimal prime divisors \( P_{i_j} \). Then each \( P_{i_j} (j = 1, \ldots, n_i) \) is a divisor of some \( P_{i-1, k} (k = 1, \ldots, n_{i-1}) \). Now, we call an ascending chain \( (R \ni) Q_i \subseteq Q_{i+1} \subseteq \cdots \) of which each \( Q_i \) is some \( P_{i_j} \) or \( R \) itself a branch of the chain \( C_1 \subseteq C_2 \subseteq \cdots \), and \( Q_1 \) the starting point of the branch. In case \( Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n \subseteq Q_{n+1} = R \) we say that the length of the branch is \( n \). In the other case, it is infinite. A prime divisor \( P_{i_j} \) of \( C_i \) is called trivial if \( P_{i_j} \) is some \( P_{i_h} \) for each \( h > i \). If the lengths of all branches with the starting point \( P_{i_j} \) of the sub-chain \( C_i \subseteq C_{i+1} \subseteq \cdots \) are bounded, we say that \( P_{i_j} \) is finite, and in the other case it is infinite.

Here, without loss of generality, we may assume that each \( P_{i_j} \) is non-trivial. As \( C_1 \subseteq C_2 \subseteq \cdots \) is infinite, there exists at least one infinite \( P_{i_r} \) for each \( r \).

1) If an ideal is represented as the intersection of a finite number of prime ideals, then there exists a unique short representation [5, Theorem 3].
Let $P_{t_1}$ be infinite. Then there exists a branch $Q_1 = \cdots = Q_{m,t_1}$ with the starting point $P_{t_1}$, such that $Q_{m,t_1} \supseteq \cdots$. As $P_{m_1,t_1}$ is a minimal prime divisor of $C_{m_1}$, for each branch $Q'_1 \subseteq Q'_2 \subseteq \cdots$ with the starting point $Q'_1 = P_{t_1}$, we have $Q'_1 \subseteq Q'_m$. Hence, there exists at least one infinite $P_{m_1,t_2}$ properly containing $P_{t_1}$. We can repeat the above argument for $P_{m_1,t_2}$ instead of $P_{t_1}$ and obtain an infinite $P_{m_2,t_3}$ properly containing $P_{m_1,t_2}$, and so on. Thus, we obtain an infinite ascending chain of prime ideals $P_{t_1} \subseteq P_{m_1,t_2} \subseteq P_{m_2,t_3} \subseteq \cdots$. But this is a contradiction.

References


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