Compactification of Topological Spaces

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COMPACTIFICATION OF TOPOLOGICAL SPACES

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One of the writers\textsuperscript{1} has given the compactification of a $T$-space $R$ as follows:

Let us denote by $R^*$ the totality of all ultrafilters in $R$. Then it is proved that the family

\[ \{U^* \mid U \text{ is an open set in } R\} \]

can be taken as a basis of open sets in $R^*$ and $R^*$ becomes a compact\textsuperscript{2} $T$-space containing the set:

\[ \widehat{R} = \{ \mathcal{F} \mid x \in R; \mathcal{F} \text{ is the ultrafilter containing } x\} \]

as a dense subset, moreover, $\widehat{R}$ is homeomorphic with $R$ by the mapping $\varphi$ defined by

\[ \varphi(x) = \mathcal{F}_x. \]

It is the purpose of this note to make clear some relations among our compactification and Wallman's of a $T_\gamma$-space and Čech's of a completely regular space.

§1. An extension theorem of continuous functions. First of all we shall prove the

Theorem 1. Let $f$ be a real valued bounded continuous function defined on $R$. Then there exists a real valued bounded continuous function $f^*$ defined on $R^*$ such that

\[ f^*(\mathcal{F}) = \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathcal{F}} \inf_{x \in A} f(x) \text{ and } f^*(\mathcal{F}_x) = f(x). \]

\textsuperscript{2) A $T$-space is a topological space which satisfies the conditions:
1) $\emptyset = \emptyset$; 2) $M \supseteq M$; 3) $M \cup N = M \cup N$; 4) $\overline{M} = \overline{M}$, where $M$ and $N$ are subsets of $R$ and $\emptyset$ is the null set as usual.
3) $M^*$ is the totality of all ultrafilters in $R$, which contains $M$.
4) The notation $[A \mid B]$ means the totality of all sets $A$ satisfying the condition $B$.
5) A $T$-space $R$ is called compact if each filter in $R$ has at least one cluster point.
6) In future, we shall denote by $\mathfrak{F}$ a ultrafilter in $R$ and by $\mathfrak{F}_x$ the ultrafilter containing $x$.}
Since \( \varphi \) is the homeomorphism of \( R \) on \( \tilde{R} \), if we regard \( \tilde{R} \) as the space \( R \), then the theorem says that \( f \) can be extended on \( R^* \).

**Proof.** Let \( \mathcal{F} \subseteq R^* \) and let

\[
t = \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x) \quad \text{and} \quad t' = \sup_{A \in \mathcal{F}} \inf_{x \in A} f(x).
\]

Then it is clear that \( t > t' \), but we can show that \( t = t' \). In fact, by definition of \( t \), for every positive number \( \varepsilon \), there exists a set \( A_i \in \mathcal{F} \) such that \( t < \sup_{x \in A_i} f(x) < t + \varepsilon \), and hence for some point \( x_i \in A_i \), \( t - \varepsilon < f(x_i) < t + \varepsilon \). Therefore, if we denote by \( B \) the set \( \{ x \mid t - \varepsilon < f(x) < t + \varepsilon \} \), then \( B \neq \emptyset \). We now show that \( B \in \mathcal{F} \). Since \( \mathcal{F} \) is an ultrafilter, if \( B \in \mathcal{F} \), then \( CB \in \mathcal{F} \) and \( A_i \cap CB \in \mathcal{F} \). Let \( x \) be a point of \( A_i \cap CB \), then \( f(x) < t + \varepsilon \) by definition of \( A_i \), and \( f(x) < t - \varepsilon \) or \( t + \varepsilon < f(x) \) by definition of \( B \). Therefore, we can say that if \( x \in A_i \cap CB \), then \( f(x) < t - \varepsilon \), and hence \( \sup_{x \in A_i \cap CB} f(x) < t - \varepsilon \). This contradicts with the definition of \( t \). Thus we have \( B \in \mathcal{F} \) and \( f(x) < t - \varepsilon \). Therefore, by definition of \( t' \), we have \( t - \varepsilon < t' \), and this shows that \( t < t' \); hence \( t = t' \).

From what we have just proved above, we can define a function \( f^* \) defined on \( R^* \) by the equality

\[
f^*(\mathcal{F}) = \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathcal{F}} \inf_{x \in A} f(x).
\]

It follows evidently that \( f^* \) is bounded and \( f^*(\mathcal{F}) = f(x) \).

To prove that \( f^* \) is continuous, let \( t = f^*(\mathcal{F}) \). In proving that \( t < t' \) above, we have shown that for every positive number \( \varepsilon \), the set \( G = \{ x \mid t - \varepsilon < f(x) < t + \varepsilon \} \) belongs to \( \mathcal{F} \). Since \( f \) is continuous, \( G \) is open in \( R \) and so \( G^* \) is open in \( R^* \). Now, it follows from the definition of \( f^* \) that \( f^*(G) \subseteq [t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}] \subseteq (t - \varepsilon, t + \varepsilon) \). This shows that \( f^* \) is continuous, Q.E.D.

**Remark.** As it is easily seen, if \( f \) is continuous, then for any set \( A \), \( \sup_{x \in A} f(x) = \sup_{x \in A^*} f(x) \) and \( \inf_{x \in A} f(x) = \inf_{x \in A^*} f(x) \); therefore the function \( f^* \) can be defined by the equality

\[
f^*(\mathcal{F}) = \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathcal{F}} \inf_{x \in A} f(x).
\]

1) \( CB \) denote the complement of \( B \), that is, \( CB = R - B \).

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§2. Compactification of a $T_0$-space. In this place, we suppose that $R$ is a $T_0$-space. For a point $\bar{x} \in R^*$ we define $\bar{x}$ by

$$\bar{x} = \{F \mid F \in \mathcal{F} \text{ and } F \text{ is closed in } R\}.$$ 

For two points $\bar{x}_1$ and $\bar{x}_2$ of $R^*$, we write $\bar{x}_1 \sim \bar{x}_2$ if $\bar{x}_1 = \bar{x}_2$. Obviously the relation $\sim$ is an equivalence relation, hence the relation $\sim$ divides $R^*$ into disjoint classes of equivalent points.

We introduce the notations:

- $R^o$ = the totality of all classes of equivalent points;
- $[\bar{x}] = \{\bar{x}_1 \mid \bar{x}_1 \sim \bar{x}\}$, the class of equivalent points, which contains $\bar{x}$.

It is important to remark that if $x \neq y$ then $[\bar{x}] \neq [\bar{y}]$. For, since $x \neq y$ and $R$ is a $T_0$-space, at least one of $x \in y$ and $y \in x$ holds, from which $\bar{x} \neq \bar{y}$; hence it is not hard to see that $\bar{x} \neq \bar{y}$.

We define the mapping $\phi_1$ of the set $R^*$ on the set $R^o$ such that

$$[\bar{x}] = \phi_1(\bar{x}).$$

Then it is evident that $\phi_1$ is one-to-one mapping between $\tilde{R}$ and $\tilde{R}$, by setting

$$\tilde{R}^o = \phi_1(\tilde{R}).$$

Now it is not difficult to see that the family

$$\mathcal{F} = \{F^o \mid \phi_1^{-1}(F^o) = F^*, \text{ where } F \text{ is closed in } R\}$$

can be taken as a basis of closed sets in $R^o$, and, moreover, $R^o$ becomes a $T$-space and $\phi_1$ is continuous.

From this definition, we can prove the

**Theorem 2.** $R^o$ is a compact $T_0$-space and contains a dense subset $\tilde{R}$ which is homeomorphic with $R$.

**Proof.** $R^o$ is compact, because $R^*$ is compact, and $\phi_1$ is continuous in the topology introduced in $R^o$.

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1) A $T_0$-space is a $T$-space such that, for any two different points $x$ and $y$, at least one of $x \not\sim y$ or $y \not\sim x$ holds.
To show that $R^o$ is a $T_v$-space, we shall prove first that
\[ \phi_1^{-1}(\phi_1(U^*)) = U^* \text{ for any open set } U \text{ in } R. \]
In fact, if $[\xi] \in \phi_1^{-1}(\phi_1(U^*))$, there exists a point $\xi_1 \in U^*$ such that $[\xi] = [\xi_1]$. Hence $U \in [\xi_1]$ and $CU \in [\xi_1]$. Since $CU$ is closed, we have $CU \subseteq [\xi_2]$; and so $U \subseteq [\xi_2]$. This shows that $\phi_1^{-1}(\phi_1(U^*)) \subseteq U^*$. Since it is evident that $\phi_1^{-1}(\phi_1(U^*)) \Rightarrow U^*$, we have $\phi_1^{-1}(\phi_1(U^*)) = U^*$.

Under this remark, we shall show that $R^o$ is a $T_v$-space. Let $[\xi_1]$ and $[\xi_2]$ be two different points of $R^o$, then $[\xi_1] \neq [\xi_2]$. Hence, at least one of $[\xi_1]$ and $[\xi_2]$, say $[\xi_1]$, contains a closed set $F$ such that $F \subseteq [\xi_1]$, and so $CF \subseteq [\xi_1]$. Since $CF$ is open, we get $\phi_1^{-1}(\phi_1(CF^*)) = (CF^*)$, and clearly $[\xi_1] \in \phi_1(CF^*)$ and $[\xi_2] \in \phi_1(CF^*)$. Hence it follows that $R^o$ is a $T_v$-space.

In order to show that $R$ and $R^o$ are homeomorphic, as it is readily seen, it is sufficient to prove that the mapping $\phi_1 \varphi$ sends an open set in $R$ to an open set in $R^o$. Now let $G$ be an open set in $R$, then
\[ \phi_1 \varphi(G) = \phi_1(G^* \cap \tilde{R}) = \phi_1( \bigcup_{[\xi] \subseteq G^*} [\xi] ) = \bigcup_{[\xi] \subseteq G^*} \phi_1([\xi]) = \tilde{R}^o \cap \phi_1(G^*), \]
from which $\phi_1 \varphi(G)$ is open in $R^o$, because $\phi_1(G^*)$ is open in $R^o$, Q.E.D.

**Theorem 3.** Let $f$ be a real valued bounded continuous function defined on $R$. Then there exists a real valued bounded continuous function $f^o$ defined on $R^o$ such that
\[ f^o([\xi]) = f^*(\xi). \]

**Proof.** Since $f$ is continuous, the function $f^*$ defined by the equation (1) in §1 takes the same value at each point which belongs to an equivalence class in $R^*$. Thus we can define a function $f^o$ such that
\[ f^o([\xi]) = f^*(\xi). \]

To prove that $f^o$ is continuous, let $[\xi]$ be a point of $R^o$ and $t = f^o([\xi]) = f^*(\xi)$. Since $f^*$ is continuous, for every neighborhood $U(t)$ of $t$, there exists an open set $G$ in $R$ such that $G \subseteq [\xi]$ and $U(t) \Rightarrow f^*(G^*)$. On the other hand, as we proved in the proof of Theorem 2, $G \subseteq [\xi]$ implies $[\xi] \subseteq G^*$ and $\phi_1(G^*)$ is open in $R^o$. Hence
Let \( U(t) \supseteq f^*(G^*) = f^*(\phi_t(G^*)) \), this shows that \( f^* \) is continuous, Q.E.D.

Let us denote by \( \alpha(R) \) the totality of all dual prime ideals in the lattice \( \mathcal{L} \) composed of all closed sets in \( R \), then we have the

**Lemma 1.** There is an one-to-one mapping of the set \( R^\circ \) on \( \alpha(R) \).

**Proof.** To a point \( [\overline{X}] \in R^\circ \) we correspond the set \( \overline{X} \), and we write

\[
\overline{X} = \psi([\overline{X}]).
\]

First of all we prove that \( \overline{X} \) is a dual prime ideal. It is evident that \( \overline{X} \) is a dual ideal. In order to show that \( \overline{X} \) is prime, let \( F_1 \) and \( F_2 \) be closed sets in \( R \) such that \( F_1 \cup F_2 \in \overline{X} \). If we suppose that \( F_1 \in \overline{X} \), then, since \( F_1 \) is closed and \( \overline{X} \) is an ultrafilter in \( R \), we have \( F_1 \in \overline{X} \) and so \( CF_1 \in \overline{X} \). Hence it follows that \( CF_1 \cap (F_1 \cup F_2) \in \overline{X} \), from which \( F_1 \in \overline{X} \) and \( F_2 \in \overline{X} \), since \( CF_1 \cap (F_1 \cup F_2) \subseteq F_2 \) and \( F_2 \) is closed. This shows that \( \overline{X} \) is a dual prime ideal, and hence \( \overline{X} \in \alpha(R) \). Moreover, it is evident that if \( [\overline{X}_1] \neq [\overline{X}_2] \), then \( \psi([\overline{X}_1]) \neq \psi([\overline{X}_2]) \).

We shall now prove that \( \psi(R^\circ) = \alpha(R) \). In fact, let \( M \) be a dual prime ideal in \( \mathcal{L} \) and let

\[
\mathcal{R} = \{ G \mid G \text{ is open in } R \text{ and } CG \subseteq M \}.
\]

Since \( R \in \mathcal{M} \), we have \( \phi \in \mathcal{R} \). Next, in order to show that \( \mathcal{R} \) has the finite intersection property, take two sets \( G_1 \) and \( G_2 \) of \( \mathcal{R} \). Then \( CG_1 \in \mathcal{M} \), \( CG_2 \in \mathcal{M} \) and \( \mathcal{M} \cap CG_1 \cup CG_2 = C(G_1 \cap G_2) \), since \( \mathcal{M} \) is prime. Therefore, \( G_1 \cap G_2 \in \mathcal{R} \) and \( G_1 \cap G_2 \neq \phi \), from which we say that \( \mathcal{R} \) has the finite intersection property. Now let \( F \in \mathcal{R} \) and \( G \in \mathcal{R} \). If we suppose that \( F \cap G = \phi \), then \( F \subseteq CG \) and so \( CG \in \mathcal{R} \). This contradicts with \( CG \in \mathcal{M} \), and hence \( F \cap G \neq \phi \). From what we have proved above, we can say that the totality of all sets \( F \cap G \), where \( F \in \mathcal{M} \) and \( G \in \mathcal{R} \), forms a basis of a filter. Hence there exists an ultrafilter \( \overline{X} \) which contains the above basis: \( M \cup R \subseteq \overline{X} \). For this ultrafilter \( \overline{X} \), we can show that \( \overline{X} = \mathcal{R} \). In fact, if \( F \in \overline{X} \), then \( F \subseteq \overline{X} \) and \( CF \subseteq \overline{X} \), from which \( CF \subseteq \mathcal{R} \). Hence \( F \subseteq \mathcal{R} \) by definition of \( \mathcal{R} \), and so \( \overline{X} \subseteq \mathcal{R} \).

From what we have just proved, it follows that \( \psi(R^\circ) = \alpha(R) \) and \( \psi \) is an one-to-one mapping between \( R^\circ \) and \( \alpha(R) \), Q.E.D.

We note here that a set \( F^\circ \subseteq R^\circ \) belongs to the closed basis \( \mathcal{F} \) of \( R^\circ \), if and only if there exists a closed set \( F \) in \( R \) such that \( \psi(F^\circ) = \alpha(F) \) by setting

\[
\alpha(F) = \{ M \mid M \text{ is a dual prime ideal in } \mathcal{L} \text{ and } F \in \mathcal{M} \}.
\]
In fact, if $F^\circ$ belongs to $\mathcal{F}$, there exists a closed set $F$ in $R$ such that $\phi^{-1}_I(F^\circ) = F^*$. Hence $\psi_1(F^\circ) = \psi_1(\phi^{-1}_I(F^*)) = \psi_1(\{F \mid F \in \mathcal{F}_I\}) = \alpha(F)$.

Conversely, let $F$ be a closed set in $R$ and $\psi(F^\circ) = \alpha(F)$. Then, since $\psi$ is one-to-one, we have $\phi^{-1}_I(F^\circ) = \phi^{-1}_I(\psi^{-1}(\psi(F^\circ))) = \phi^{-1}_I(\psi^{-1}(\alpha(F))) = \phi^{-1}_I(\{F \mid F \in \mathcal{F}_I\}) = F^*$, and hence $F^\circ$ is contained in the closed basis $\mathcal{F}'$ of $R^\circ$.

Thus we have the

**Lemma 2.** If we introduce the topology in $\alpha(R)$ such that the family $\{\alpha(F) \mid F \text{ is closed in } R\}$ is a closed basis of $\alpha(R)$, then the mapping $\psi(\mathcal{F}_I) = \mathcal{F}_I$ is a homeomorphism of the space $R^\circ$ on the space $\alpha(R)$.

§ 3. Compactification of a $T_1$-space. In this section, we suppose that $R$ is a $T_1$-space. Now let

$$
\beta^\circ(R) = \{\mathcal{F}_I \mid \mathcal{F}_I \text{ is a maximal dual ideal in } \mathcal{F}\},
$$

$$
\beta(R) = \psi(\beta^\circ(R)).
$$

Then, it is evident that $\beta(R)$ is a subset of $\alpha(R)$ and consists of all maximal dual prime ideals in $\mathcal{F}$. Moreover, since $\beta^\circ(R)$ and $\beta(R)$ are regarded as the subspaces of $R^\circ$ and $\alpha(R)$ respectively, $\beta^\circ(R)$ and $\beta(R)$ are homeomorphic with each other.

Since $R$ is a $T_1$-space, it is important to remark that $[\mathcal{F}_I]$ is $\mathcal{F}_I$ itself and hence $\tilde{R} = R^\circ \subset \beta^\circ(R)$.

Under these remarks, we have the well known

**Wallman's Theorem.** The space $\beta(R)$ is a compact $T_1$-space and contains a dense subset $\tilde{R}$ which is homeomorphic with $R$.

But we give a proof of this theorem for the purpose to make clear the relation among the spaces considered in this note.

**Proof.** Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be any two distinct points of $\beta(R)$. Then, since they are maximal ideals in $\mathcal{F}$, any one of them, say $\mathcal{W}_1$, contains a closed set $F$ in $R$ such that $F \in \mathcal{W}_2$. However, $\beta(R) - \alpha(F)$ is open in $\beta(R)$ and contains $\mathcal{W}_2$ and not $\mathcal{W}_1$, hence $\beta(R)$ is a $T_1$-space.

To show that $\beta(R)$ is compact, we take an ultrafilter $F$ in $\beta(R)$. Obviously, since $F$ is a filter in $\alpha(R)$ which is compact, there is a cluster point $\mathcal{F}_I$ of $F$. As we know, there is an ultrafilter $\mathcal{F}_I$ in $R$ such that $\mathcal{F}_I \in \beta^\circ(R)$ and $\mathcal{F}_I \subset \mathcal{F}_I$. Since $\mathcal{F}_I \subset \mathcal{F}_I$, for any closed set $F$ in $R$, $\alpha(R) - \alpha(F) \ni \mathcal{F}_I$ implies $\alpha(R) - \alpha(F) \ni \mathcal{F}_I$, and hence we can say...
that in the space $\alpha(R)$ each neighborhood of $\mathcal{F}_1$ is also a neighborhood of $\mathcal{F}$. Hence $\mathcal{F}_1$ is a cluster point of $F$, and therefore $\beta(R)$ is compact.

It is almost evident that the subset $\mathcal{R}^c = \mathcal{R}$ of $\beta(R)$ is dense and homeomorphic with $R$, Q.E.D.

By using Theorem 3, we can prove the

**Theorem 4.** A real valued bounded continuous function $f$ defined on $R$ is extendable to a real valued bounded continuous function $f_\beta$ defined on $\beta(R)$ such that

$$f_\beta(\mathcal{R}) = f^*(\mathcal{R}).$$

Finally we give the

**Theorem 5.** In order that $\beta(R)$ be normal, it is necessary and sufficient that $R$ be normal.

*Proof.* Suppose that $\beta(R)$ is normal, and let $F_1$ and $F_2$ be two disjoint closed sets in $R$. Then, the sets $F_\beta = \beta(R) \cap \alpha(F_1)$ and $F_\beta = \beta(R) \cap \alpha(F_2)$ are disjoint closed sets in $\beta(R)$. Hence there exists a continuous function $f_\beta$ defined on $\beta(R)$ such that $f_\beta = 0$ on $F_\beta$, $f_\beta = 1$ on $F_\beta$ and $0 < f_\beta < 1$ on $\beta(R)$. If we define a function $f$ by the equality $f(x) = f_\beta(\mathcal{F}_1)$, it is clear that $f$ is continuous and $0 < f < 1$ on $R$. Moreover, if $x \in F_1$, then $\mathcal{F}_1 \in \beta(R) \cap \alpha(F_1)$ and hence $f(x) = f_\beta(\mathcal{F}_1) = 0$. Similarly, if $x \in F_1$, then $f(x) = 1$. This shows that $R$ is normal.

Conversely, let $R$ be normal and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two distinct points in $\beta(R)$. Then, as $\mathcal{M}_1$ and $\mathcal{M}_2$ are maximal dual ideals in $\mathcal{F}$, there exist disjoint closed sets $F_1$ and $F_2$ such that $F_1 \in \mathcal{M}_1$ and $F_2 \in \mathcal{M}_2$. Since $R$ is normal, there exists a continuous function $f$ such that $f(x) = 0$ on $F_1$, $f(x) = 1$ on $F_2$ and $0 < f < 1$ on $R$. Let $f_\beta$ be the function extended from $f$ by Theorems 1 and 4, then it is clear that $f_\beta(\mathcal{M}_1) = 0$ since $F_1 \in \mathcal{M}_1$, and similarly $f_\beta(\mathcal{M}_2) = 1$. This shows that $\beta(R)$ is a Hausdorff space, and hence, as $\beta(R)$ is compact, $\beta(R)$ is normal, Q.E.D.

§ 4. Compactification of a completely regular space. In this section, we suppose first that $R$ is a complete Hausdorff space. By considering the remark in §1, it is easily seen that, for two points

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1) We means by a complete Hausdorff space the Hausdorff space such that, for any two distinct points $x$ and $y$, there exists a real valued bounded continuous function taking different values at $x$ and $y$. 

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The following propositions are equivalent:

(α). There is no real valued continuous function $f^*$ defined on $\mathbb{R}^*$ such that $f^*(\mathcal{F}_1) = 0$, $f^*(\mathcal{F}_2) = 1$ and $0 < f^* < 1$ on $\mathbb{R}^*$.

(β). Any real valued bounded continuous function $f^*$ defined on $\mathbb{R}^*$ takes the same value at $\mathcal{F}_1$ and $\mathcal{F}_2$.

If two points $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfies the proposition, we write $\mathcal{F}_1 \sim \mathcal{F}_2$.

Evidently the relation $\sim$ is an equivalence relation, hence the relation divides $\mathbb{R}^*$ into disjoint classes of equivalent points.

We introduce the notations:

\[ \tau(\mathbb{R}) = \text{the totality of all classes of equivalent points}; \]
\[ \{\mathcal{F}\} = \text{the class which contains } \mathcal{F}. \]

Moreover, we define the mapping $\phi_2$ of $\mathbb{R}^*$ on $\tau(\mathbb{R})$ such that

\[ \phi_2(\mathcal{F}) = \{\mathcal{F}\}. \]

We shall give the

**Lemma 3.** If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $f^*(\mathcal{F}_1) = f^*(\mathcal{F}_2)$, for every real valued bounded continuous function $f^*$ on $\mathbb{R}^*$.

**Proof.** Let $f(x) = f^*(\mathcal{F}_1)$, then $f$ is a real valued bounded continuous function and equality (1) in §1 holds. On the other hand, since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, it follows that $\inf A \sup f(x) \geq \inf A \sup f(x)$ and therefore, it is clear that $f^*(\mathcal{F}_1) = f^*(\mathcal{F}_2)$. Q.E.D.

From the lemma, it is not difficult to see that:

(a). $[\mathcal{F}] \subseteq \{\mathcal{F}\}$;

(b). $\{\mathcal{F}\}$ contains a point $\mathcal{F}$, such that $[\mathcal{F}] \in \beta^0(\mathbb{R})$.

Since $\mathbb{R}$ is a complete Hausdorff space, for two distinct points $x$ and $y$, there exists a continuous function $f$ such that $f(x) = 0$, $f(y) = 1$ and $0 \leq f \leq 1$ on $\mathbb{R}$. Then, by the equality (1) in §1, $f^*(\mathcal{F}_1) = 0$, $f^*(\mathcal{F}_2) = 1$ and $0 \leq f^* \leq 1$. Thus we have

(c). If $x \neq y$, then $\phi_2(\mathcal{F}_1) \neq \phi_2(\mathcal{F}_2)$.

Hence, if we put

\[ \tilde{\tau}(\mathbb{R}) = \phi_2(\tilde{\mathbb{R}}), \]

then $\phi_2$ gives an one-to-one correspondence between $\tilde{\mathbb{R}}$ and $\tilde{\tau}(\mathbb{R})$.

Moreover, if we take the set:

\[ \tilde{\tau}(\mathbb{R}) \]
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{Fγ | φ⁻¹(γ) is closed in \( R^* \)},

as the totality of all closed sets in \( r(R) \), then \( r(R) \) is a T-space and \( φ_2 \) is a continuous mapping.

Thus we can prove more precisely the

**Theorem 6.** The space \( r(R) \) is a compact Hausdorff space and contains a dense subset \( \overline{r}(R) \). Moreover, a real valued bounded continuous function \( f \) defined on \( R \) can be extended to the function \( f_r \) defined on \( r(R) \) such that

\[
f_r([G]) = f^*(G).
\]

**Proof.** Since \( φ_2 \) is continuous, it follows that \( r(R) \) is compact.

Let \( \{G_1\} \) and \( \{G_2\} \) be two distinct points in \( r(R) \). Then \( G_1 \neq G_2 \) and, therefore, there exists a continuous function \( f^* \) defined on \( R^* \) such that \( f^*(G_1) = 0, f^*(G_2) = 1 \) and \( 0 \leq f^* \leq 1 \) on \( R^* \). Hence, if we put

\[
U_i^* = f^{-1}(\left[ 0, \frac{1}{2} \right]) \quad \text{and} \quad U_i^* = f^{-1}(\left( \frac{1}{2}, 1 \right]),
\]

then \( U_i^* \) and \( U_i^* \) are disjoint open sets in \( R^* \). Moreover, it is evident, from the definition of the relation \( \approx \), that if \( G \in U_i^* \) then \( \{G\} \in U_i^* \), from which it is easily seen that \( U_i^* = φ_2^{-1}(φ_i(U_i^*)) \), \( i = 1, 2 \). Hence, it follows that \( φ_i(U_i^*) \) and \( φ_i(U_i^*) \) are disjoint open sets in \( r(R) \) such that \( \{G_i\} \in φ_i(U_i^*) \) and \( \{G_2\} \in φ_2(U_i^*) \). Thus \( r(R) \) is a Hausdorff space.

To prove that \( f_r \) is the function extended from \( f \), it is sufficient to verify that \( f_r \) is continuous. Now, let \( \{G\} \) be a point of \( r(R) \), \( t = f_r(\{G\}) \), and let \( U(t) \) be an open set containing \( t \). Then, since \( f_r(\{G\}) = f^*(G) \), the set \( U^* = f^{-1}(U(t)) \) is open in \( R^* \) which contains \( G \). As it is easily seen, from \( U^* = f^{-1}(U(t)) \) and the definition of \( f_r \), that \( f_r^{-1}(U(t)) = φ_i(f^*(U(t))) = φ_i(U_i^*) \). As we proved above, \( φ_i(U_i^*) \) is open in \( R(R) \), which contains \( \{G\} \), thus \( f_r \) is continuous, Q.E.D.

**Remark.** In the same manner as we used above, we can define two spaces \( \overline{r}(R) \) and \( \overline{r}^o(R) \) from \( R^o \) and \( β^o(R) \) respectively. That is, if \( φ \) and \( φ_i \) are the mapping of \( R^o \) on \( \overline{r}(R) \) and that of \( β^o(R) \) on \( \overline{r}^o(R) \) respectively, then

\[
φ_i([G]) = φ_i(φ_i^{-1}([G])), \quad G \in R^o; \quad \text{and} \quad \phi_i([G]) = φ_i(φ_i^{-1}([G])), \quad G \in β^o(R).
\]

Therefore, if we denote by \( ψ \) and \( ψ_i \) the mapping of \( r(R) \) on \( \overline{r}(R) \) and that of \( r(R) \) on \( r^o(R) \) respectively, such that \( ψ_i(\{G\}) = φ_i(\{G\}) \),
\( \psi_r(\{\beta\}) = \phi_r(\{\beta\}) \), then by considering the properties (a) and (b), we can prove that \( \tau(R) \), \( \tau_1(R) \) and \( \tau^o(R) \) are homeomorphic with each other.

The space \( \tau(R) \) in the Theorem 6 is a continuous image of \( \bar{R} \), but not necessarily homeomorphic with \( \bar{R} \). As the condition of that \( \tau(R) \) and \( \bar{R} \) be homeomorphic with each other, we have the

**Lemma 4.** In order that \( \tau(R) \) and \( \bar{R} \) be homeomorphic, it is necessary and sufficient that \( R \) be a completely regular space.

**Proof.** The necessity is evident.

Conversely, suppose that \( R \) be completely regular and we shall show that the mapping \( \phi^{-1}_2 \) of \( \tau(R) \) on \( \bar{R} \) is continuous.

Let \( F \) be a closed set of \( R \) and \( \{\gamma_0\} \) be a point of \( \tau(R) - \phi_2(F* \cap \bar{R}) \). Then \( x \) does not belong to \( F \), and, since \( R \) is completely regular, there exists a real valued continuous function defined on \( R \) such that \( f(x) = 0, f(y) = 1 \) for every point \( y \in F \) and \( 0 \leq f \leq 1 \). Let \( f_\gamma \) be the function extended from \( f \) by the Theorems 1 and 6, then it is clear that \( f_\gamma(\{\gamma_0\}) = 0 \) and \( f_\gamma(\{\gamma\}) = 1 \) for every point \( \{\gamma\} \in \phi_2(F*) \). This implies that the open set \( \{\gamma\} \cap [0, \frac{1}{2}] \) of \( \tau(R) \) contains \( \{\gamma_0\} \) and does not intersect with \( \phi_2(F*) \), and hence the open set \( f_\gamma^{-1}(0, \frac{1}{2}) \cap \tau(R) \) of \( \tau(R) \) contains \( \{\gamma_0\} \) and is contained in \( \tau(R) - \phi_2(F* \cap \bar{R}) \). This shows that \( \tau(R) - \phi_2(F* \cap \bar{R}) \) is open in \( \tau(R) \) and so \( \phi_2(F* \cap \bar{R}) \) is closed in \( \tau(R) \), Q.E.D.

Thus, as we know, there is the well known

Čech's Theorem. For a completely regular space \( R \), there is a space \( W \) satisfying the following conditions:

1. \( W \) is a compact Hausdorff space;
2. \( R \subseteq W \) and \( \bar{R} = W \);
3. Any real valued bounded continuous function defined on \( R \) can be extended on \( W \).

Moreover, the spaces which satisfies the three conditions given above are homeomorphic with each other.

In this place, we will give a proof of this theorem for the purpose to make clear the structure of the space \( W \).

**Proof.** The space \( \tau(R) \) is certainly a space satisfying the conditions (1), (2) and (3), thinking \( \tau(R) = \bar{R} \) be \( R \). Hence the existence is true.
Let $W$ be a space satisfying the conditions (1), (2) and (3), and we will prove that $W$ and $\tau(R)$ are homeomorphic, by dividing the proof into seven parts.

(a). Let $M$ be a subset of $W$. Then, by using the condition (1), it is not difficult to see that an ultrafilter $\mathcal{F}(M)$ in $M$ converges to only one point $w$ which belongs to $\bar{M}$, and conversely, if $w \in \bar{M}$, then there exists an ultrafilter $\mathcal{F}(M)$ in $M$ such that $\mathcal{F}(M)$ converges to $w$.

(b). Let $g(w)$ be a real valued bounded continuous function defined on $W$. Let $\mathcal{F}$ be an ultrafilter in $R$ such that $\mathcal{F}$ converges to a point $w$ and let $g(w) = t$. Since $g(w)$ is continuous, for every neighborhood $U(t)$ of $t$ there exists a neighborhood $V(w)$ such that $U(t) \ni g(V(w))$, and from which we have:

$$g(w) = \inf_A \sup_{x} g(x) = \sup_A \inf_{x} g(x).$$

If we define a function $f$ such that

$$f(x) = g(x), \ x \in R \subset W,$$

then $f(x)$ is continuous and the function $f^*$ defined by

$$f^*(\mathcal{F}) = \inf_A \sup_{x} f(x) = \sup_A \inf_{x} f(x)$$

is the function obtained in the Theorem 1. Hence it is evident that if $\mathcal{F}$ converges to $w$, then

$$g(w) = f^*(\mathcal{F}).$$

Conversely, let $f^*$ be a continuous function defined on $R^*$. Then the function $f$ defined by $f(x) = f^*(\mathcal{F}_w)$ is continuous on $R$. On the other hand, by the condition (3), there exists a continuous function $g(w)$ defined on $W$ such that $f(x) = g(x)$ for $x \in R$. Since $g(w)$ is continuous, from what we have proved above, we have

$$g(w) = f^*(\mathcal{F}),$$

for any ultrafilter $\mathcal{F}$ in $R$, which converges to $w$.

(c). For a point $w \in W$, if we denote by $\{\mathcal{F}\}_w$ the family of all ultrafilters $\mathcal{F}$ in $R$ which converges to $w$ in $W$, then $R^*$ is divided into disjoint classes $\{\mathcal{F}\}_w$. From (b), it follows that $\{\mathcal{F}\}_w \subset \{\mathcal{F}\}$. 

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(d). Let $w_1$ and $w_2$ be two distinct points of $W$, then there exists a real valued bounded continuous function $g(w)$ defined on $W$ such that $g(w_1) \neq g(w_2)$. On the other hand, by (a) and the condition (2), there exist two ultrafilters $\mathcal{F}_1$ and $\mathcal{F}_2$ in $R$ such that $\mathcal{F}_1$ converges to $w_1$ and $\mathcal{F}_2$ converges to $w_2$. Therefore, by (b), it follows that $f^*(\mathcal{F}_1) \neq f^*(\mathcal{F}_2)$, from which $\mathcal{F}_1 \neq \mathcal{F}_2$. This shows that $\{\mathcal{F}\} = \{\mathcal{F}_m\}$ for every point $w \in W$. Then we have, for $\{\mathcal{F}\}$ there exists a point $w \in W$ such that $\{\mathcal{F}\} = \{\mathcal{F}_m\}$.

(e). By (d), we can define the function $\phi_2$ of $W$ on $\tau(R)$ such that

$$\phi_2(w) = \{\mathcal{F}\}.$$

It is evident that $\phi_2$ is one-to-one.

(f). We shall prove that $\phi_2$ is continuous. Let $\phi_2(w_0) = \{\mathcal{F}_0\}$ and $U_\gamma$ be an open set of $\tau(R)$ containing $\{\mathcal{F}_0\}$. By the normality of $\tau(R)$, there exists a real valued continuous function $f_\gamma$ defined on $\tau(R)$ such that $f_\gamma(\{\mathcal{F}_0\}) = 0$, $f_\gamma(\{\mathcal{F}\}) = 1$ on $\tau(R) - U_\gamma$ and $0 \leq f_\gamma \leq 1$ on $\tau(R)$. If we define the function $f^*$ on $\tau(R)$ such that $f^*(\{\mathcal{F}\}) = f_\gamma(\{\mathcal{F}\})$, then $f^*$ is a real valued bounded continuous function on $\tau(R)$, and, by (b), we get a continuous function $g$ defined on $W$.

We shall show that the open set $V = g^{-1}(\{0, \frac{1}{2}\})$ of $W$ contains $w_0$ and $\phi_2(V) \subseteq U_\gamma$. By (b), $g(w_0) = f^*(\mathcal{F}_0) = f_\gamma(\{\mathcal{F}_0\}) = 0$ and so $V$ contains $w_0$. Let $w$ be a point of $V$, then $f_\gamma(\phi_2(w)) = f^*(\phi_2^{-1}(\phi_2(w))) = g(w) \in (0, \frac{1}{2})$ and hence $\phi_2(w)$ does not belong to $\tau(R) - U_\gamma$ and this shows that $\phi_2(w) \in U_\gamma$. Thus the proof of the continuity of $\phi_2$ is established.

(g). Since $W$ is compact and $\tau(R)$ is a Hausdorff space, from (e) and (f), the mapping $\phi_2$ is a homeomorphism. Thus the theorem is completely proved, Q.E.D.

Remark. Finally, if $\tau(R)$ is homeomorphic with $\beta(R)$, then $\beta(R)$ is normal. Then, by Theorem 5, the space $R$ is normal.

Conversely, let $R$ be normal and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two distinct points of $\beta(R)$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are maximal dual ideals in $\mathcal{L}$, there exists two distinct closed sets $F_1$ and $F_2$ in $R$ such that $F_1 \in \mathcal{F}_1$, $F_1 \in \mathcal{F}_2$, $F_2 \in \mathcal{F}_1$, and $F_2 \in \mathcal{F}_2$. Hence it is evident that $\beta(R) - \alpha(F_1)$ and $\beta(R) - \alpha(F_2)$ are disjoint open sets in $\beta(R)$ and the former contains $\mathcal{F}_2$, the latter $\mathcal{F}_1$. Then $\beta(R)$ is a Hausdorff space. Hence $\beta(R)$
satisfies the three conditions (1), (2) and (3) given in the Čech's Theorem, and, therefore, $r(R)$ is homeomorphic with $\beta(R)$.

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